Chapter 1

Fundamentals

1.1 Newton's laws of motion

The three laws of motion, which were formulated by I. Newton (1643-1727) in his masterpiece *Philosophiae Naturalis Principia Mathematica*, usually abbreviated to Principia, in 1687, read in modern terminology:

- First law: Every particle continues in its state of rest or uniform motion in a straight line relative to an inertial reference frame, unless it is compelled to change that state by forces acting upon it.
- Second law: The time rate of change of linear momentum of a particle relative to an inertial reference frame is proportional to the resultant of all forces acting upon that particle and is collinear with and in the direction of the resultant force.
- Third law: If two particles exert forces on each other, these forces are equal in magnitude and opposite in direction (action = reaction).

1.2 Inertial reference frames

The formal definition of an inertial reference frame can be derived from Newton's first law: *"An inertial reference frame is a reference frame with respect to which a particle remains at rest or in uniform rectilinear motion if no resultant force acts upon that particle."*

1.3 Newton's law of gravitation

Partially based on the observed motions of the planets around the Sun, Newton formulated his law of gravitation and published it also in his Principia:

- Two particles attract each other with a force directly proportional to their masses and inversely proportional to the square of the distance between them.

Mathematically, this law can be expressed as follows:

$$F = G\frac{m_1 m_2}{r^2} \quad ; \quad \overline{F}_2 = G\frac{m_1 m_2}{r_2^3}\overline{r}_2 \tag{1.1}$$

The gravitational acceleration generated by m_1 is given below. Introducing the **gravitational potential** U, the gravitational acceleration can be rewritten as the derivative (or gradient) of the graviational potential. With $U_{2_0} = 0$, the potential is defined to be negative and equal to zero at an infinite distance of the attractor.

$$\overline{g}_2 = -G\frac{m_1}{r_2^3}\overline{r}_2 \quad ; \quad U_2 = -G\frac{m_1}{r_2} + U_{2_0} \quad ; \quad \overline{g}_2 = -\overline{\nabla}_2 U_2 \quad with \quad U_{2_0} = 0 \tag{1.2}$$

1.4 Maneuvers with rocket thrust

If the maneuver is assumed to be an impulsive shot, thus an instant increase in velocity, the change in velocity can be written as:

$$\overline{V}_1 = \overline{V}_0 + \Delta\overline{V} \tag{1.3}$$

The change in angular momentum and change in orbital energy (per unit of mass) can be written as:

$$\Delta\overline{H} = \overline{r}_0 \times \overline{V}_1 - \overline{r}_0 \times \overline{V}_0 = \overline{r}_0 \times \Delta\overline{V} \tag{1.4}$$

$$\Delta E = \frac{1}{2}\left(V_1^2 - V_0^2\right) = \frac{1}{2}\left(\Delta V\right)^2 + \overline{V}_0 \bullet \Delta\overline{V} \tag{1.5}$$

From the above equations some interesting conclusions can be drawn:

- For a given magnitude of $\Delta\overline{V}$, the maximum change in orbital angular momentum is achieved if the impulsive shot is executed when the spacecraft is farthest away from Earth and if is perpendicular to $\overline{r}_0$.
- If the direction of the orbital angular momentum vector should not be changed, $\Delta\overline{V}$ should be directed in the initial orbital plane. If the direction of the angular momentum vector should be changed, a component of $\Delta\overline{V}$ should be directed perpendicular to the initial orbital plane.
- For a given magnitude of $\Delta\overline{V}$, the maximum change in (total) orbital energy is achieved if the impulsive shot is executed at the point in the orbit where the velocity reaches a maximum value, and if $\Delta\overline{V}$ is directed along the velocity vector $\overline{V}_0$, i.e. tangentially to the (initial) orbit.

A finite burn time can be taken into account by adding a *gravity loss* and a drag loss, resulting in:

$$\Delta V = \Delta V_{id} - \Delta V_G - \Delta V_D \quad with \quad \Delta V_G = \int_{t_0}^{t_e} g \sin\gamma dt \tag{1.6}$$

1.5 Kepler's Laws

Kepler's Laws of planetary motion are:

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Chapter 2

The Many-Body Problem

Let us consider a system composed of n bodies, which may be considered as point masses. When it is assumed that outside the system of n bodies no other bodies exist, that no external forces act on the system, and that within the system of n bodies only gravitational forces occur, then, applying Newton's second law of motion and Newton's law of gravitation, the motion of body i with respect to the inertial reference frame can be written as:

$$m_i \frac{d^2\bar{r}_i}{dt^2} = \sum_{j \neq i} G \frac{m_i m_j}{r_{ij}^3} \bar{r}_{ij} \tag{2.1}$$

The equation of motion of body i may be written as three scalar second-order differential equations. Similarly, for the motion of n bodies 3n second-order differential equations can be written. Generally, this set cannot be solved analytically and one has to rely on numerical integration techniques to determine the motion of the bodies. However, some general characteristics of the many-body problem can be derived. These characteristics are known as the ten integrals of motion, which will be derived in the following Section.

2.1 Integrals of motion

To determine the first 6 integrals of motion, a summation over all bodies i of equation 2.1 is taken. Because all bodies exert graviational forces on each other, the net graviational force is zero. Substituting the defintion of the center of mass in the equation results in:

$$\sum_i m_i \frac{d^2\bar{r}_i}{dt^2} = \sum_i \sum_{j \neq i} G \frac{m_i m_j}{r_{ij}^3} \bar{r}_{ij} = 0 \qquad \rightarrow \qquad \frac{d^2\bar{r}_{cm}}{dt^2} = 0 \tag{2.2}$$

This shows that the center of mass is in constant motion. Integrating the differential equation results in the first 6 integrals of motion, written in cartesian coordinates.

$$\frac{d\bar{r}_{cm}}{dt} = \bar{a} \qquad ; \qquad \bar{r}_{cm} = \bar{a}t + \bar{b} \tag{2.3}$$

Three more integrals of motion can be found by taking the vector product of 2.1 and subsequently applying a summation for all i.

$$\sum_i m_i \bar{r}_i \times \frac{d^2\bar{r}_i}{dt^2} = \sum_i \sum_{j \neq i} G \frac{m_i m_j}{r_{ij}^3} \bar{r}_i \times \bar{r}_j = 0 \tag{2.4}$$

Resulting three integrals of motion, where $\overline{H}$ denotes the *total angular momentum* of the many-body system.

$$\frac{d}{dt}\left(\sum_i m_i \bar{r}_i \times \frac{d\bar{r}_i}{dt}\right) = 0 \qquad ; \qquad \overline{H} = \sum_i m_i \bar{r}_i \times \frac{d\bar{r}_i}{dt} = \bar{c} \tag{2.5}$$

The vector $\overline{H}$ defines an invariable plane that passes through the center of mass of the n bodies and that is perpendicular to the angular momentum vector. This plane is called the **invariable plane of Laplace** and can be used as a reference plane for describing the motion of the n bodies.

