The panel method

A nice method to simulate the flow around an airfoil, is the so-called panel method. We'll examine this method in this chapter. But before we do that, we first have to examine some basic flow types.

1 Flow types

1.1 Incompressible flow

Let's examine a steady incompressible flow. Based on these assumptions, the continuity equation reduces to $\nabla \cdot \mathbf{u} = 0$, with $\mathbf{u}$ the velocity field. This is nice to know, but by itself it isn't very handy. We therefore also examine the **vorticity** $\omega = \nabla \times \mathbf{u}$. Let's assume that this vorticity is $\omega = 0$. If this is the case, then it can be shown that there is a function Φ such that

$$\mathbf{u} = \nabla\Phi. \tag{1.1}$$

Φ is called the **(total) potential function**. Using our earlier relation for $\mathbf{u}$, we find that Φ must satisfy $\nabla^2\Phi = 0$. (This is called the **Laplace equation**.) From it, one can attempt to solve for Φ, using given boundary conditions. Once Φ is known, the pressure p in the flow can also be found. For that, we can use the momentum equation, being

$$\frac{1}{2}\rho|\nabla\Phi|^2 + p = \text{constant}. \tag{1.2}$$

What might be more useful is to find the pressure coefficient C_p. For that, we can use

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 1 - \frac{|\nabla\Phi|^2}{V_\infty^2}. \tag{1.3}$$

1.2 Compressible flow

For incompressible flows, we can not use the above equations. So we have to use different ones. Before we examine them, we first define the **disturbance potential** ϕ, such that $\Phi = V_\infty x + \phi$. It can be shown that this disturbance potential satisfies

$$\rho_\infty(1 - M_\infty^2)\frac{\partial^2\phi}{\partial^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{1.4}$$

The factor $(1 - M_\infty^2)$ is slightly annoying. To get rid of it, we can change our coordinate system. Let's define $x' = x/\sqrt{1 - M_\infty^2}$, $y' = y$ and $z' = z$. We then get

$$\frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi}{\partial z'^2}. \tag{1.5}$$

And this is simply the Laplace equation! So once more, we need to solve a Laplace equation to find ϕ. Once we have found ϕ, we can again find the pressure coefficient C_p. However, this time we have to use the approximation

$$C_p \approx -2\frac{\partial\phi/\partial x}{V_\infty} \tag{1.6}$$

1.3 Boundary conditions

We see that we need to solve the Laplace equation. However, to solve it, we need boundary conditions.

We can, for example, set the value of ϕ along the boundary. This is a so-called **Dirichlet boundary conditions**. If we do this, then there is a unique solution for ϕ. (We say that the problem is **well-posed**.)

However, we can also set the value of $\partial\phi/\partial\mathbf{n}$ at the boundary. We now have **Neumann boundary conditions**. This time, there is no unique solution for ϕ. We can only determine ϕ up to a certain unknown constant.

Of course we can also use combinations of Dirichlet and Neumann boundary conditions. However, as long as ϕ is set somewhere, then there is always a unique solution for ϕ.

2 Solving the Laplace equation using singularity distributions

2.1 Singularity distributions

Sometimes it is very hard to solve the Laplace equation for given boundary conditions. Luckily the Laplace equation is a linear differential equation. So let's suppose that we have some solutions of the Laplace equation. Any linear combination of these solutions will then also be a solution. It thus also satisfies the Laplace equation!

So the first thing we need to do is find some **elementary solutions** for the Laplace equation. Examples of such solutions are sources, sinks, dipoles, vortices and such. We will discuss some of them now.

2.2 Sources and sinks

Now it is time to examine some elementary solutions. One such solution is the so-called **source**. Let's suppose we have a source with strength $\sigma(Q)$, positioned at some point Q. The potential ϕ caused by this source at some point P then is

$$\phi(P) = \frac{\sigma(Q)}{2\pi}\ln\left(r(P,Q)\right). \tag{2.1}$$

By the way, if the source strength σ is negative, the source is often called a **sink**.

