Solutions to UFQETH-10-1 EXAM 2009 – 10

Question 1

(a) Part (i)

$$r=\sqrt{a^2+b^2}=\sqrt{25+16}=\sqrt{41}=6.40$$

$$\tan\theta=\frac{b}{a}\Rightarrow\theta=\tan^{-1}\left(\frac{4}{5}\right)=0.67 \text{ radians}$$

$$z=6.4\angle 0.67$$ [2]

Part (ii)

$$r=\sqrt{a^2+b^2}=1$$

$$\theta=-\frac{\pi}{2} \text{ radians}$$

$$z=\angle-\frac{\pi}{2}$$ [1]

Part (iii)

$$z=\frac{1}{j}=-j$$

$$r=\sqrt{a^2+b^2}=1$$

$$\theta=-\frac{\pi}{2} \text{ radians}$$

$$z=\angle-\frac{\pi}{2}$$ [1]

Part (iv)

$$z=(-2+j)(4-3j)=-8+6j+4j+3=-5+10j$$

$$r=\sqrt{a^2+b^2}=\sqrt{25+100}=\sqrt{125}=11.18$$

$$\tan\theta=\frac{b}{a}\Rightarrow\theta=\tan^{-1}\left(\frac{10}{-5}\right)=\tan^{-1}(-2)+\pi=2.03 \text{ radians}$$

$$z=11.18\angle 2.03$$ [2]

Question 1

(b) Part (i)

$$\frac{w_1^2}{w_2} = \frac{36e^{0.5j}}{2e^{-0.75j}} = 18e^{1.25j} \; : \quad r = 18,\ \theta = 1.25 \text{ radians}$$ [2]

Part (ii)

$$\frac{1}{w_1 w_2} = \frac{1}{\left(6e^{0.5j}\right)\left(2e^{-0.75j}\right)} = \frac{1}{12e^{-0.5j}} = \frac{1}{12}e^{0.5j}$$

$$r = \frac{1}{12},\ \theta = 0.5 \text{ radians}$$ [2]

(c) Poles: $x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}j,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}j$

Zeros: $x + 4 = 0 \Rightarrow x = -4$

[4]

Question 2

(a) Solve $\frac{dy}{dt}+4y=4t+1,\quad y(0)=1$

y = complementary function + particular integral [1]

Complementary function

Solve: $\frac{dy}{dt}+4y=0 \;\Rightarrow y_{CF}=Ae^{-4t}$ [2]

Particular integral

Let $y=at+b \Rightarrow \frac{dy}{dt}=a$

Solve $a+4(at+b)=4t+1$ for *a* and *b*

$4at+(a+4b)=4t+1 \Rightarrow a=1,\; b=0$

And so $y_{PI}=t$. [3]

General solution: $y=Ae^{-4t}+t$

If $y(0)=1 \Rightarrow 1=A+0 \Rightarrow A=1$

Solution: $y=e^{-4t}+t$ [2]

Question 2

(b) Forced response

Let

$$y = a\sin(2t) + b\cos(2t)$$
$$\Rightarrow \frac{dy}{dt} = 2a\cos(2t) - 2b\sin(2t),\ \frac{d^2y}{dt^2} = -4a\sin(2t) - 4b\cos(2t)$$

[2]

Solve
$-4a\sin(2t) - 4b\cos(2t) + 2(2a\cos(2t) - 2b\sin(2t)) + 3(a\sin(2t) + b\cos(2t)) = 6\cos(2t)$ fo
r *a* and *b*

$$(-a - 4b)\sin(2t) + (4a - b)\cos(2t) = 6\cos(2t)$$

[1]

Equating sin terms: $-a - 4b = 0 \Rightarrow a = -4b$

Equating cosine terms: $4a - b = 6 \Rightarrow -17b = 6 \Rightarrow b = -\frac{6}{17}$

Hence $a = \frac{24}{17}$

Forced response: $y_{PI} = \frac{24}{17}\sin(2t) - \frac{6}{17}\cos(2t)$

[3]