The force field is **non-central** because the potential is dependent on the relative positions r_{ij} and not the position with respect to an inertial reference frame r_i. Secondly, the value of the potential at a fixed position relative to the inertial reference frame will vary with time, because the bodies j are moving. For such a time-varying potential the sum of kinetic and potential energy of body mi is not constant. Therefore, we are dealing with a non-central, **non**-conservative force field.

The last integral of motion can be found by taking the scalar product of $d\overline{r}_i/dt$ and 2.1 and subsequently applying a summation for all i. With E_k representing *the total kinetic energy* and E_p representing *the total potential energy.*

$$\sum_i \frac{1}{2} m_i V_i^2 - \frac{1}{2} G \sum_i \sum_{j \neq i} \frac{m_i m_j}{r_{ij}} = C \qquad \rightarrow \qquad \varepsilon_k + \varepsilon_p = C \tag{2.6}$$

2.2 Moment of inertia

The polar moment of inertia is given by:

$$I = \sum_i m_i r_i^2 \tag{2.7}$$

The derivatives of the moment of inertia are found to be:

$$\frac{dI}{dt} = 2 \sum_i m_i \overline{r}_i \frac{d\overline{r}_i}{dt} \qquad ; \qquad \frac{d^2 I}{dt^2} = 4\varepsilon_k + 2\varepsilon_p = 4C - 2\varepsilon_p = 2C + 2\varepsilon_k \tag{2.8}$$

A stable system is a system for which $d^2I/dt^2 < 0$, because even if the derivative is positive, eventually the moment of inertia starts decreasing. So, a necessary, but not sufficient, condition for a stable system is: $C < 0$. That this is a non-sufficient condition follows directly from the fact that, according to the above equation, for given values of ε_k and ε_p, negative values of C can exist that still yield: $d^2I/dt^2 > 0$.

2.3 Stable systems

To investigate stable systems, the average of the second derivative of the polar moment of inertia is taken.

$$\frac{1}{t_e} \int_0^{t_e} \frac{d^2 I}{dt^2} = \frac{4}{t_e} \int_0^{t_e} \varepsilon_k + \frac{2}{t_e} \int_0^{t_e} \varepsilon_p \qquad \rightarrow \qquad \frac{1}{t_e} \left(\frac{dI}{dt} \right)_0^{t_e} = 4\overline{\varepsilon}_k + 2\overline{\varepsilon}_p \tag{2.9}$$

In a stable systems no collisions and no escapes occur. In other words: all bodies stay within a finite distance from the origin and the velocities of all bodies remain finite. In that case, the value of the expression between brackets in will remain finite. Therefore, if the time interval is chosen large enough, the left-hand side of will approach zero. So, for a sufficiently long averaging period, we find for a stable system:

$$4\overline{\varepsilon}_k + 2\overline{\varepsilon}_p = 0 \qquad \rightarrow \qquad \overline{\varepsilon}_k = -\frac{1}{2}\overline{\varepsilon}_p = -C \tag{2.10}$$

Chapter 3

The Three Body Problem

3.1 Equations of motion

The general equation of motion for a body i in a Three body problem is given by:

$$\frac{d^2\overline{r}_i}{dt^2} = G\frac{m_j}{r_{ij}^3}\overline{r}_{ij} + G\frac{m_k}{r_{ik}^3}\overline{r}_{ik} \quad \{i,j,k\} = \{1,2,3\} \qquad where \quad \overline{r}_{ij} = \overline{r}_j - \overline{r}_i \quad ; \quad r_{ij} = |\overline{r}_{ij}| \tag{3.1}$$

These equations represent the *classical or Euler* formulation of the three body problem. When the position of the bodies is written in the rectangular coordinates x, y, z, we arrive at a set of first-order differential equations of the order eighteen.

In the *Lagrange formulation* of the three body problem, the variables are $\overline{r}_{12}$, $\overline{r}_{23}$ and $\overline{r}_{31}$. Resulting in the following equations of motion.

$$\frac{d^2\overline{r}_{12}}{dt^2} = G\left(m_3\left(\frac{\overline{r}_{23}}{r_{23}^3} + \frac{\overline{r}_{31}}{r_{31}^3}\right) - (m_1+m_2)\frac{\overline{r}_{12}}{r_{12}^3}\right) \tag{3.2}$$

$$\frac{d^2\overline{r}_{23}}{dt^2} = G\left(m_1\left(\frac{\overline{r}_{31}}{r_{31}^3} + \frac{\overline{r}_{12}}{r_{12}^3}\right) - (m_2+m_3)\frac{\overline{r}_{23}}{r_{23}^3}\right) \tag{3.3}$$

$$\frac{d^2\overline{r}_{31}}{dt^2} = G\left(m_2\left(\frac{\overline{r}_{12}}{r_{12}^3} + \frac{\overline{r}_{23}}{r_{23}^3}\right) - (m_3+m_1)\frac{\overline{r}_{31}}{r_{31}^3}\right) \tag{3.4}$$

The Jacobi method uses the vector $\overline{r}_{12}$ and the vector $\overline{R}$ from the center of mass of P_1 and P_2 to P_3. This vector , of course, passes through the center of mass of the entire system. In the figure below these centers of mass are indicated by O_{12} and O, respectively.

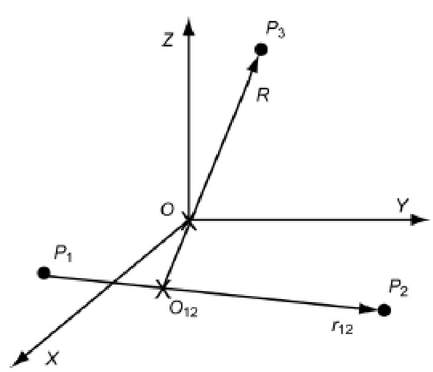

Figure 3.1: Jacobi method.

