Aircraft Stress Analysis and Structural Design Summary

1. Trusses

1.1 Determinacy in Truss Structures

1.1.1 Introduction to determinacy

A **truss structure** is a structure consisting of members, connected by joints. Truss members are only subject to tension/compression.

Suppose we have a 2-dimensional truss structure. n is the number of **joints** (nodes) in the structure, m the number of **members** and r is the number of **reaction forces** acting on the structure. (So for a clamped beam $r = 3$, for a hinge support $r = 2$ and for a hinge on a roller support $r = 1$.) Whether the structure is **kinematically determinate** and **statically determinate** can be derived from figure 1.1.

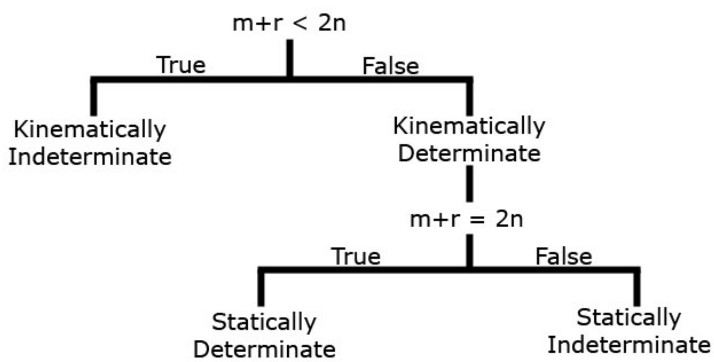

Figure 1.1: Overview of kinematical/statical determinacy

1.1.2 Kinematical determinacy

But what does it mean? A structure that is kinematically indeterminate can move (or parts of it can move), while kinematically determinate structures can not move at all. Statical determinacy is a bit more difficult to explain.

1.1.3 Statical determinacy

A statically determinate structure has just the right amount of members to make the structure also kinematically determinate. Remove 1 member, and the structure becomes kinematically indeterminate. Add 1 member, and the structure becomes statically indeterminate. The result of this is the following.

Let's suppose we have a statically determinate structure with a given applied load and truss geometry. The stress in any beam depends only on the cross-sectional area of that beam: $\sigma_i = f(A_i)$. Now suppose we have a statically indeterminate structure. The stress depends on the cross-sectional areas of all the beams: $\sigma_i = f(A_1 + \ldots + A_m)$. Also, to find the stresses, additional boundary conditions are required.

1.2 Stress Analysis in Truss Structures

1.2.1 Statically determinate trusses

Suppose at a later time we will receive data about a truss structure and want to calculate the forces in each member. To calculate this, we could write a computer program. But how can we let a program calculate the forces in every member?

For statically determinate structures, we can take the sum of forces in every node, both in x-direction and y-direction. We then get $2n$ equations. But what are our unknowns? The m internal forces in the members are unknown, and the r reaction forces are unknown as well. But since $m + r = 2n$ (the structure is statically determinate), we have the same amount of unknowns as equations. It can therefore be solved by a computer program.

When all the forces are known, it is easy to calculate the stresses present in the structure. To calculate the stress in beam i, just use

$$\sigma_i = \frac{F_i}{A_i}, \tag{1.2.1}$$

where F_i is the normal force in that beam and A_i is the cross-sectional area.

1.2.2 Statically indeterminate trusses

If the structure is statically indeterminate, then $m + r > 2n$, meaning that there are more unknowns then equations. Therefore additional boundary conditions are needed to calculate the stresses. These boundary conditions usually involve displacements.

Displacements are generally difficult to calculate. They don't only depend on the truss configuration. They also depend on the cross-sectional areas of the members. But there is a method with which this can be done, called the dummy load method. Let's take a look at that method now.

1.3 Dummy Load Method Derivation

1.3.1 Displacements and energy

A force on a structure always causes **displacements**. Energy is stored in such displacements. If P is the applied force on a structure and δ is the displacement, then the work done in the structure (and thus the energy stored in the structure) is

$$U = \int P \, d\delta. \tag{1.3.1}$$

In an **elastic (conservative) structure** the energy can be recovered completely. In a **plastic (non-conservative) structure** part of the energy is lost, causing permanent deformations.

It is usually assumed that δ is a linear combination of P, meaning that

$$P = k\delta, \tag{1.3.2}$$

where k is the **stiffness**. For truss members with uniform cross-section $k = \frac{EA}{L}$. This implies that

$$\frac{dU}{d\delta} = P, \qquad \frac{dU}{dP} = \delta. \tag{1.3.3}$$

This relation is called **Castigliano's theorem**. So for one truss member, the internal energy is given by

$$U = \int_0^l P\,d\delta = \int_0^l \frac{P}{k}dP = \frac{P^2}{2k} = \frac{P^2 L}{2AE}. \tag{1.3.4}$$

To derive the dummy load method for other load cases than just tension/compression, we can use table 1.3.1.

