Calculus - Period 1

Differentiation and Integration

Chain Rule:

$$(f(g(x)))' = f'(g(x))g'(x) \tag{1}$$

Implicit Differentiation:
When applying implicit differentiation for a function y of x, every term with a y should, after normal differentiation (often involving the product rule), be multiplied by y' because of the chain rule. After that, the equation should be solved for y'.

Lineair Approximations:

$$f(x) - f(a) \approx f'(a)(x-a) \tag{2}$$

Mean Value Theorem:
If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a c in (a, b) such that:

$$f(b) - f(a) = f'(c)(b-a) \tag{3}$$

Integration:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{4}$$

Where F is any antiderivative/primitive function of f, that is, $F' = f$.

Substitution Rule:
If $u = g(x)$ then:

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{5}$$

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \tag{6}$$

Integration By Parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \tag{7}$$

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx \tag{8}$$

Improper Integrals:

$$\int_1^\infty \frac{1}{x^p}dx \tag{9}$$

This function is convergent for $p > 1$ and divergent for $p \leq 1$.

Comparison Theorem:
If $f(x) \geq g(x) \geq 0$ for $x \geq a$ then:

- If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
- If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Complex Numbers

Complex Number Notations:

$$i^2 = -1 \tag{10}$$

$$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{11}$$

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta \tag{12}$$

$$\begin{gathered} |z| = r = \sqrt{a^2 + b^2} \\ \theta = \arctan\tfrac{b}{a} \quad \text{or} \quad \theta = \arctan\tfrac{b}{a} + \pi \end{gathered} \tag{13}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{14}$$

Complex Number Calculation:

$$\begin{gathered} (a + bi) + (c + di) = (a + c) + (b + d)i \\ (a + bi)(c + di) = (ac - bd) + (ad + bc)i \end{gathered} \tag{15}$$

$$\begin{gathered} z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\ r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{gathered} \tag{16}$$

Complex Conjugates:

$$z = a + bi \quad \Rightarrow \quad \overline{z} = a - bi \tag{17}$$

$$\begin{gathered} \overline{z + w} = \overline{z} + \overline{w} \\ \overline{zw} = \overline{z}\,\overline{w} \\ \overline{z^n} = \overline{z}^n \\ z\overline{z} = |z|^2 \end{gathered} \tag{18}$$

Differential Equations

Separable Differential Equations:

$$\text{Form}: \ \frac{dy}{dx} = y' = P(x)Q(y)$$

$$\text{Solution}: \int \frac{1}{Q(y)}dy = \int P(x)dx \tag{19}$$

First-Order Differential Equations

$$\text{Form}: \; y' + P(x)y = Q(x)$$

Let $\Upsilon(x)$ be any integral of $P(x)$. Solution is:

$$y = e^{-\Upsilon(x)}\left(\int e^{\Upsilon(x)}Q(x)dx + C\right) \tag{20}$$

Homogeneous second-order linear differential equations:

$$\text{Form}: \; ay'' + by' + cy = 0$$

- $b^2 - 4ac > 0$:
  Define r such that $ar^2 + br + c = 0$.

  $$\text{Solution}: \; y = c_1e^{r_1x} + c_2e^{r_2x} \tag{21}$$

- $b^2 - 4ac = 0$:
  Define r such that $ar^2 + br + c = 0$.

  $$\text{Solution}: \; y = c_1e^{rx} + c_2xe^{rx} \tag{22}$$

- $b^2 - 4ac < 0$:
  Define $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$. Solution:

  $$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \tag{23}$$

Nonhomogeneous second-order linear differential equations:

$$\text{Form}: \; ay'' + by' + cy = P(x)$$

First solve $ay_c'' + by_c' + cy_c = 0$. Then use an auxiliary equation to find one solution y_p for the given differential equation. The solution is:

$$y = y_c + y_p \tag{24}$$