Question 3

(a) $z = 4x^2 + e^{xy} + 2y^2$

$$\frac{\partial z}{\partial x} = 8x + ye^{xy} \qquad [1]$$

$$\frac{\partial z}{\partial y} = xe^{xy} + 4y \qquad [1]$$

$$\frac{\partial^2 z}{\partial x^2} = 8 + y^2 e^{xy} \qquad [1]$$

$$\frac{\partial^2 z}{\partial y^2} = x^2 e^{xy} + 4 \qquad [1]$$

(b) $u = \dfrac{3xy}{1+y^2}$

Generating partial derivatives

$$\frac{\partial u}{\partial x} = \frac{3y}{(1+y^2)} \qquad [1]$$

$$\frac{\partial u}{\partial y} = \frac{(1+y^2)3x - 3xy(2y)}{(1+y^2)^2} = \frac{3x - 3xy^2}{(1+y^2)^2} \qquad [1]$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{(1+y^2)3 - 3y(2y)}{(1+y^2)^2} = \frac{3 - 3y^2}{(1+y^2)^2} \qquad [1]$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{3 - 3y^2}{(1+y^2)^2} \qquad [1]$$

which proves the required result.

Question 3

(c) $V = \sqrt{x^2 + y^2}$

Generating partial derivatives

$$\frac{\partial V}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}$$ [1]

$$\frac{\partial V}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}$$ [1]

$$\frac{\partial^2 V}{\partial x^2} = \frac{\sqrt{x^2 + y^2} - x\left(\frac{x}{\sqrt{x^2 + y^2}}\right)}{x^2 + y^2}$$

$$= \frac{x^2 + y^2 - x^2}{\left(x^2 + y^2\right)^{3/2}}$$

$$= \frac{y^2}{\left(x^2 + y^2\right)^{3/2}}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\sqrt{x^2 + y^2} - y\left(\frac{y}{\sqrt{x^2 + y^2}}\right)}{x^2 + y^2}$$ [2]

$$= \frac{x^2}{\left(x^2 + y^2\right)^{3/2}}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{y^2}{\left(x^2 + y^2\right)^{3/2}} + \frac{x^2}{\left(x^2 + y^2\right)^{3/2}}$$

$$= \frac{x^2 + y^2}{\left(x^2 + y^2\right)^{3/2}}$$ [2]

$$= \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{V}$$

Question 4

(a) Part(i): $3\begin{pmatrix}5\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\2\\-3\end{pmatrix}-2\begin{pmatrix}3\\-1\\4\end{pmatrix}=\begin{pmatrix}10\\1\\-11\end{pmatrix}$ [1]

Part (ii): $\hat{\mathbf{a}}=\frac{1}{\sqrt{26}}\begin{pmatrix}5\\-1\\0\end{pmatrix}$ [1]

Part (iii): $\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}5\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-3\end{pmatrix}=5-2+0=3$ [1]

Part (iv): $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 2 & -3\\3 & -1 & 4\end{vmatrix}=5\mathbf{i}-13\mathbf{j}-7\mathbf{k}$ [2]

(b) $\mathbf{a}=\begin{pmatrix}1\\1\\5\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\0\\3\end{pmatrix}$, $\mathbf{c}=\begin{pmatrix}4\\1\\1\end{pmatrix}$

Identify two vectors that lie in the plane

$$\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}1\\0\\3\end{pmatrix}-\begin{pmatrix}1\\1\\5\end{pmatrix}=\begin{pmatrix}0\\-1\\-2\end{pmatrix}$$

$$\overrightarrow{AC}=\mathbf{c}-\mathbf{a}=\begin{pmatrix}4\\1\\1\end{pmatrix}-\begin{pmatrix}1\\1\\5\end{pmatrix}=\begin{pmatrix}3\\0\\-4\end{pmatrix}$$ [2]