Type	Displacement	Energy
Tension/Compression Bar	$d\delta = \frac{F\,dx}{AE}$	$dU = \frac{1}{2}F\,d\delta = \frac{F^2\,dx}{2AE}$
Torsion Bar	$d\phi = \frac{T\,dx}{GJ}$	$dU = \frac{1}{2}T\,d\phi = \frac{T^2\,dx}{2GJ}$
Bending Beam	$d\theta = \frac{M\,dx}{EI}$	$dU = \frac{1}{2}M\,d\theta = \frac{M^2\,dx}{2EI}$
Shear Beam	$\gamma = \frac{VQ}{ItG} \approx \frac{V}{AG}$	$dU = \frac{1}{2}V\gamma dx = \frac{V^2Q\,dx}{2ItG} \approx \frac{V^2 dx}{2AG}$

Table 1.1: Basic Structural Deformations

1.3.2 Multiple members and loads

Now look at a complete structure with n members with internal forces $F_1, \dots F_n$, caused by m applied loads $P_1, \dots, P_m$. The total energy stored in the system is

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_iE_i}. \tag{1.3.5}$$

Suppose P_j is a virtual (nonexisting) force acting on some point j in the structure. The displacement of j in the direction of P_j now is

$$\delta_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{\partial\left(\frac{F_i^2 L_i}{2A_iE_i}\right)}{\partial P_j} = \sum_{i=1}^{n} \frac{\frac{\partial\left(F_i^2\right)}{\partial P_j} L_i}{2A_iE_i} = \sum_{i=1}^{n} \frac{2F_i \frac{\partial F_i}{\partial P_j} L_i}{2A_iE_i} = \sum_{i=1}^{n} \frac{F_i f_i L_i}{A_iE_i}, \tag{1.3.6}$$

where f_i is defined as $f_i = \frac{\partial F_i}{\partial P_j}$. Let's take a closer look at this f_i. What is it? In fact, there is a linear relation between F_i and P_j. So we can say that $F_i = P_j f_i$. To find f_i, simply set $P_j = 1$ and calculate F_i.

1.3.3 Dummy load method

The **dummy load method** is a method to calculate displacements. It uses the relation that was just found (equation 1.3.6).

It can be used for statically determinate structures to calculate the displacements. This is the subject of the next part. It can also be used for statically indeterminate structures. In a statically indeterminate structure, displacements are necessary to calculate the forces in the structure.

1.4 Displacements in Statically Determinate Trusses

1.4.1 Step 1 - Calculate all the internal forces

Suppose we have a statically determinate structure and want to find the displacement of some point j. To use the dummy load method, we have to calculate the internal forces $F_1, \dots, F_n$ first.

1.4.2 Step 2 - Calculate the force derivatives

Now we know F_i for every i. Of course we also know the shape of the structure, so we know L_i, A_i and E_i for every i. The only unknowns are $f_1, \ldots, f_n$. Use the following steps to find them.

- Remove all external forces $P_1, \ldots P_m$ from the structure.
- Place a load $P_j = 1$ at point j, in the direction of which you want to know the displacement. Note that if you want to find the actual displacement vector of a point, you need to perform this method twice. Once for each direction.
- Calculate the force in all the members, due to this load P_j. The value of f_i is now the internal force of member i that results from this calculation.

1.4.3 Step 3 - Calculate the displacement

Now F_i and f_i are known for every i. To calculate the displacement in the specified direction, you must use

$$\delta_j = \sum_{i=1}^{n} \frac{F_i f_i L_i}{A_i E_i}. \tag{1.4.1}$$

With this method the displacement of every node in the structure can be calculated.

1.5 Statically Indeterminate Trusses

1.5.1 Superposition

In the last chapter we saw that a structure with m members, n nodes and r reaction forces is statically indeterminate if $m + r > 2n$. The **degree of statical indeterminacy** is defined as $d = m + r - 2n$. This is also the amount of members you can remove until the structure becomes statically determinate.

The dummy load method for indeterminate structures uses the principle of **superposition**. This states that for multiple loads, the displacements caused by the individual loads can be added up. So if there is a structure with multiple loads acting on it, you can calculate the displacement caused by every load individually, and eventually add them all up.

1.5.2 Step 1 - Remove members until statical determinacy is reached

But how can we use this on a statically indeterminate structure? To keep things simple, we assume that the degree of statical indeterminacy is 1. Let's suppose that member j is between nodes A and B. Also suppose that if we remove member j, the structure becomes statically determinate.

1.5.3 Step 2 - Calculate displacement due to external forces

We now have a statically determinate structure. Therefore, we can calculate the force in every member $F_1^{ext}, \ldots F_m^{ext}$ (except member j, of course) due to the externally applied loads. Since member j is removed, it can't carry any loads. So we assume that $F_j^{ext} = 0$. Keep in mind that these are not the loads that are actually present in the structure, since we removed a member!

To use the dummy load method, we want to calculate the change of the distance AB. So we assume that there is a unit load (of size $1N$) acting on both A and B, pointing inward (where member j was).

The forces in every member $f_1, \ldots, f_m$ (due to this unit load) can now be calculated. The shortening of distance AB in the determinate structure, due to the external forces, is

$$\delta_{AB}^{ext} = \sum_{i=1}^{m} \frac{F_i^{ext} f_i L_i}{A_i E_i}. \tag{1.5.1}$$

In the above summation, we have $F_j^{ext} = 0$. So the entire j-term will vanish.

1.5.4 Step 3 - Calculate displacement due to the removed member

We have assumed member j wasn't present. But of course it is present. The member is actually causing a force F_j on the structure at points A and B. And to calculate the displacement due to member j, we need to take into account this force.