The following equations are used to derive the equations of motion for this representation.

$$\alpha = \frac{m_1}{m_1 + m_2} \quad ; \quad \overline{r}_{13} = \overline{R} + (1-\alpha)\overline{r}_{12} \quad ; \quad \overline{r}_{23} = \overline{R} - \alpha\overline{r}_{12} \quad ; \quad \overline{R} = \alpha\overline{r}_{13} + (1-\alpha)\overline{r}_{23} \tag{3.5}$$

The following equations can be derived:

$$\frac{d^2\overline{R}}{dt^2} = -GM\left[\alpha\frac{\overline{r}_{13}}{r_{13}^3} + (1-\alpha)\frac{\overline{r}_{23}}{r_{23}^3}\right] \tag{3.6}$$

$$\frac{d^2\overline{r}_{12}}{dt^2} = -G\left[(m_1+m_2)\frac{\overline{r}_{12}}{r_{12}^3} + m_3\left(\frac{\overline{r}_{13}}{r_{13}^3} - \frac{\overline{r}_{23}}{r_{23}^3}\right)\right] \tag{3.7}$$

form the Jacobi set of equations for the three-body problem. It is emphasized that these equations constitute a twelfth-order system; the reduction from eighteenth order to twelfth order was essentially achieved by the explicit use of the center-of-mass integrals. A further reduction is, of course, possible using the remaining integrals of motion, the invariable plane of Laplace as reference plane and an angular coordinate to replace time.

As an application of the Jacobi set of equations, we consider the so-called lunar case and planetary case. In the *lunar case*, where P_1 is the Earth, P_2 the Moon and P_3 the Sun, we know that:

$$\frac{d^2\overline{R}}{dt^2} = -GM\left(\alpha\frac{\overline{R}}{R^3}\right) \qquad \frac{d^2\overline{r}_{12}}{dt^2} = -G(m_1+m_2)\frac{\overline{r}_{12}}{r_{12}^3} \qquad with \quad \alpha \approx 1 \quad ; \quad \overline{r}_{13} \approx \overline{r}_{23} \approx \overline{R} \tag{3.8}$$

For the planetary case, with P_1 the Sun, P_2 the Earth and P_3 a planet we arrive at the same approximative equations of motion, using the following assumptions:

$$\alpha \approx 1 \quad ; \quad \frac{m3}{m_1+m_2} << 1 \quad ; \quad \overline{r}_{13} \approx \overline{R} \tag{3.9}$$

3.2 Central configuration solutions

Lagrange has found a particular case of three-body motion in which the mutual distances between the bodies remain constant, and Euler has extended this class of motion and has found solutions in which the ratios of the mutual distances remain constant. These classes of solutions refer to cases where the geometrical shape of the three-body configuration does not change, although the scale may change and the configuration may rotate. Lagrange and Euler showed that for three bodies of arbitrary mass such solutions are possible if:

- The resultant force on each body passes through the center of mass of the system.
- The resultant force is proportional to the distance of a body from the center of mass of the system.
- The magnitudes of the initial velocity vectors are proportional to the respective distances of the bodies from the center of mass of the system, and these velocity vectors make equal angles with the radius vectors to the bodies from the center of mass of the system.

Because of these requirements, the solutions are generally referred to as *central configurations*.

An extinsive derivation beginning with the equations of motion leads to the following equations:

$$m_2\overline{r}_1 \times \overline{r}_2\left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3}\right) = 0 \quad ; \quad m_3\overline{r}_2 \times \overline{r}_3\left(\frac{1}{r_{23}^3} - \frac{1}{r_{12}^3}\right) = 0 \quad ; \quad m_1\overline{r}_3 \times \overline{r}_1\left(\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3}\right) = 0 \tag{3.10}$$

This set of equations can be satisfied by the **equilateral triangle solution**:

$$r_{12} = r_{23} = r_{13} = r \tag{3.11}$$

And This set of equations can also be satisfied by a solution which puts the bodies on a **straight line**:

$$\overline{r}_1 \times \overline{r}_2 = \overline{r}_2 \times \overline{r}_3 = \overline{r}_3 \times \overline{r}_1 = 0 \tag{3.12}$$

3.3 Circular restricted three-body problem

A partial solution of a special *curcular restricted three-body problem* is obtained by Lagrange. For this special three-body problem, the following assumptions are made:

- The mass of two bodies is much larger than the mass of the third body. Then, the third body moves in the gravity field of the two massive bodies, and the effect of the gravitational attraction by the third body on the motion of these massive bodies can be neglected.
- The two massive bodies move in circular orbits about the center of mass of the system.

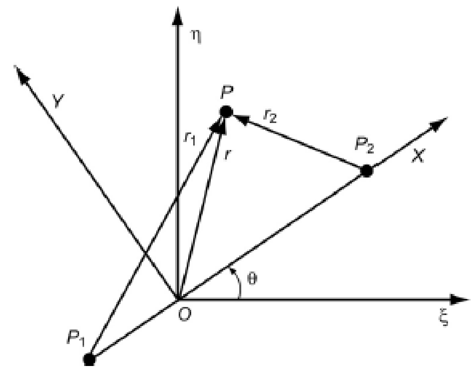

Figure 3.2: Inertial and rotating reference frames in the circular restricted three-body problem

The orbits of the two massive bodies (P_1, P_2) being known, the problem is to determine the motion of the third body P. The general three-body problem is thus reduced from nine second-order differential equations to three second-order ones. This means a reduction from order eighteen to order six. Since the mass of the third body is assumed to be negligible, the two main bodies move as if they form a two-body system.

A reference frame is chosen with its origin at the center of mass of the system of three bodies of which the X-axis coincides with P_1P_2. This reference frame rotates with a constant angular velocity ω. Since both massive bodies move in circular orbits about the center of mass O, we may conclude that the distances OP_1 and OP_2 are constant.

An extensive analysis can be performed resulting in the following equations:

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \tag{3.13}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} \tag{3.14}$$

$$\ddot{z} = \frac{\partial U}{\partial z} \tag{3.15}$$

With:

$$U = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{3.16}$$

From these equations we conclude that U is a potential function that accounts both for the gravitational forces and for the centrifugal force. The potential function can, of course, not account for the Coriolis force, because this force is a function of velocity components. The force field described by the potential U is clearly *non-central*. Because the bodies P_1 and P_2 have fixed positions with respect to the rotating reference frame, U is not explicitly a function of time, which means that the force field is conservative.

