# Calculus - Period 4

## Three-Dimensional Integrals

**Cylindrical Coordinates:**

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{1}$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z \tag{2}$$

**Integrating Over Cylindrical Coordinates:**

$$\begin{gathered}\int\int\int_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \dots \\ \dots \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} r f(r\cos\theta, r\sin\theta, z) dz dr d\theta\end{gathered} \tag{3}$$

**Spherical Coordinates:**

$$x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi \tag{4}$$

$$\rho^2 = x^2 + y^2 + z^2 \tag{5}$$

**Integrating Over Spherical Coordinates:**
If $E$ is the spherical wedge given by $E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$, then:

$$\begin{gathered}\int\int\int_E f(x,y,z)dV = \int_a^b \int_\alpha^\beta \int_c^d \rho^2 \sin\phi \dots \\ \dots f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) d\rho d\theta d\phi\end{gathered} \tag{6}$$

**Change of Variables:**
The Jacobian of the transformation $T$ given by $x = g(u,v)$ and $y = h(u,v)$ is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{7}$$

If the Jacobian is nonzero and the transformation is one-to-one, then:

$$\iint_R f(x,y)dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \tag{8}$$

This method is similar to the one for triple integrals, for which the Jacobian has a bigger matrix and the change-of-variable equation has some more terms.

## Basic Vector Field Theorems

**Definitions**

- A piecewise-smooth curve - A union of a finite number of smooth curves.
- A closed curve - A curve of which its terminal point coincides with its initial point.
- A simple curve - A curve that doesn't intersect itself anywhere between its endpoints.
- An open region - A region which doesn't contain any of its boundary points.
- A connected region - A region $D$ for which any two points in $D$ can be connected by a path that lies in $D$.
- A simply-connected region - A region $D$ such that every simple closed curve in $D$ encloses only points that are in $D$. It contains no holes and consists of only one piece.
- Positive orientation - The positive orientation of a simple closed curve $C$ refers to a single counterclockwise traversal of $C$.

**Vector Field:**
A vector field on $\mathbb{R}^n$ is a function $\mathbf{F}$ that assigns to each point $(x,y)$ in an $n$-dimensional set an $n$-dimensional vector $\mathbf{F}(x,y)$. The gradient $\nabla f$ is defined by:

$$\nabla f(x, y, \dots) = f_x\,\mathbf{i} + f_y\,\mathbf{j} + \dots \tag{9}$$

and is called the gradient vector field. A vector field $\mathbf{F}$ is called a conservative vector field if it is the gradient of some scalar function.

**Line Integrals:**
The line integral of $f$ along $C$ is:

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{10}$$

The line integral of $f$ along $C$ with respect to $x$ is:

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t)) \frac{dx}{dt}\, dt \tag{11}$$

The line integral of a vector field $\mathbf{F}$ along $C$ is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{12}$$

Where $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent vector.

**Conservative Vector Fields:**
If $C$ is the curve given by $\mathbf{r}(t)$ $(a \leq t \leq b)$, then:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{13}$$

The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$ if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path $C$ in $D$.
If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, then:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{14}$$

Also, if $D$ is an open simply-connected region, and if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\mathbf{F}$ is conservative in $D$.

# Surfaces

**Parametric Surfaces:**
A surface described by $\mathbf{r}(u, v)$ is called a parametric surface. $\mathbf{r_u} = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r_v} = \frac{\partial \mathbf{r}}{\partial v}$. For smooth surfaces ($\mathbf{r_u} \times \mathbf{r_v} \neq \mathbf{0}$ for every $u$ and $v$) the tangent plane is the plane that contains the tangent vectors $\mathbf{r_u}$ and $\mathbf{r_v}$, and the vector $\mathbf{r_u} \times \mathbf{r_v}$ is the normal vector to the tangent plane.

**Surface Areas:**
For a parametric surface, the surface area is given by:

$$A = \iint_D |\mathbf{r_u} \times \mathbf{r_v}| dA \tag{15}$$

For a surface graph of $g(x, y)$, the surface area is given by:

$$A = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dA \tag{16}$$

**Surface Integrals:**
For a parametric surface, the surface integral is given by:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r_u} \times \mathbf{r_v}| dA \tag{17}$$

For a surface graph of $g(x, y)$, the surface integral is given by:

$$\iint_S f(x, y, z)\, dS =$$

$$\iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA \tag{18}$$

**Normal Vectors:**
For a parametric surface, the normal vector is given by:

$$\mathbf{n} = \frac{\mathbf{r_u} \times \mathbf{r_v}}{|\mathbf{r_u} \times \mathbf{r_v}|} \tag{19}$$

For a surface graph of $g(x, y)$, the normal vector is given by:

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}} \tag{20}$$

**Flux:**
If $\mathbf{F}$ is a vector field on a surface $S$ with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over $S$ is:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \tag{21}$$

This integral is also called the flux of $\mathbf{F}$ across $S$. For a parametric surface, the flux is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r_u} \times \mathbf{r_v})\, dA \tag{22}$$

For a surface graph of $g(x, y)$, the flux is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \tag{23}$$

# Advanced Vector Field Theorems

**Curl:**
If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then the curl of $\mathbf{F}$, denoted by curl $\mathbf{F}$ or also $\nabla \times \mathbf{F}$, is defined by:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \tag{24}$$

If $f$ is a function of three variables, then:

$$\text{curl}(\nabla f) = \mathbf{0} \tag{25}$$

This implies that if $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$. The converse is only true if $\mathbf{F}$ is defined on all of $\mathbb{R}^n$. So if $\mathbf{F}$ is defined on all of $\mathbb{R}^n$ and if curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

**Divergence:**
If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then the divergence of $\mathbf{F}$, denoted by div $\mathbf{F}$ or also $\nabla \cdot \mathbf{F}$, is defined by:

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{26}$$

If $\mathbf{F}$ is a vector field on $\mathbb{R}^n$, then div curl $\mathbf{F} = 0$. If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be incompressible. Note that curl $\mathbf{F}$ returns a vector field and div $\mathbf{F}$ returns a scalar field.

**Green's Theorem:**
Let $C$ be a positively oriented piecewise-smooth simple closed curve in the plane and $D$ be the region bounded by $C$. Now:

$$\int_C P\ dx + Q\ dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \qquad (27)$$

This can also be useful for calculating areas. To calculate an area, take functions $P$ and $Q$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ and then apply Green's theorem.
In vector form, Green's theorem can also be written as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k}\ dA \qquad (28)$$

$$\int_C \mathbf{F} \cdot \mathbf{n}\ ds = \iint_D \text{div } \mathbf{F}(x, y)\ dA \qquad (29)$$

**Stoke's Theorem:**
Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive orientation. Let $\mathbf{F}$ be a vector field that contains $S$. Then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \qquad (30)$$

**The Divergence Theorem:**
Let $E$ be a simple solid region and let $S$ be the boundary surface of $E$, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field on an open region that contains $E$. Then:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\ dV \qquad (31)$$