So we assume there is a force F_j acting on both A and B, pointing inward (where member j was). The forces in all members due to this internal load, $F_1^{int}, \ldots, F_m^{int}$, can now be calculated. In fact, we can take a short cut using

$$F_i^{int} = F_j f_i, \tag{1.5.2}$$

for every member $i \neq j$. Note that F_j^{int} isn't really defined. But if we define $F_j^{int} = F_j$ and $f_j = 1$, then equation 1.5.2 also holds for $i = j$.

Using equation 1.5.2 and $f_j = 1$, we can derive that the change of distance AB, due to the force caused by member j, is

$$\delta_{AB}^{int} = \sum_{i=1}^{m} \frac{F_i^{int} f_i L_i}{A_i E_i} = F_j \sum_{i=1}^{m} \frac{f_i^2 L_i}{A_i E_i}. \tag{1.5.3}$$

1.5.5 Step 4 - Equate displacements to zero

We now know the change of distance AB due to external forces (being δ_{AB}^{ext}) and the change of distance AB due to internal forces (being δ_{AB}^{int}). To find F_j, we can use the simple relation

$$\delta_{AB}^{ext} + \delta_{AB}^{int} = 0. \tag{1.5.4}$$

The only unknown in this relation is F_j, so it can be solved. By the way, we will not show why this relation is true. (The explanation is too long for a summary.)

Now we finally know F_j. We still don't know the other forces in the structure. To find the actual forces, we just have to sum up the part caused by external loads and the part caused by internal forces. So the actual forces in every member can be calculated using

$$F_i^{act} = F_i^{ext} + F_i^{int} = F_i^{ext} + F_j f_i. \tag{1.5.5}$$

1.6 Actuation

1.6.1 Actuated members

Truss members can be actuated by external effects. One of the most common effects is a change in temperature. A material subject to a change in temperature is assumed to elongate/shorten according to

$$\varepsilon_T = \alpha \, \Delta T, \tag{1.6.1}$$

where α is the **coefficient of thermal expansion** (CTE). Other ways of actuation can be treated similarly, so we will only examine actuation by temperature. The total strain now is

$$\varepsilon = \varepsilon_T + \varepsilon_M = \alpha\,\Delta T + \frac{P}{AE}, \tag{1.6.2}$$

where ε_M is the strain due to mechanical forces.

1.6.2 Actuation effects

But what effect does this have on the dummy load method? In equation 1.5.1 we saw the part $F_i^{ext}/A_i E_i$. This is equal to the mechanical strain ε_M. We should replace this by the new strain. So instead of using equation 1.5.1 we use

$$\delta_{AB}^{ext} = \sum_{i=1}^{m} f_i \varepsilon_i L_i = \sum_{i=1}^{m} f_i L_i \left(\alpha\,\Delta T + \frac{F_i^{ext}}{A_i E_i}\right). \tag{1.6.3}$$

2. Beams

2.1 Stress Analysis in Statically Determinate Beams

2.1.1 Beam rules

In a truss structure only **normal forces** are present in the members. In a beam, also **shear forces** and **bending moments** can be present. This has effects on the deformation of those beams. Two general rules apply for that.

- Plane sections normal to the longitudinal axis remain normal after deformation.
- The thickness of the beam is unchanged.

When a beam has three support reactions acting on it, the beam is **statically determinate**. The reaction forces can then be solved in an easy way. If there are, however, more reaction forces acting on it, the beam is called **statically indeterminate**.

2.1.2 Normal stress

To calculate the **normal stress** σ at a certain point in the beam, we can use

$$\sigma = \sigma_N + \sigma_M = \frac{F}{A} - \frac{My}{I}, \tag{2.1.1}$$

where M is the bending moment at the specified point, I is the second area moment of inertia and y is the vertical distance from the COG of the cross-section. Note that $\sigma_N = \frac{F}{A}$ is the part due to normal forces and $\sigma_M = \frac{My}{I}$ is the part due to bending moments. The minus sign is present due to sign convention.

2.1.3 Shear stress

To calculate the **shear stress** at a certain point, we can use

$$\tau = \frac{VQ}{It}, \tag{2.1.2}$$

where V is the shear force present, Q is the first area moment of inertia, I is the second area moment of inertia and t is the thickness at the part where the shear stress is calculated.

2.1.4 Rotations and displacements

The rotations and displacements of a beam can be calculated using the so-called forget-me-nots. If a beam of length L, E-modulus E and moment of inertia I is subject to either a bending moment M, a load P or a distributed load q, then the rotation θ and the displacement δ can be found by using

$$\theta = \frac{ML}{EI}, \qquad \theta = \frac{PL^2}{2EI}, \qquad \theta = \frac{qL^3}{6EI}, \tag{2.1.3}$$

$$\delta = \frac{ML^2}{2EI}, \qquad \delta = \frac{PL^3}{3EI}, \qquad \delta = \frac{qL^4}{8EI}. \tag{2.1.4}$$

For complicated structures, applying these equations isn't always very easy. So there exists another method to find the rotation and displacement of a beam.