3.4 Jacobi's integral

Using the potential function as described in the *circular restricted problem* an integral can be derived with integration constant C. Where the value of C is determined by the position and velocity of body P.

$$V^2 = 2U - C \tag{3.17}$$

3.5 Surfaces of Hill

A special case occurs when the velocity of the small body P is zero.

$$2U = C \tag{3.18}$$

This equation describes the *surfaces of Hill.* These are surfaces in XYZ-space on which the velocity of the third body is zero.

Since for any real body $V^2 > 0$, the region in space where the third body can move is given by:

$$2U \geq C \tag{3.19}$$

So, although we cannot determine the orbit of the third body, we can determine which part of the XYZ-space is accessible to the third body for a given value of $C \rightarrow$ *initial conditions.* In the figure below, this unaccesible area is hatch for several values of C.

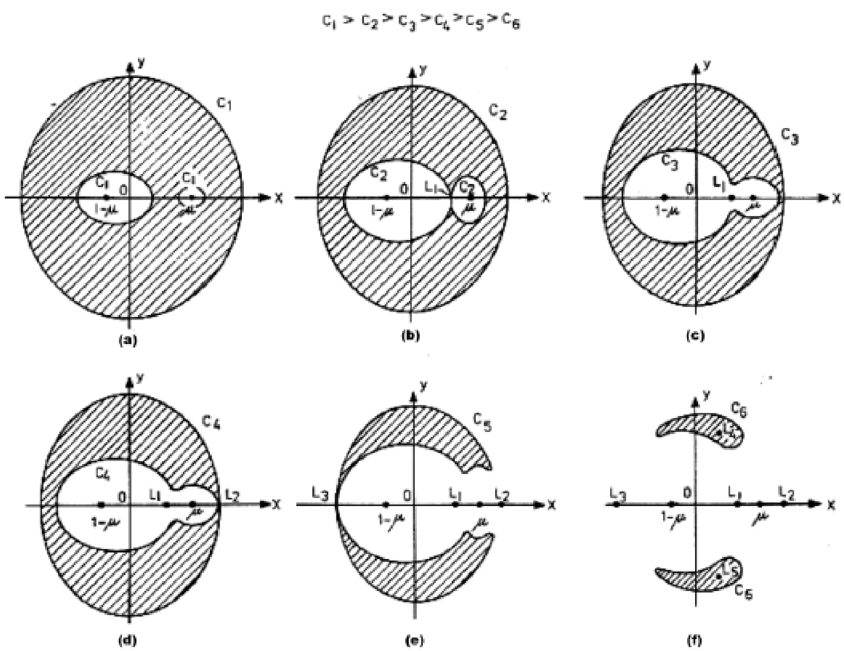

Figure 3.3: Schematic picture of the surfaces of HIll for creasing values of C

3.6 Lagrange libration points

The figure above shows the accesible area for a body in the co-rotating reference frame for several values of C (representing the energy). The figure shows that for decreasing values of C (increasing velocity), the surfaces open up at certain points. These points are the *Lagrange libration points*. A minimum amount of energy is needed to go from P_1 to P_2 through the Lagrange point $L1$.

Another way to describe the Lagrange libration points is to look at the potential function U which is described in paragraph 3.3. The Lagrange libration points are the points where there is a local minimum in the potential function U.

$$\left|\frac{\partial U}{\partial x}\right|_L = \left|\frac{\partial U}{\partial y}\right|_L = \left|\frac{\partial U}{\partial z}\right|_L = 0 \tag{3.20}$$

Chapter 4

Relative Motion in the Many Body Problem

An Equation of motion can be derived for the motion of body i under the influence of the gravitational attraction of bodies j and a body k:

$$\frac{d^2\overline{r}_{ki}}{dt^2} = -G\frac{m_1 + m_2}{r_{ki}^3}\overline{r}_{ki} + \sum_{j\neq i,k} m_j \left(\frac{\overline{r}_{kj} - \overline{r}_{ki}}{r_{ij}^3} - \frac{\overline{r}_{kj}}{r_{kj}^3}\right) \tag{4.1}$$

When the origin of the reference frame is body k, the equation becomes:

$$\frac{d^2\overline{r}_{i}}{dt^2} = -G\frac{m_1 + m_2}{r_{i}^3}\overline{r}_{i} + \sum_{j\neq i,k} m_j \left(\frac{\overline{r}_{j} - \overline{r}_{i}}{r_{ij}^3} - \frac{\overline{r}_{j}}{r_{j}^3}\right) \tag{4.2}$$

4.1 Influence of perturbing accelerations

Figure 4.1 shows the geometry of the problem. It is assumed that body i moves in a circular orbit about body k. Neglecting the mass of body i with respect to the mass of body k, gives the magnitude of the main acceleration of body i:

$$a_m = G\frac{m_k}{r_i^2} \tag{4.3}$$

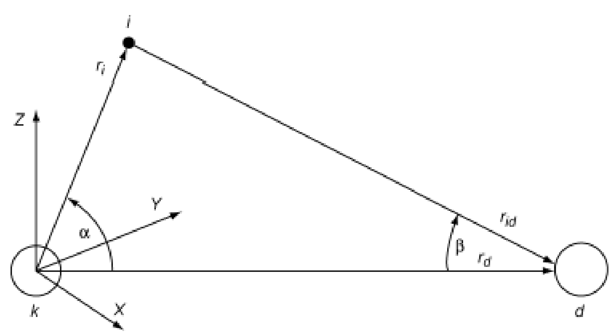

Figure 4.1: Jacobi method.

