# Applications of the Mach number

## 1 Definitions

$\rho =$ Air density ($kg/m^3$)
$A =$ Area ($m^2$)
$a =$ Speed of sound ($m/s$)
$p =$ Pressure ($Pa = N/m^2$)
$R =$ Gas constant (Value is: $287J/(kgK)$) ($J/(kgK)$)
$T =$ The temperature of a certain amount of gas ($K$)
$V =$ Velocity ($m/s$)
$M =$ Mach number (dimensionless)
$\gamma =$ Ratio of specific heats (Value for normal air is: 1.4) (dimensionless)
$c_v =$ Specific heat at constant volume (For air: $c_v = 717J/(kgK)$) ($J/(kgK)$)
$c_p =$ Specific heat at constant pressure (For air: $c_p = 1004J/(kgK)$) ($J/(kgK)$)
$V_{cal} =$ Calibrated flight velocity ($m/s$)

## 2 Speed of sound

The speed of sound is an important thing in aerodynamics. A formula to calculate it can be derived. Therefore the continuity equation is used on a sound wave traveling through a tube with constant area:

$$\rho A a = (\rho + d\rho)A(a + da) \Rightarrow a = -\rho \frac{da}{d\rho}$$

Assuming an isentropic flow, it is allowed to fill in the speed of sound in the Euler equation, which gives $dp = -\rho\, a\, da$. Combining this with the previous equation gives:

$$a = \sqrt{\frac{dp}{d\rho}} \tag{2.1}$$

This is not a very easy formula to work with though. But by using the isentropic flow relations, it can be derived that $\frac{dp}{d\rho} = \gamma \frac{p}{\rho}$, and therefore:

$$a = \sqrt{\gamma \frac{p}{\rho}} \tag{2.2}$$

Using the equation of state, this can be transformed to:

$$a = \sqrt{\gamma R T} \tag{2.3}$$

So the speed of sound only depends on the temperature. But when the speed of sound is known, also the Mach number can be calculated:

$$M = \frac{V}{a} \tag{2.4}$$

There are three important regions of Mach numbers:

1. $M < 1$: Subsonic flow
2. $M = 1$: Sonic flow
3. $M > 1$: Supersonic flow

These regions can be split up into more detailed, but less important regions, as can be seen in table 1.

| Mach number: | $M < 0.3$ | $0.3 < M < 1$ | $M = 1$ | $M \approx 1$ | $1 < M < 5$ | $5 < M$ |
|---|---|---|---|---|---|---|
| Name: | Low subsonic | High subsonic | Sonic | Transonic | Supersonic | Hypersonic |

Table 1: Mach regions and corresponding names.

## 3 Isentropic flow relations including Mach number

Let point 0 be the stagnation point, and let point 1 be a point in the undisturbed flow. The stagnation speed $V_0$ is 0. So $p_0$ is equal to the total pressure in the flow. The energy equation implies that:

$$c_p T_1 + \frac{1}{2} V_1^2 = c_p T_0 \Rightarrow \frac{T_0}{T_1} = 1 + \frac{V_1^2}{2 c_p T_1}$$

It is known that $R = c_p - c_v$ and $\gamma = \frac{c_p}{c_v}$. From that it can be derived that:

$$c_p = \frac{\gamma R}{\gamma - 1}$$

Combining these two equations results in:

$$\frac{T_0}{T_1} = 1 + \frac{\gamma - 1}{2} \frac{V_1^2}{\gamma R T_1}$$

It is also known that $a_1^2 = \gamma R T_1$ and $M_1 \frac{V_1}{a_1}$. Using that data, the previous formula can be transformed to:

$$\frac{T_0}{T_1} = 1 + \frac{\gamma - 1}{2} M_1^2 \tag{3.1}$$

And by using the isentropic flow relations, also the following can be deduced:

$$\frac{p_0}{p_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{\gamma}{\gamma - 1}} \tag{3.2}$$

$$\frac{\rho_0}{\rho_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{1}{\gamma - 1}} \tag{3.3}$$

But do keep in mind that $p_0$ and $\rho_0$ are the pressure and density, respectively, at the stagnation point (because $V_0 = 0$).

## 4 Calculating flight velocity

Calculating flight velocity for low subsonic flows is not difficult, because incompressibility can be assumed. Therefore only the formula for the flight velocity at high subsonic speeds is shown. The equations derived in the previous paragraph can be used to find those formulas. But to do that, the Mach number should first be isolated:

$$M_1^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

And using equation 2.4, the following relation is evident:

$$V_1^2 = \frac{2a_1^2}{\gamma - 1}\left[\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \tag{4.1}$$

However, it is not easy to measure a pressure ratio $\frac{p_0}{p_1}$. Most measuring devices measure a pressure difference $p_0 - p_1$ instead. Therefore the previous equation can be transformed.

$$V_1^2 = \frac{2a_1^2}{\gamma - 1}\left[\left(\frac{p_0 - p_1}{p_1} + 1\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

This formula still has a pressure value $p_1$ (which is not a pressure difference), and the speed of sound (which depends on $T_1$, the static temperature in the air around the airplane). Both are rather difficult to measure accurately. Therefore the calibrated airspeed $V_{cal}$ is introduced, and defined as follows:

$$V_{cal}^2 = \frac{2a_s^2}{\gamma - 1}\left[\left(\frac{p_0 - p_1}{p_s} + 1\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \tag{4.2}$$

## 5 Supersonic wind tunnels

From the continuity equation, the following can be derived:

$$\rho A V = c \Rightarrow \ln \rho A V = \ln \rho + \ln A + \ln V = \ln c \Rightarrow \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

Where $c$ is any constant mass flow. First filling in Euler's equation $\rho = -\frac{dp}{V\,dV}$ and then filling in equation 2.1 results in the following:

$$0 = -\frac{d\rho\; V\,dV}{dp} + \frac{dA}{A} + \frac{dV}{V} = -\frac{V\,dV}{a^2} + \frac{dA}{A} + \frac{dV}{V} = -\frac{M^2\,dV}{V} + \frac{dA}{A} + \frac{dV}{V}$$

A final transformation results in the so-called area-velocity relation:

$$\frac{dA}{A} = \left(M^2 - 1\right)\frac{dV}{V} \tag{5.1}$$

This formula has interesting implications. When looking at the three previously discussed Mach number regions, the following three phenomena can occur:

1. Subsonic flows ($M < 1$): For the velocity to increase, the area must decrease (and vice-versa).
2. Sonic flows ($M = 1$): This is a rather strange case, because it implies $\frac{dA}{A}$ to be 0. Therefore this only occurs in the tunnel where the cross-sectional area is minimal (this point is called the throat of the tunnel).
3. Supersonic flows ($M > 1$): For the velocity to increase, the area must increase (and vice-versa).