Calculus - Period 3

Functions of Multiple Variables

Definitions:
The domain D is the set (x, y) for which $f(x, y)$ exists. The range is the set of values z for which there are x, y such that $z = f(x, y)$. The level curves are the curves with equations $f(x, y) = k$ where k is a constant.

Checking for Limits:
If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a path C_2, where $L_1 \neq L_2$ then $\lim_{(x,y)\rightarrow(a,b)} f(x, y)$ does not exist. Also f is continuous at (a, b) if $\lim_{(x,y)\rightarrow(a,b)} f(x, y) = f(a, b)$

Partial Derivatives:
The partial derivative of f with respect to x at (a, b) is:

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b) \tag{1}$$

In words, to find f_x, regard y as constant and differentiate $f(x, y)$ with respect to x. f_y is defined similarly. If f_{xy} and f_{yx} are both continuous on D, then $f_{xy} = f_{yx}$.

Tangent Planes:
For points close to $z_0 = f(x_0, y_0)$ the curve of $f(x, y)$ can be approximated by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{2}$$

The plane described by this equation is the plane tangent to the curve of $f(x, y)$ at (x_0, y_0).

Differentials:

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \tag{3}$$

If $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ then:

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds} \tag{4}$$

Directional Derivatives:
The directional derivative of f at (x_0, y_0) in the direction of a unit vector (meaning, $|\mathbf{u}| = 1$) $\mathbf{u} = \langle a, b\rangle$ is:

$$D_u f(x_0, y_0) = f_x(x, y)a + f_y(x, y)b = \nabla f \cdot \mathbf{u} \tag{5}$$

$$\textbf{grad } f = \nabla f = \langle f_x(x, y), f_y(x, y)\rangle \tag{6}$$

The maximum value of $D_u f(x, y)$ is $|\nabla f(x, y)|$ and occurs when the vector $\mathbf{u} = \langle a, b\rangle$ has the same direction as $\nabla f(x, y)$.

Local Maxima and Minima:
If f has a local maximum or minimum at (a, b), then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. If $f_x(a, b) = 0$ and $f_y(a, b) = 0$ then (a, b) is a critical point. If (a, b) is a critical point, then let D be defined as:

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2 \tag{7}$$

- If $D > 0$ then:
  - If $f_{xx}(a, b) > 0$, then $f(a, b)$ is a minimum.
  - If $f_{xx}(a, b) < 0$, then $f(a, b)$ is a maximum.
- If $D < 0$, then $f(a, b)$ is a saddle point.

Absolute Maxima and Minima:
To find the absolute maximum and minimum values of a continuous function f on a closed bounded set D, first find the values of f at the critical points of f in D. Then find the extreme values of f on the boundary of D. The largest of these values is the absolute maximum. The lowest is the minimum.

Multiple Integrals

Integrals over Rectangles:
If R is the rectangle such that $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, then:

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx \tag{8}$$

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy \tag{9}$$

Integrals over Regions:
If D_1 is the region such that $D_1 = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then:

$$\iint_{D_1} f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{10}$$

If D_2 is the region such that $D_2 = \{(x, y)|a \leq y \leq b, h_1(y) \leq x \leq h_2(y)\}$, then:

$$\iint_{D_2} f(x, y)\, dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{11}$$

Integrating over Polar Coordinates

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \tag{12}$$

$$x = r\cos\theta \qquad y = r\sin\theta \tag{13}$$

If R is the polar rectangle such that $R = \{(r,\theta)|0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$ where $0 \leq \beta - \alpha \leq 2\pi$, then:

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta \tag{14}$$

If D is the polar rectangle such that $D = \{(r,\theta)|0 \leq h_1(\theta) \leq r \leq h_2(\theta), \alpha \leq \theta \leq \beta\}$ where $0 \leq \beta - \alpha \leq 2\pi$, then:

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta \tag{15}$$

Applications:

If m is the mass, and $\rho(x,y)$ the density, then:

$$m = \iint_D \rho(x,y)\ dA \tag{16}$$

The x-coordinate of the center of mass is:

$$\overline{x} = \frac{\iint_D x\ \rho(x,y)\ dA}{\iint_D \rho(x,y)\ dA} \tag{17}$$

The moment of inertia about the x-axis is:

$$I_x = \iint_D y^2 \rho(x,y)\ dA \tag{18}$$

The moment of inertia about the origin is:

$$I_0 = \iint_D (x^2+y^2)\rho(x,y)\ dA = I_x + I_y \tag{19}$$

Triple Integrals

If E is the volume such that $E = \{(x,y,z)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x,y) \leq z \leq h_2(x,y)\}$, then:

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dzdydx \tag{20}$$