For the magnitude of the perturbing acceleration follows from 4.1:

$$a_d = Gm_d \sqrt{\left(\frac{\overline{r}_{id}}{r_{id}^3} - \frac{\overline{r}_d}{r_d^3}\right) \cdot \left(\frac{\overline{r}_{id}}{r_{id}^3} - \frac{\overline{r}_d}{r_d^3}\right)} \tag{4.4}$$

which gives:

$$a_d = Gm_d \sqrt{\left(\frac{1}{r_{id}^4} + \frac{1}{r_d^4} - \frac{2r_{id}r_d\cos\beta}{r_d^3 r_{id}^3}\right)} \qquad using \qquad A \cdot B = |A|\,|B|\cos\theta \tag{4.5}$$

Using to the figure the following equations can be derived:

$$\cos\beta = \frac{r_d - r_i\cos\alpha}{r_{id}} \qquad ; \qquad r_{id}^2 = r_i^2 + r_d^2 - 2r_i r_d \cos\alpha \tag{4.6}$$

Using the following equation:

$$\left(1 - (1-\gamma)^{-2}\right)^2 = 1 - (1-\gamma)^{-2} + (1-\gamma)^{-4} \tag{4.7}$$

The magnitude of the perturbing acceleration can be derived:

$$a_d = G\frac{m_d}{r_d^2}\sqrt{1 + \frac{1}{(1 - 2\gamma\cos\alpha + \gamma^2)^2} - \frac{2\,(1 - \gamma\cos\alpha)}{(1 - 2\gamma\cos\alpha + \gamma^2)^{3/2}}} \tag{4.8}$$

For the maximum perturbing acceleration ($\alpha = 0$) we find:

$$a_d = G\frac{m_d}{r_d^2}\left|\left(\left(\frac{1}{1-\gamma}\right)^2 - 1\right)\right| \tag{4.9}$$

Chapter 5

Two body problem

The Two body problem can be mathemally described by:

$$\frac{d^2\overline{r}}{dt^2} = -\frac{\mu}{r^3}\overline{r} \quad where \quad \mu = G(m_k + m_i) \tag{5.1}$$

5.1 Conservation laws

To derive the first conservation law we take the scalar product of the equation of motion with dr/dt:

$$\frac{d\overline{r}}{dt} \cdot \frac{d^2\overline{r}}{dt^2} + \frac{\mu}{r^3}\frac{d\overline{r}}{dt} \cdot \overline{r} = \frac{1}{2}\frac{d}{dt}\left(\frac{d\overline{r}}{dt} \cdot \frac{d\overline{r}}{dt}\right) - \frac{d}{dt}\left(\frac{\mu}{r}\right) = 0 \quad \rightarrow \quad \frac{1}{2}V^2 - \frac{\mu}{r} = \varepsilon \tag{5.2}$$

The second conservation law is derived by taking the vector product:

$$\overline{r} \times \overline{V} = \overline{H} \quad ; \quad r^2\dot{\varphi} = H = constant \tag{5.3}$$

The area for a small surface element defined by the vectors $\overline{r}$ and $\overline{r} + \Delta\overline{r}$ is given in the equation below. From this equation Kepler's second law can be derived, which says that equal areas are swept out in equal intervals of time.

$$\Delta A = \frac{1}{2}r^2\Delta\varphi + O(r\Delta r\Delta\varphi) \quad \rightarrow \quad \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\varphi}{dt} \quad \rightarrow \quad \frac{dA}{dt} = \frac{1}{2}H \quad ; \quad \frac{d\varphi}{dt} = \frac{H}{r^2} \tag{5.4}$$

5.2 The orbital equation

The orbital equation is derived by taking the scalar product of equation 5.1 with $\overline{r}$.

$$\overline{r} \cdot \frac{d^2\overline{r}}{dt^2} + \frac{\mu}{r^3}\overline{r} \cdot \overline{r} = \frac{d}{dt}\left(\overline{r} \cdot \frac{d\overline{r}}{dt}\right) - \frac{d\overline{r}}{dt} \cdot \frac{d\overline{r}}{dt} + \frac{\mu}{r} = 0 \tag{5.5}$$

Substituting the following definitions:

$$\overline{r} \cdot \frac{d\overline{r}}{dt} = r \cdot \frac{dr}{dt} \quad ; \quad \frac{d\overline{r}}{dt} \cdot \frac{d\overline{r}}{dt} = V^2 \quad ; \quad V^2 = (\dot{r})^2 + (r\dot{\varphi})^2 \tag{5.6}$$

Gives the equation below. This equation is a non-linear differential equation coupled with equation 5.4.

$$\left(\frac{dr}{dt}\right)^2 + r\frac{d^2r}{dt^2} - (\dot{r})^2 - (r\dot{\varphi})^2 = -\frac{\mu}{r} \quad \rightarrow \quad \ddot{r} - r\dot{\varphi}^2 = -\frac{\mu}{r^2} \tag{5.7}$$

The following derivatives are used:

$$\dot{r} = \frac{\partial r}{\partial \varphi}\frac{\partial \varphi}{\partial t} = \frac{H}{r^2}\frac{\partial r}{\partial \varphi} \quad ; \quad \ddot{r} = \frac{\partial \dot{r}}{\partial \varphi}\frac{\partial \varphi}{\partial t} = \frac{H}{r^2}\frac{\partial \dot{r}}{\partial \varphi} \tag{5.8}$$

The parameter u is introduced for simplicity, giving:

$$u = \frac{1}{r} \quad ; \quad \frac{\partial r}{\partial u} = -\frac{1}{u^2} \quad ; \quad \frac{\partial r}{\partial \varphi} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial \varphi} = -\frac{1}{u^2}\frac{\partial u}{\partial \varphi} \quad \rightarrow \quad \dot{r} = Hu^2\left(-\frac{1}{u^2}\right)\frac{\partial u}{\partial \varphi} = -H\frac{\partial u}{\partial \varphi} \tag{5.9}$$

The second derivative of r is found by:

$$\ddot{r} = Hu^2\frac{\partial \dot{r}}{\partial \varphi} = Hu^2\frac{\partial Hu^2\frac{\partial r}{\partial \varphi}}{\partial \varphi} = Hu^2\frac{\partial Hu^2\left(-\frac{1}{u^2}\right)\frac{\partial u}{\partial \varphi}}{\partial \varphi} = -H^2u^2\frac{\partial^2 u}{\partial \varphi^2} \tag{5.10}$$

Substituting equations 5.10 and 5.9 into equation 5.6, results in the following differential equation:

$$\frac{\partial^2 u}{\partial \varphi^2} + u = \frac{\mu}{H^2} \tag{5.11}$$

The solution to this differential equation is:

$$u = \frac{\mu}{H^2} + c_1\cos\varphi + c_2\sin\varphi = \frac{\mu}{H^2}\left(1 + c_3\cos\left(\varphi + \omega\right)\right) \tag{5.12}$$