Now take cross product

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\0 & -1 & -2\\3 & 0 & -4\end{vmatrix}=4\mathbf{i}-6\mathbf{j}+3\mathbf{k}$$ [2]

Unit vector: $\frac{1}{\sqrt{61}}\begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix}$ [1]

(c) $\mathbf{a} = \begin{pmatrix} 3 \\ -4t \\ -8 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} t^2 \\ -5 \\ 2 \end{pmatrix}$

For two vectors to be at right angles to each other $\mathbf{a} \cdot \mathbf{b} = 0$ [1]

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 3 \\ -4t \\ -8 \end{pmatrix} \cdot \begin{pmatrix} t^2 \\ -5 \\ 2 \end{pmatrix} = 3t^2 + 20t - 16 = 0$$

$$\Rightarrow t = \frac{-20 \pm \sqrt{400+192}}{6}$$

$$t = \frac{-20 \pm \sqrt{592}}{6}$$

$$t = 0.72, -7.39$$

[3]

Question 5

(a) **AB** produces a 2x2 matrix whereas **BA** produces a 3x3 matrix so $\mathbf{AB} \neq \mathbf{BA}$.
Alternatively students may calculate the first element of each product to show these terms are different.

[3]

(b) **Part (i)**

$$p = 0.35x + 0.6y$$
$$q = 0.65x + 0.4y$$
$$\Rightarrow \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0.35 & 0.6 \\ 0.65 & 0.4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

[2]

$$\mathbf{M}^{-1} = -4 \begin{pmatrix} 0.4 & -0.6 \\ -0.65 & 0.35 \end{pmatrix} = \begin{pmatrix} -1.6 & 2.4 \\ 2.6 & -1.4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1.6 & 2.4 \\ 2.6 & -1.4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$
$$= \begin{pmatrix} -1.6 & 2.4 \\ 2.6 & -1.4 \end{pmatrix} \begin{pmatrix} 155 \\ 145 \end{pmatrix}$$
$$= \begin{pmatrix} 200 \\ 100 \end{pmatrix}$$

[3]

Question 5

(c).

$$\begin{pmatrix} 2 & 1 & 5 & 12 \\ 3 & 15 & 4 & 20 \\ 1 & 12 & 8 & 8 \end{pmatrix} \rightarrow \quad R_1 \leftrightarrow R_3 \begin{pmatrix} 1 & 12 & 8 & 8 \\ 3 & 15 & 4 & 20 \\ 2 & 1 & 5 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 12 & 8 & 8 \\ 3 & 15 & 4 & 20 \\ 2 & 1 & 5 & 12 \end{pmatrix} \rightarrow \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \begin{pmatrix} 1 & 12 & 8 & 8 \\ 0 & -21 & -20 & -4 \\ 0 & -23 & -11 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 12 & 8 & 8 \\ 0 & -21 & -20 & -4 \\ 0 & -23 & -11 & -4 \end{pmatrix} \rightarrow \begin{matrix} \\ \\ 21R_3 - 23R_2 \end{matrix} \begin{pmatrix} 1 & 12 & 8 & 8 \\ 0 & -21 & -20 & -4 \\ 0 & 0 & 229 & 8 \end{pmatrix}$$

[3]

Back substitution

$$\begin{aligned} I_1 + 12I_2 + 8I_3 &= 8 \qquad (1) \\ -21I_2 - 20I_3 &= -4 \qquad (2) \\ 229I_3 &= 8 \qquad (3) \end{aligned}$$

Equation (1) implies $I_3 = \frac{8}{229} = 0.03 \ \ (2dp)$

Equation (2) implies $-21I_2 - 20\left(\frac{8}{229}\right) = -4 \Rightarrow I_2 = \frac{36}{229} = 0.16 \ \ (2dp)$

Equation (3) implies $I_1 + 12\left(\frac{36}{229}\right) + 8\left(\frac{8}{229}\right) = 8 \Rightarrow I_1 = \frac{1336}{229} = 5.83 \ \ (2dp)$

[3]