2.2 Dummy Load Method for Beams

2.2.1 Derivation for Rotations

The **dummy load method for beams** makes use of an equation we saw earlier. Slightly rewritten, this equation was

$$U = \int_{beam} \frac{M^2}{2EI} dx, \tag{2.2.1}$$

where we integrate over the entire beam. Just like in the dummy load method, we need to differentiate U. But now we differentiate with respect to a moment T. What we find is the rotation θ in the direction of T. So we get

$$\theta = \frac{\partial U}{\partial T} = \int_{beam} \frac{M \frac{\partial M}{\partial T}}{EI} dx = \int_{beam} \frac{Mm}{EI} dx, \tag{2.2.2}$$

where we have defined m as $\frac{\partial M}{\partial T}$.

2.2.2 Derivation for Displacements

We can also find displacements with this method. In that case we shouldn't differentiate U with respect to a moment T, but with respect to a force P. We then get

$$\delta = \frac{\partial U}{\partial P} = \int_{beam} \frac{M \frac{\partial M}{\partial P}}{EI} dx = \int_{beam} \frac{Mm}{EI} dx. \tag{2.2.3}$$

Note that m is now defined as $\frac{\partial M}{\partial P}$. This displacement is in the direction of the force P.

2.2.3 Using the Method

Now let's take a look at how to use this method. We have a beam and want to find the displacement at some point B. First we need to find $M(x)$. This is simply given by the moment diagram over the beam, caused by all the external forces.

We then need to find $m(x) = \frac{\partial M}{\partial P}$. It can be shown that the moment M depends linearly on P. So $M = mP$. To find m, we need to set $P = 1$. So we apply a unit load P at point B, perpendicular to the beam. We then derive the moment diagram, and we've got m.

Now all that is left for us to do is to apply the integral given by equation 2.2.3. That gives us the displacement we were looking for.

2.2.4 Avoiding the Integral

The integral in equation 2.2.3 can sometimes be very hard to evaluate. Therefore it is often allowed to make an approximation. For that, we split the beam up in a number of n segments. For every segment, we calculate the average values $M_{i_{ave}}$ and $m_{i_{ave}}$ using

$$M_{i_{ave}} = \frac{M_{i_{left}} + M_{i_{right}}}{2} \qquad \text{and} \qquad m_{i_{ave}} = \frac{m_{i_{left}} + m_{i_{right}}}{2}. \tag{2.2.4}$$

So what does this mean? We still need to find the moment diagrams for both M and m. Then, for every segment, we take the values for M on the left and right side of the segment, and take their average. In this way we find $M_{i_{ave}}$. We do the same for m to find $m_{i_{ave}}$.

In the end, when we have found all the average values, we simply apply

$$\delta = \sum \frac{M_{i_{mean}} m_{i_{mean}} L_i}{E_i I_i}. \tag{2.2.5}$$

Here L_i is the length segment i, and E_i and I_i are its E-modulus and moment of inertia. By the way, this equation also works to find the rotation θ. In that case the other definition of m needs to be applied. (The one with the unit moment.)

2.2.5 Beams and Bars

Sometimes a structure doesn't consist of only a bending beam. If the beam is supported by bars, then those bars deform as well. In that case the expression for U we have used earlier isn't complete. So (for simplicity of this example) let's suppose there's only 1 (vertical) bar supporting the (horizontal) beam. We then get

$$U = \int \frac{M^2}{2EI} dx + \frac{F^2 L}{2EA}. \tag{2.2.6}$$

Differentiating with respect to a load P now once more gives the displacement. We will find

$$\delta = \frac{\partial U}{\partial P} = \int \frac{Mm}{EI} dx + \frac{FfL}{EA}, \tag{2.2.7}$$

where the coefficient f is the force in the bar due to the applied unit load. If there are more beams or bars that deform, they also need to be considered. All parts that store energy need to be added to the above equation. It's as simple as that.

2.3 Statically Indeterminate Beams

2.3.1 Maxwell's theorem

Let's discuss statically indeterminate beams now. Statically indeterminate beams are usually difficult to analyze, as you need to use compatibility equations to solve the reaction forces. While finding these compatibility equations, the so-called **flexibility coefficients** can come in handy. The flexibility coefficient f_{BA} is the displacement of point B due to a unit load at point A. Now **Maxwell's Theorem** states that

$$f_{BA} = f_{AB}. \tag{2.3.1}$$

In words, the displacement of point B due to a unit load in A is the same as the displacement of A due to a unit load in B.

2.3.2 Other flexibility coefficients

It is also possible to calculate the flexibility coefficient m_{BA}, being the displacement of point B due to a unit moment at point A. Using these moment flexibility coefficients is identical as using the force flexibility coefficients, except that they involve moments and not forces.

Next to finding the displacement, you can also involve rotations in flexibility coefficients. For example, you can define the rotational flexibility coefficient f_{BA} as the rotation of a point B due to a unit force at point A. The same goes for unit moments.

2.3.3 Step 1 - Making the structure determinate

Suppose we have a statically indeterminate beam. To analyze the beam - finding all the reaction forces - we first have to remove supports until it becomes statically determinate. For every support, ask yourself: "If I remove it, will the structure be able to move?" If the answer is no, remove it.