Rewriting the solution gives the *orbital equation*:

$$r = \frac{H^2/\mu}{1 + c_3\cos\left(\varphi + \omega\right)} \tag{5.13}$$

Equation 5.13 is the general equation for a **Conic section**, which can be rewritten in the form given below. With e as the eccentricity of the conic section and p as the semi-latus rectum. An example of a conic section is an ellipse, therefore it can be concluded that this is the proof for Kepler's first law.

$$r = \frac{p}{1 + e\cos\left(\varphi + \omega\right)} = \frac{p}{1 + e\cos\theta} \tag{5.14}$$

The type of conic section that is described by this equation depends on the *eccentricity* e, the following cases can be distinguised:

$e = 0$	:	circle
$0 < e < 1$	:	ellipse
$e = 1$	:	parabola
$e > 1$	:	hyperbola

Finally returning to the equation of motion. From this relation it can be concluded that the radial acceleration of body i is equal to the difference between the centrifugal acceleration and the gravitational acceleration.

$$\ddot{r} = r\dot{\varphi}^2 - \frac{\mu}{r^2} \tag{5.15}$$

5.3 Velocity components

When The *flight path angle* γ is introduced the velocity components can be rewritten. With $\dot{r}$ as the radial velocity and $r\dot{\theta}$ as the normal velocity.

$$\dot{r} = V \sin\gamma \quad ; \quad r\dot{\theta} = V\cos\gamma \tag{5.16}$$

Using the definition of the angular momentum H, the velocity components can be rewritten:

$$\dot{r} = \frac{\mu}{H} e \sin\theta \quad ; \quad r\dot{\theta} = \frac{\mu}{H}(1 + e\cos\theta) \quad with \quad H = rV\cos\gamma \tag{5.17}$$

The flight path angle can be derived from the above equations, yielding:

$$\tan\gamma = \frac{\dot{r}}{r\dot{\theta}} = \frac{e\sin\theta}{1 + e\cos\theta} \tag{5.18}$$

The velocity components V_n and V_l can be derived using the following figure:

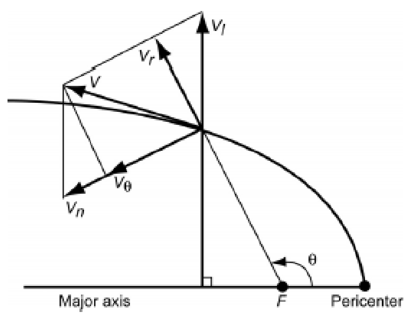

Figure 5.1: Velocity components

$$V_l = \frac{\dot{r}}{\sin\theta} = \frac{\mu e}{H} \quad ; \quad V_n = r\dot{\theta} - \frac{\dot{r}}{\tan\theta} = \frac{\mu}{H} \tag{5.19}$$

These velocity components have a constant magnitude and the component V_l also has a constant orientation. The component V_l is always oriented perpendicular to the axis of symmetry of the conic section. The component V_n is always oriented in the direction of $V_\theta = r\dot{\theta}$, which doesn't have a constant orientation, but is always perpedicular to the radius vector.

Using the figure and the velocity components V_n and V_l, the following equation can be derived:

$$\dot{r}^2 + \left(r\dot{\theta} - \frac{\mu}{H}\right)^2 = \left(\frac{\mu e}{H}\right)^2 \tag{5.20}$$

This equation can be used to construct *Velocity hodographs*. These are displayed in the figure below:

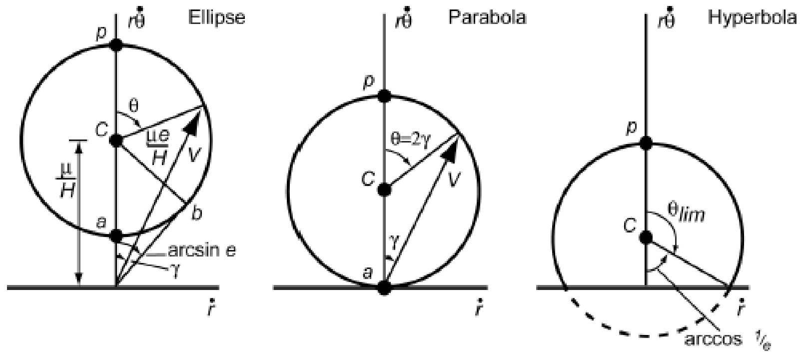

Figure 5.2: Velocity components

The eccentricity vector is:

$$e^2 = 1 - \frac{rV^2}{\mu}\left(2 - \frac{rV^2}{\mu}\right)\cos^2\gamma \tag{5.21}$$

5.4 Roche limit

The *Roche limit* is defined by the transit from 'stability' to 'disintegration' of a moon. In other words it defines if a bould that is placed on a moon will stay on the moon or accelerate away from the moon, due to centrifugal accelerations. It can be calculated by:

$$a_{rel} = a_{cen,orb} + a_{cen,rot} + a_{tide} - a_{grav} \quad \rightarrow \quad a_rel = G\left(4\frac{m_1 R_2}{r_2^3} - \frac{m_2}{R_2^2}\right) \quad a_{rel} < 0 \;\rightarrow\; stable \tag{5.22}$$

5.5 Relativistic effects

The major effect of *Relativity* is the rotation of the semi major axis. The difference per revolution in the *argument of perigee* ω is given by:

$$\Delta\omega = 6\pi\frac{\mu^2}{H^2c^2} \;\; [rad/rev] \quad ; \quad \omega = \omega_0 + 3\frac{\mu^2}{c^2H^2} \tag{5.23}$$

5.6 Poynting-Robertson effect

The force per unit of mass by radiation of the sun on a body i can be modelled by:

$$\frac{F}{m} = \frac{3}{4}\frac{C_R W_S R_S^2}{c\rho R}\frac{1}{r^2} = \frac{\alpha}{r^2} \tag{5.24}$$

When applying 5.24 for analyzing the motion of body i, two phenomena should be taken into account. First, if the body has a radial velocity component relative to the Sun, then the frequency ν of the incoming sunlight is shifted through the *Doppler effec*t to a new frequency ν' given by:

$$\nu' = \nu\left(1 - \frac{\dot{r}}{c}\right) \tag{5.25}$$

The energy and thus momentum of a photon is dependent on the frequency ν. Therefore, the power density W of the radiation intercepted by body i changes to:

$$W' = W\left(1 - \frac{\dot{r}}{c}\right) \quad ; \quad W \backsim \nu \tag{5.26}$$

A second phenomenon is related to the finite speed of light. This means that the sunlight intercepted by body i at t_0 is actually emitted by the Sun at $t_0 - \Delta t$. During the time interval Δt light travels the distance $c\Delta t$, while the body has moved over a distance $r\dot{\varphi}\Delta t$ in the direction normal to the direction to the Sun (Figure 5.12). This leads to the so-called aberration of the incoming sunlight; the aberration angle γ is given by:

$$\gamma \approx \frac{r\dot{\varphi}\Delta t}{c\Delta t} = \frac{r\dot{\varphi}}{c} \tag{5.27}$$

These equations show that the action of sunlight effectively reduces the central gravitational attraction force by the Sun, but also produces two additional terms proportional to the radial and circumferential velocities of the body. The second additional term corresponds to a drag-type of force. These adjusted equations of motion were first formulated by H.P. Robertson.