What can we do with these sources? Well, we can put a lot of them on a curve S. We then get a **source distribution**. To find the velocity potential at some point P, caused by this distribution, we use an integral. This velocity potential thus will be

$$\phi(P) = \int_S \frac{\sigma(Q)}{2\pi}\ln\left(r(P,Q)\right)ds. \tag{2.2}$$

A problem might occur if the point P lies on the source distribution itself. Such a situation should always be considered separately.

2.3 Doublets

Another elementary solution is the **doublet**. The potential at some point P, caused by a doublet at Q, is given by

$$\phi(P) = \mu(Q)\frac{\partial}{\partial\mathbf{n_Q}}\left(\frac{1}{2\pi}\ln\left(r(P,Q)\right)\right). \tag{2.3}$$

Here $\mu(Q)$ is the strength of the doublet and $\mathbf{n_Q}$ is the direction of the doublet.

Once more, we can put a lot of doublets in a row. We then get a **doublet distribution**. To find the velocity potential at P, we now have to use

$$\phi(P) = \int_S \mu(Q) \frac{\partial}{\partial \mathbf{n_Q}} \left(\frac{1}{2\pi} \ln \left(r(P, Q) \right) \right) ds. \tag{2.4}$$

Once more, the case $Q \in S$ should be considered separately.

3 The panel method

3.1 Using source distributions

We know that any combination of sources, doublets, vortices and such satisfies the Laplace equation. However, which combination describes our flow? To find that out, we have to use the boundary conditions. But how should we use them? We just use the panel method! Simply apply the following steps.

- First we take the edge of our airfoil. We split it up in N panels.
- We then place a source distribution (or similarly, a doublet distribution or vortex distribution) of constant strength σ_i on every panel. ($1 \leq i \leq N$.)
- At every panel, we can find the velocity potential ϕ (as a function of the source strengths σ). We can also find the flow velocity $\partial\phi/\partial\mathbf{n}$ normal to the airfoil. This flow velocity should of course be 0. This gives us N equations. Using these conditions, we can solve for the source strengths σ_i.

Solving for the source strengths σ might seem like a difficult task. However, these equations can simply be put in a matrix. This goes according to

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \cdots & A_{N,N} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}. \tag{3.1}$$

The left part of the above equation calculates the velocities caused by the source distributions. The right part of the equation then takes into account the parameters V_∞ and the angle of attack α.

3.2 Adding doublets/vortices

There is, however, one slight problem. Sources and sinks don't create any drag/lift. In other words, if we only place sources and sinks on our airfoil, then $c_l = c_d = 0$. To solve this problem, we also need to add doublets/vortices to the wing.

It is, for example, possible to add a constant vortex distribution with strength μ along the entire airfoil. This would give our airfoil lift. However, it would also give us an extra unknown. We thus also need an extra boundary condition. A condition that is often used, is the **Kutta condition**. It demands that the flow leaves the trailing edge smoothly. But what does that mean? To find that out, let's examine the velocity at the lower part of the trailing edge V_t^l. We also examine the velocity at the upper part of the trailing edge V_t^u. The Kutta condition claims that these velocities are equal: $V_t^l = V_t^u$.

Now that we have an extra unknown, and an extra equation, our system of equations also expands. We thus get

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,N} & A_{1,\mu} \\ \vdots & \ddots & \vdots & \cdots \\ A_{N,1} & \cdots & A_{N,N} & A_{N,\mu} \\ A_{\mu,1} & \cdots & A_{\mu,N} & A_{\mu,\mu} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \\ \mu \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \\ f_\mu \end{bmatrix}. \tag{3.2}$$

From this the values of σ and also μ can be solved.

3.3 Finding the lift

But the question still remains, how can we find the lift? For that, we can use the **Kutta-Joukowski theorem**. This theorem gives the relation between the circulation Γ and the lift L. It is

$$L = \rho_\infty V_\infty \Gamma. \tag{3.3}$$

If we want to find the lift coefficient, we can use

$$c_l = \frac{2\Gamma}{Vc}. \tag{3.4}$$

By the way, by adding vortices/doublets, you do not effect the drag coefficient. This coefficient is still zero. And that is one of the major downsides of the panel method — you can't really use it to calculate drag.