2.3.4 Step 2 - Calculate displacements

Suppose we have removed supports at points $A_1, \dots A_n$. We can now calculate the displacement ΔA_i of every point A_i for the new statically determinate beam, due to the external loads. The dummy load method for beams is suited for this rather well.

2.3.5 Step 3 - Calculate flexibility coefficients

Next to the displacements, we can also calculate the flexibility coefficients for the statically indeterminate beam. First remove all the external loads from the structure.

To find f_{AB} for some points A and B, just put a unit load at B and calculate the displacement at A. For the dummy load method for beams, you have to find $f_{A_iA_j}$ for every combination of i and j. So if n reaction forces are removed, you need to find n^2 flexibility coefficients (or about $\frac{1}{2}n^2$ is you use Maxwell's theorem to save time).

2.3.6 Step 4 - Formulate and solve compatibility equations

Now, using flexibility coefficients, several **compatibility equations** can be determined. For any node A_i, the total displacement in the original (statically indeterminate) structure is

$$\delta_{A_i} = \Delta A_i + R_{A_1} f_{A_iA_1} + R_{A_2} f_{A_iA_2} + \dots + R_{A_n} f_{A_iA_n} = 0. \tag{2.3.2}$$

So we have n unknown reaction forces and n compatibility equations. All the reaction forces can be solved. And if all the reaction forces are known, it is relatively easy to calculate any displacement in the structure. For that, simply use the dummy load method for beams again.

2.4 Beams of Multiple Materials

2.4.1 Introduction to multiple material beams

Sometimes beams are made up out of layers of different materials. Usually each layer has a different E-modulus E. To be able to make calculations on these beams, we make a few assumptions. We assume that there is perfect bonding between the layers of material. So there can't be any slipping. We also assume that a line normal to the beam remans normal and perpendicular to the mid plane (MP) of the beam.

2.4.2 Weighted cross-sectional area

We want to be able to do calculations with beams consisting of multiple materials. So we define the weighted cross-sectional area A^* such that

$$dA^* = \frac{E}{E_{ref}} dA \qquad \Rightarrow \qquad A^* = \int_{A^*} dA^* = \int_A \frac{E}{E_{ref}} dA, \tag{2.4.1}$$

where E_{ref} is a reference E-modulus (usually taken to be the E-modulus of one of the materials in the beam). By examinig this definition, you see that stiff parts (with high E) contribute more to A^* than flexible parts.

2.4.3 Centroids and moment of inertia

The position of the weighted centroid can also be found by using the definition for the weighted area. The x-position of this centroid is

$$\bar{x}^* = \frac{1}{A^*} \int_{A^*} x \, dA^* = \frac{1}{A^*} \int_A x \frac{E}{E_{ref}} dA. \tag{2.4.2}$$

The y-position of the weighted centroid can be found identically. And once the position of the centroid is found, we can find the weighted moment of inertia I^*. The weighted moment of inertia about the x-axis is

$$I^* = \int_{A^*} y^2 dA^* = \int_A y^2 \frac{E}{E_{ref}} dA, \tag{2.4.3}$$

where y is the vertical distance between the current point dA that is examined, and the position of the weighted centroid $\bar{y}^*$.

2.4.4 Stresses

The normal stress in a beam at some point (x, y) (with respect to the weighted centroid) due to a normal force P can be found using

$$\sigma(x, y) = \frac{P}{A^*} \frac{E(x, y)}{E_{ref}}. \tag{2.4.4}$$

When the beam is subject to a bending moment M, the stress at the point (x, y) can be found using

$$\sigma(x, y) = -\frac{My}{I^*} \frac{E(x, y)}{E_{ref}}. \tag{2.4.5}$$

Once more, the minus sign is present due to sign convention. When there is a combined normal force and bending moment, the above stresses can simply be added up. (By the way, the calculation of shear stress isn't different than for normal beams.)

2.4.5 Rotations and displacements

For beams of multiple materials, the forget-me-nots ought to be adjusted slightly. The new versions are

$$\theta = \frac{ML}{E_{ref} I^*}, \qquad \theta = \frac{PL^2}{2E_{ref} I^*}, \qquad \theta = \frac{qL^3}{6E_{ref} I^*}, \tag{2.4.6}$$

$$\delta = \frac{ML^2}{2E_{ref} I^*}, \qquad \delta = \frac{PL^3}{3E_{ref} I^*}, \qquad \delta = \frac{qL^4}{8E_{ref} I^*}. \tag{2.4.7}$$

2.4.6 Stiffness

For beams it's often interesting to know the stiffness. There are two kinds of stiffness. There is the **elongation stiffness** A and the **bending stiffness** D, defined as

$$A = \frac{N}{\epsilon}, \qquad \text{and} \qquad D = \frac{M}{\kappa}, \tag{2.4.8}$$

where κ is the curvature of the beam. If the beam consists of only layers of different materials, the values of A and D can be easily calculated. Let's suppose we have n layers $1 \ldots n$, each layer i, starting at z_{i-1}, ending at z_i and having E-modulus E_i. Here the values z are the vertical distance, measured from the centroid (with downward being negative). It can then be determined that

$$A = \sum_{k=1}^{n} E_k \left(z_k - z_{k-1} \right), \qquad \text{and} \qquad D = \sum_{k=1}^{n} E_k \left(\frac{z_k^3 - z_{k-1}^3}{3} \right). \tag{2.4.9}$$