Chapter 6

Elliptical and circular orbits

In general the elliptical orbit can be drawn as in the following figure:

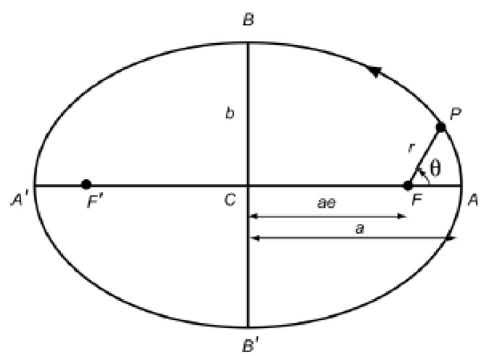

Figure 6.1: Nomenclature of an ellipse.

The semi-latus rectum is derived in the following way:

$$2a = r_a + r_p = \frac{p}{1+e} + \frac{p}{1-e} \quad \rightarrow \quad p = a(1-e^2) \tag{6.1}$$

With the above equation the semi major axis a can be related to the total energy. This is can be derived by using the definition of the semi latus rectum p and the eccentricity vector. From this equation it can be shown that the total energy is smaller than zero. $\varepsilon < 0$

$$a = -\frac{\mu}{2\varepsilon} \quad ; \quad \frac{V^2}{2} - \frac{\mu}{r} = \varepsilon = -\frac{\mu}{2a} \tag{6.2}$$

The semi-minor axis b can be derived by differentiating vertical distance with respect to θ and setting the result equal to zero.

$$b = a\sqrt{1-e^2} \tag{6.3}$$

By using the area equation of an ellipse, the orbital period T can be derived. The *mean motion* n is defined by the average angular velocity.

$$\pi ab = \frac{1}{2}HT \quad ; \quad T = 2\pi\sqrt{\frac{a^3}{\mu}} \quad ; \quad n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}} \tag{6.4}$$

These equations give the proof of the third law of Kepler.

6.1 Circular velocity

The circular velocity is derived by using the definition of a circular orbit. Where the eccentricity e is equal to zero and the semi major axis a is equal to the radius r. It can also be derived using the eccentricity vector. The circular velocity is given by:

$$V_c = \sqrt{\frac{\mu}{r}} \tag{6.5}$$

6.2 Kepler's equation

A relation between the position in the orbit and time is derived with the use of the *Eccentric anomaly E*. The eccentric anomaly can be derived from the figure below:

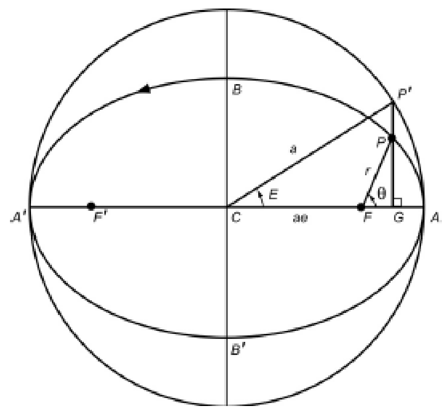

Figure 6.2: Eccentric anomaly.

Using the figure, the following equation can be derived:

$$r\cos\theta = a\cos E - ae \tag{6.6}$$

With the use of the definitions of an ellipse the following equation can be derived:

$$\frac{PG}{P'G} = \frac{b}{a} = \sqrt{1-e^2} \quad ; \quad r\sin\theta = a\sqrt{1-e^2}\sin E \tag{6.7}$$

Using the equations above, the following equation can be derived by substituting:

$$r = a(1 - e\cos E) \tag{6.8}$$

Combining all the equations and using a trigonometric expression, the relationship between E and θ can be derived:

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \quad ; \quad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} \tag{6.9}$$

Differentiating Equation 6.8 and substituting the earlier found equations result in:

$$E - e\sin E = n(t-\tau) = M \quad \textit{with} \quad n = \sqrt{\frac{\mu}{a^3}} \tag{6.10}$$

Chapter 7

Parabolic orbits

The parabolic orbit is characterized as follows:

$$e = 1 \qquad ; \qquad a = \infty \tag{7.1}$$

With these relations the general equation for a parabola can be derived:

$$r = \frac{p}{1 + \cos\theta} \tag{7.2}$$

A parabolic orbit escapes the gravitation of the attracting body. The escape velocity can thus be derived by using $e = 1$ and the eccentricity vector. Or by using the energy equation and substituting the earlier given conditions:

$$V_{esc} = \sqrt{\frac{2\mu}{r}} = \sqrt{2}V_c \tag{7.3}$$

In other words the escape velocity can be computed by multiplying the local circular velocity with $\sqrt{2}$. It can be derived that the total energy of a body in a parabolic orbit is always equal to zero.

$$\varepsilon = 0 \tag{7.4}$$

7.1 Barker's equation

For a hyperbolic orbit is not needed to introduce a new variable to obtain an equation that relates position and time.

$$\frac{d\theta}{dt} = \frac{H}{r^2} = \frac{\sqrt{\mu p}}{r^2} \qquad ; \qquad dt = \sqrt{\frac{p^3}{\mu}}\frac{d\theta}{(1+\cos\theta)^2} \tag{7.5}$$

Integrating this equation gives Barker's equation:

$$\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2} = 2\sqrt{\frac{\mu}{p^3}}(t - \tau) \tag{7.6}$$

Introducing the angular velocity $\overline{n}$ and a kind of mean anomaly $\overline{M}$, gives:

$$\overline{n} = \sqrt{\frac{\mu}{p^3}} \qquad ; \qquad \overline{M} = \overline{n}(t - \tau) \tag{7.7}$$

Chapter 8

Hyperbolic orbits

The same equations as for the elliptical orbit hold for the hyperbolic orbit. The following definitions hold for the hyperbolic orbit:

$$e > 1 \quad ; \quad a < 0 \tag{8.1}$$

With the orbit equation, the limit value of θ can be derived by setting the radius r to ∞.