3. Plates

3.1 Basic Stress Analysis in Plates

3.1.1 Uniaxial stress state

Suppose we have a plate with thickness t, width w (in the x-direction) and length L (in the y-direction). Also suppose that the plate is rigidly connected at the bottom and loaded by a force P at the top, such that the stresses are uniformly divided over the plate. Now we have a plate with a **uniaxial stress state**. The stresses are now given by

$$\sigma_y = \frac{P}{wt}, \qquad \sigma_x = 0. \tag{3.1.1}$$

From these stresses we can derive the strains. If ν is **Poisson's ratio** of the material, then

$$\varepsilon_y = \frac{\sigma_y}{E} = \frac{P}{Ewt}, \qquad \varepsilon_x = -\nu\varepsilon_y = -\frac{\nu P}{Ewt}. \tag{3.1.2}$$

3.1.2 Biaxial stress state

What if we constrain the plate of the last paragraph on the left and the right side? We then actually set ε_x to be zero. This causes the stresses in the plate to change. Let's call the strain in x-direction ε_x and the strain in y-direction ε_y. We can now solve this problem, using the basic equations

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} = \frac{P}{Ewt} - \nu\frac{R_x}{ELt}, \qquad \varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{R_x}{ELt} - \nu\frac{P}{Ewt}, \tag{3.1.3}$$

where R_x is the reaction force in horizontal direction. Since $\varepsilon_x = 0$, we can derive that

$$R_x = \nu\frac{PL}{w}. \tag{3.1.4}$$

Using this, the stresses can be found. They are

$$\sigma_y = \frac{P}{wt}, \qquad \sigma_x = \frac{R_x}{Lt} = \nu\frac{P}{wt} = \nu\sigma_y. \tag{3.1.5}$$

3.1.3 Multiple materials

When a plate consists of individual parts of different materials, the situation is more complicated. Still the equations of the previous paragraphs can be applied, but several other compatibility equations are necessary. There are no basic equations for that, but there are a few tricks that often need to be used.

- The stresses in the individual plate parts, in both x and y-direction can be expressed as a force using $F = \sigma A$, where A is the cross-sectional area. Using "sum of the forces is zero" for individual plate parts can give several compatibility equations.
- Often a plate is constrained in horizontal direction. In that case you can use the rule: "The sum of the horizontal displacements is zero."
- Finally, a plate is often constrained at the bottom and stressed uniformly at the top. In that case you can use the rule: "The vertical displacement of every part is equal."

3.2 Stress Analysis in Plates Using Mohr's Circle

3.2.1 Present stresses

In the previous chapter, we have only considered uniform load cases on rectangular uniform plates. Such simple geometries are often not present. For plates, stress analysis is therefore usually too complex. In that case stresses can not be calculated directly. Nevertheless, the stresses are present.

In a 2-dimensional plate, three kinds of stresses are present, being σ_x (normal stress in x-direction), σ_y (normal stress in y-direction) and τ_{xy} (shear stress in the xy-plane). In 3-dimensional plates, six kinds of stresses are present, being σ_x, σ_y, σ_z, τ_{xy}, τ_{xz} and τ_{yz}. However, we only consider 2-dimensional plates from now on.

3.2.2 Mohr's circle

The stresses in a plate can not be calculated, but they can be measured. Suppose we measure σ_x, σ_y and τ_{xy} at a certain point. We get certain values. Suppose we now change (rotate) our coordinate system and measure σ_x, σ_y and τ_{xy} at the same point again. We now get different values. If we keep doing this for different (rotated) coordinate systems, and plot all the data we find in a $\sigma - \tau$ coordinate system, we get a circle, as can be seen in figure 3.1. This circle is called **Mohr's circle**.

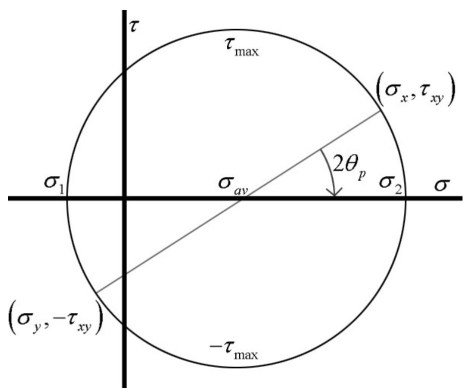

Figure 3.1: Mohr's Circle

3.2.3 Circle properties

Now let's take another look at figure 3.1. Suppose we have done a measurement and gotten the values σ_x, σ_y and τ_{xy}. We mark them in a graph and then draw a line between them. Where the line crosses the σ-axis is the **average stress** σ_{av}, which can also be calculated using

$$\sigma_{av} = \frac{\sigma_x + \sigma_y}{2}. \tag{3.2.1}$$

Also the **radius** of the circle can be calculated, using

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \tag{3.2.2}$$

3.2.4 Maximum stress

If we take measurements for different coordinate axes, the stresses will be different. What we are interested in, are the maximum stresses. The **maximum shear stress** is

$$\tau_{max} = R. \tag{3.2.3}$$

To find the **minimum and maximum normal stresses**, we can use

$$\sigma_1 = \sigma_{av} - R, \qquad \sigma_2 = \sigma_{av} + R. \tag{3.2.4}$$

Note that σ_1 and σ_2 can be both positive (in case of tension in both directions), both negative (in case of compression in both directions), or it is possible that $\sigma_2 > 0$ and $\sigma_1 < 0$. For figure 3.1 this means that the circle can move to the left and to the right for different load cases. However, the circle can not move upward or downward - the circle center is always at the σ-axis.

3.2.5 Stress directions

It is often handy to know in which direction maximum stresses occur. This can also be derived from Mohr's circle. Let's start rotating our coordinate system and do measurements. At some moment, when we have rotated the coordinate system by an angle θ_p, we measure maximum normal stress. In figure 3.1 this angle θ_p is visualized. From this figure we can see that

$$\tan 2\theta_p = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}. \tag{3.2.5}$$

Let's define θ_s to be the angle at which maximum shear stress occurs. We can now see that

$$\theta_s = \theta_p \pm 45°. \tag{3.2.6}$$

4. Designing

4.1 Designing considerations

4.1.1 Design Types

There are three types of design. In **routine design** familiar parts and devices are used. In **variant design** parts are modified in form and function. In **creative design** new parts and artifacts are designed. Next to deciding the design type, a designer also has to decide whether to use truss structures, beams or plates in his design.

4.1.2 Design parameters and variables

In a design, there are usually certain given **design parameters** $\mathbf{p} = [p_1 \ldots p_m]^T$ and **design variables** $\mathbf{x} = [x_1 \ldots x_m]^T$ which still need to be determined. The more design variables, the more decisions need to be made, and the more complex the designing process can be. If there are n design variables, then the **dimensionality of the design space** is said to be n-dimensional. That is, you can draw an n-dimensional space, in which every point represents a possible design.

4.1.3 Design constraints

Next to the variables, there are also certain **design constraints**. These constraints can be on the design variables (for example, the width of a beam may not be bigger than ...) or on the design outcome. However, to put constraints on the design outcome, there must be some function $g(\mathbf{x}, \mathbf{p})$ whose value is constrained to a certain value. Constraints can be drawn in the design space, to visualize which options are possible and which options are not.

In a design, stresses σ will always have bounds. Displacements δ usually also have bounds.

4.1.4 Design objectives

Naturally a design is not made without a reason. Certain **design objectives** are present. An example of an objective is to minimize weight.

The "goodness" of the design, being the amount in which the objective has been satisfied, must be computable. So there must be a function $f(\mathbf{x}, \mathbf{p})$ indicating how much the objective has been fulfilled. If there are more than one objectives, there is a multi-objective formulation $(f_1, f_2, \ldots, f_j)$.

4.1.5 Design efficiency

For a safe structure, the stresses in every member may not be bigger than the maximum allowable stress: $\sigma \leq \sigma_{all}$. But for an efficient design, all members should have $\sigma = \sigma_{all}$ for at least one load condition. If this is not the case, than the cross-sectional area (or in some cases the thickness) can be changed according to

$$A_{new} = A_{old} \frac{\sigma}{\sigma_{all}}. \tag{4.1.1}$$

This works for statically determinate structures, but not always for statically indeterminate structures, since the stress in a member then depends on various other parameters as well.

In this way the design can be optimized as much as possible, until the design is satisfactory. Being **satisfied** with a design means that all constraints/bounds are satisfied and all design objectives are met.

4.1.6 Sensitivity Analysis

The **sensitivity** is the response of the design due to a change in a design variable. In a way, the sensitivity is a derivative $\frac{\partial f}{\partial x}$. However, a function is not always present, so a **finite difference approximation** can be used. The sensitivity of the stress with respect to the cross-sectional area is for example

$$\frac{\partial \sigma}{\partial A} = \frac{\sigma(A_1) - \sigma(A_0)}{A_1 - A_0} = \frac{\sigma_1 - \sigma_0}{A_1 - A_0}, \tag{4.1.2}$$

where $dA = A_1 - A_0$ is very small.

4.2 Loads and failures

4.2.1 Limit load and ultimate load

The **limit load** is the maximum load which the aircraft may encounter at any time during its lifetime. No yielding/permanent deformation may occur at the limit load.

The **ultimate load** is the load which may occur once in the lifetime of an aircraft. All parts must be able to carry this load without failure. Permanent deformation may occur though.

4.2.2 Failure criteria

The structure fails when the stresses get too high. We already know how to calculate the minimum and maximum stresses σ_1 and σ_2 in a plate. These stresses must comply to the **maximum normal stress criterion**, meaning that

$$\sigma_1 \leq \sigma_{all}, \qquad \sigma_2 \leq \sigma_{all}, \tag{4.2.1}$$

where σ_{all} is the maximum allowable normal stress. There is also the **maximum shear stress criterion**, meaning that

$$|\tau_{max}| \leq \tau_{all} \qquad \Leftrightarrow \qquad |\sigma_1 - \sigma_2| \leq 2\tau_{all}, \tag{4.2.2}$$

where τ_{max} is the maximum allowable shear stress. The **von Mises criterion** also demands that

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \leq \sigma_{all}^2 \tag{4.2.3}$$

And eventually, in some cases, there is also the **maximum strain criterion**.