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad ; \quad \theta_{lim} = \arccos(-1/e) \tag{8.2}$$

Because for the hyperbolic orbit the eccentricity is negative, we find that the total energy is positive. This can be seen as that the kinetic energy of the body is larger than the potential(graviational) energy at any point in the orbit.

$$\varepsilon > 0 \tag{8.3}$$

The velocity reaches a minimum for $r = \infty$, giving the excess velocity.

$$V_\infty = \sqrt{-\frac{\mu}{a}} \tag{8.4}$$

Using the definition of the escape velocity and the excess velocity the following equation can be derived:

$$V^2 = V_{esc}^2 + V_\infty^2 \tag{8.5}$$

8.1 Relation between position and time

A relation between position and time can be derived by using the *Hyperbolic anaomaly* F, which is defined as:

$$r = a(1 - e\cosh F) \quad ; \quad F = \frac{2 \times area}{a^2} \tag{8.6}$$

The hyperbolic anomaly is related with the angle θ through the following equation:

$$\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{F}{2} \tag{8.7}$$

Giving the following equation:

$$e\sinh F - F = \overline{n}(t-\tau) = \overline{M} \quad ; \quad \overline{n} = \sqrt{\frac{\mu}{-a^3}} \tag{8.8}$$

Chapter 9

Relative motion

The motion of a satellite 2 w.r.t satellite 1 can be determined by adjusting the equations of motion. The reference frame that is used is given in the figure below. Where the X,Y and Z specify the *radial, along-track* and *cross-track* directions of the motion of satellite 2 with respect to satellite 1.

Chapter 10

Reference frames, Cooordinates and Time

A **reference system** then is the complete specification of how a celestial coordinate system is to be formed. Both the origin and the orientation of the fundamental (reference) planes (or axes) are defined. A reference system also incorporates a specification of the fundamental models needed to construct the system; that is, the basis for the algorithms used to transform between observable quantities and reference data in the system.

A **reference frame**, on the other hand, consists of a set of identifiable fiducial points on the sky along with their coordinates, which serves as the practical realization of a reference system.

10.1 Positions on Earth

- Equator: The great circle on the Earth's surface halfway between the poles.
- Meridian: Great circle passing through the poles, perpendicular to the Equator.
- Geographic longitude: Geocentric angle, measured along the equator, from the prime meridian, to the meridian of point P.
- Geocentric latitude: Geocentric angle, measured along the meridian of point P, from the equator to point P.
- Geodetic latitude: Angle between the equatorial plane and a line normal to the tangent plane at the point P.

10.2 Positions of celestial objects

- Celestial sphere: This is a sphere with an infinitely large radius, centered at an observer on Earth or at the mass center of the Earth. The remote stars appear to be set on the inner surface of this sphere.
- Ecliptic: The path of the Sun over the celestial sphere, or the intersection of the plane of the Earth's orbit about the Sun with the celestial sphere.
- Equinox line: Intersecting line of the equatorial plane and the ecliptic plane
- Vernal / autumnal equinoxes: Intersection points of the equinox line and the celestial sphere. When the Sun crosses this line, the Earth's axis of rotation is at right angles to the Sun-Earth line and, consequently, day and night have equal length, everywhere on Earth. This is the case around March

20/21 and September 22/23 each year. These crossing points are therefore called the vernal equinox and the autumnal equinox, respectively.

- Declination: Geocentric angle, measured from the celestial equator, along the object's hour circle, to the object.
- Right ascension: Geocentric angle, measured along the celestial equator, from the vernal equinox to the foot of the object's hour circle.

10.3 Time

- Solar time: Time defined by the angular distance covered by the Sun on the celestial sphere after its last crossing of the observer's celestial meridian.
- Sidereal time: Time defined by the angular distance covered by the vernal equinox on the celestial sphere after its last crossing of the observer's celestial meridian.
- Atomic time (TAI): Time based on the analysis of about 200 frequency standards (atomic clocks) maintained by several countries to keep a unit of time as close to the ideal SI second as possible (defined in terms of Caesium 133 transitions).
- Universal time (UT / UT1): A standardised mean solar time, based on a fictitious mean Sun that moves at a uniform rate eastward along the celestial equator.
- Universal time coordinated (UTC): A hybrid standard of time, of which the progression is determined by atomic time (TAI), but leap seconds are introduced, when needed, to keep up with Universal Time (UT1).

Chapter 11

Equations to remember

The orbital equation:

$$r = \frac{p}{1 + e\cos\theta} \quad ; \quad p = \frac{H^2}{\mu} \quad ; \quad p = a(1 - e^2) \tag{11.1}$$

The energy equation:

$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \quad ; \quad V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{11.2}$$

Velocity components:

$$\dot{r} = V\sin\gamma = \frac{\mu}{H}e\sin\theta \quad ; \quad r\dot{\theta} = V\cos\gamma = \frac{\mu}{H}(1 + e\cos\theta) \tag{11.3}$$

Angular momentum H:

$$H = r^2\dot{\theta} = rV\cos\gamma \tag{11.4}$$

Orbital period T and mean motion n:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \quad ; \quad n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}} \tag{11.5}$$

Circular velocity (Circular orbit):

$$V_c = \sqrt{\frac{\mu}{a}} \quad ; \quad r = a \quad ; \quad e = 0 \tag{11.6}$$

Escape velocity (Parabolic orbit):

$$V_{esc} = \sqrt{\frac{2\mu}{a}} = \sqrt{2}V_c \quad ; \quad a = \infty \quad ; \quad e = 1 \tag{11.7}$$

Excess velocity (Hyperbolic orbit):

$$V_\infty = \sqrt{\frac{\mu}{-a}} \quad ; \quad V^2 = V_\infty^2 + V_{esc}^2 \quad ; \quad r = \infty \quad ; \quad a < 0 \quad ; \quad e > 1 \tag{11.8}$$

Eccentric anomaly, E:

$$r\cos\theta = a\cos E - ae \qquad E - e\sin E = n(t - \tau) \qquad M = n(t - \tau) \tag{11.9}$$