4.2.3 Buckling

For beams under tension, only the cross-sectional area A is important. For beams under compression, also the shape matters, since buckling may occur. The maximum force which a beam can take depends partially depends on how it is connected to the structure. For example, the maximum compressive force for a simply supported column is

$$P_{max} = \frac{\pi^2 EI}{L^2}. \tag{4.2.4}$$

4.2.4 Stress Concentrations

Stress concentrations occur at discontinuities in the structure, such as holes, cracks or a change in the cross-sectional shape. At such concentrations, stresses are higher. The **stress concentration factor** K is defined such that

$$\sigma_{max} = K\sigma_{average}. \tag{4.2.5}$$

4.2.5 Load Cases

A structure could be design for just one **load case** (a way of loading it). However, some structures have multiple loads acting on them simultaneously. This is called a **combined load case**. It could also occur that multiple loads do occur, but not simultaneously. This is called a **multiple load case**.

5. Loads and Failures

5.1 Load Types

5.1.1 Limit load and ultimate load

The **limit load** is the maximum load which the aircraft may encounter at any time during its lifetime. No yielding/permanent deformation may occur at the limit load.

The **ultimate load** is the load which may occur once in the lifetime of an aircraft. All parts must be able to carry this load without failure. Permanent deformation may occur though.

5.1.2 Stress Concentrations

Stress concentrations occur at discontinuities in the structure, such as holes, cracks or a change in the cross-sectional shape. At such concentrations, stresses are higher. The **stress concentration factor** K is defined such that

$$\sigma_{max} = K\sigma_{average}. \tag{5.1.1}$$

5.1.3 Load Cases

A structure could be designed for just one **load case** (a way of loading it). However, some structures have multiple loads acting on them simultaneously. This is called a **combined load case**. It could also occur that multiple loads do occur, but not simultaneously. This is called a **multiple load case**.

5.2 Stress Criterions

5.2.1 Stress criterions in trusses

Members of truss structures are only subject to tensile/compressive stresses. If these stresses get too high, failure will occur. For the tensile case there is usually a maximum tensile stress $\sigma_{t_{max}}$. The **failure criterion for tension** then is

$$\sigma \leq \sigma_{t_{max}}. \tag{5.2.1}$$

When compression is present, things are slightly different. Sometimes a maximum compressive stress $\sigma_{c_{max}}$ is given. Sometimes **buckling** may occur. This will be discussed later.

5.2.2 Stress criterions in beams

In beams there are the same criterions as for truss members. However, there can also be the **maximum shear stress criterion**

$$\tau \leq \tau_{max}. \tag{5.2.2}$$

Besides shear stress, there can also be constraints on the bending moment present.

5.2.3 Stress criterions in plates

In plates stresses in multiple direction often occur. We already know how to calculate the minimum and maximum stresses σ_1 and σ_2 in a plate. These stresses must comply to the **maximum normal stress**

criterion, meaning that

$$\sigma_1 \leq \sigma_{all}, \qquad \sigma_2 \leq \sigma_{all}, \tag{5.2.3}$$

where σ_{all} is the maximum allowable normal stress. There is also the **maximum shear stress criterion**, meaning that

$$|\tau_{max}| \leq \tau_{all} \qquad \Leftrightarrow \qquad |\sigma_1 - \sigma_2| \leq 2\tau_{all}, \tag{5.2.4}$$

where τ_{max} is the maximum allowable shear stress. The **von Mises criterion** also demands that

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \leq \sigma_{all}^2. \tag{5.2.5}$$

And eventually, in some cases, there is also the **maximum strain criterion**.

5.3 Buckling

5.3.1 Definition

Buckling of a structure means failure due to excessive displacements or loss of stability. Buckling occurs at the so-called **critical load**. This depends on various properties of the beam/plate itself, as how it is connected to the structure.

5.3.2 Buckling of beams

Let's suppose a simply supported beam with E-modulus E, length L and moment of inertia I is loaded under compression. The compressive force at which buckling will occur is

$$P_{max} = \frac{\pi^2 EI}{L^2}. \tag{5.3.1}$$

5.3.3 Buckling of plates

Let's look at a flat plate with height h, width w and thickness t, loaded in the vertical direction. Also assume all four edges of the plate are supported. The load at which failure occurs is

$$\frac{4\pi^2}{w}\frac{Et^3}{12\left(1-\nu^2\right)}. \tag{5.3.2}$$

Plates often have **post-buckling strength**, meaning that the plate can resist increased loads even after buckling.

5.3.4 Buckling of cylinder shells

Let's examine a cylinder shell with radius R, thickness t and height h. The critical load can be found by using

$$\sigma_{cr} = \frac{1}{\sqrt{3\left(1-\nu^2\right)}}\left(\frac{Et}{R}\right), \qquad \text{and} \qquad P_{cr} = 2\pi Rt\sigma_{cr}. \tag{5.3.3}$$

This, however, is only the result of theory. In practice there are small imperfections in cylinders, which significantly reduce the strength. To compensate for this, there is the so-called **knockdown factor** γ, such that

$$\sigma_{cr_{design}} = \gamma\sigma_{cr_{theory}}. \tag{5.3.4}$$

The knockdown factor is a function of R/t. It decreases for thinner shells.