Constitutive Modeling Summary (Solid Part)

1. Forces and stresses

Consitutive modelling is mainly about the relation between things like forces and things like displacements. In this chapter we examine the forces. In the next chapter we will discuss the displacements and their relationship with forces. In the third and last chapter, we examine some methods for solving problems.

1.1 Forces and momentum

1.1.1 Types of forces

We can distinguish two important types of forces. These are the **(distributed) contact forces t** and the **(distributed) mass forces b**. They are defined as

$$\mathbf{t} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A} \qquad \text{and} \qquad \mathbf{b} = \lim_{\Delta V \to 0} \frac{\Delta \mathbf{F}}{\Delta V}. \tag{1.1.1}$$

Here $\mathbf{F}$ denotes a force, A denotes an area and V denotes a volume. Now let's examine a certain volume Ω. The **total contact force $\mathbf{F_s}$** and the **total body force $\mathbf{F_b}$** can be found using

$$\mathbf{F_s} = \int_{\partial\Omega} \mathbf{t}(\mathbf{x})\, dA \qquad \text{and} \qquad \mathbf{F_b} = \int_{\Omega} \mathbf{b}(\mathbf{x})\, dV. \tag{1.1.2}$$

(The signal $\partial\Omega$ means we integrate over the surface of the volume Ω.) Together, the total contact force F_s and the total body force form the **total external force** F_{ext}.

1.1.2 Linear momentum

Again, we examine a volume Ω. The **total linear momentum** P of the volume can be found using

$$\mathbf{P} = \int_{\Omega} \rho \mathbf{v}\, dV, \tag{1.1.3}$$

where ρ denotes the density of the volume and $\mathbf{v}$ the velocity. It can be shown that $\mathbf{F_{ext}} = d\mathbf{P}/dt$. In other words,

$$\int_{\partial\Omega} \mathbf{t}\, dA + \int_{\Omega} \mathbf{b}\, dV = \frac{d}{dt} \int_{\Omega} \rho \mathbf{v}\, dV = \int_{\Omega} \rho \mathbf{a}\, dV. \tag{1.1.4}$$

1.1.3 Moments and angular momentum

Forces also cause moments. The **moment due to surface forces $\mathbf{M_s}$** and **the moment due to body forces $\mathbf{M_b}$** can be found using

$$\mathbf{M_s} = \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}\, dA \qquad \text{and} \qquad M_b = \int_{\Omega} \mathbf{x} \times \mathbf{b}\, dV. \tag{1.1.5}$$

Together, these two moments form the **total moment of external forces** M_{ext}. (All moments are about the origin.)

We can also find the **total angular momentum** H. (Also with respect to the origin.) We do this using

$$\mathbf{H} = \int_{\Omega} \mathbf{x} \times \rho \mathbf{v} \, dV. \tag{1.1.6}$$

Similar to linear momentum, it also holds that $\mathbf{M_{ext}} = d\mathbf{H}/dt$. From this, it can be derived that

$$\int_{\partial\Omega} \mathbf{x} \times \mathbf{t} \, dA + \int_{\Omega} \mathbf{x} \times \mathbf{b} \, dV = \frac{d}{dt} \int_{\Omega} \mathbf{x} \times \rho \mathbf{v} \, dV = \int_{\Omega} \mathbf{x} \times \rho \mathbf{a} \, dV. \tag{1.1.7}$$

1.2 Stress vectors and tensors

1.2.1 The stress vector

It's time to examine **internal forces**. To examine the internal forces in an object, we make a cut along a plane. This plane has a certain **unit normal vector n**. The internal forces at a given position are now indicated by the **stress vector t(n)**. (Note that the stress vector can be seen as a surface force. That's why it is also denoted by **t**.)

Let's suppose that we know the stress vector **t** at a given point for a given normal vector **n**. We can then also find **normal component** $\mathbf{t_n}$ (the stress normal to the cutting plane) and the **tangential component** $\mathbf{t_s}$ (the stress parallel to the cutting plane). This can be done using

$$\mathbf{t_n} = (\mathbf{t} \cdot \mathbf{n})\mathbf{n} \qquad \text{and} \qquad \mathbf{t_s} = \mathbf{t} - \mathbf{t_n} = \mathbf{t} - (\mathbf{t} \cdot \mathbf{n})\mathbf{n}. \tag{1.2.1}$$

1.2.2 The stress tensor

There is, however, one small problem. The stress vector **t** depends on the on the cutting plane normal vector **n**. To know the exact stress distribution, we need to know **t** for every **n**. This may seem like a lot of work. Luckily, there is a trick (originating from the balance of momentum) called the stress tensor.

The **stress tensor** $[\sigma_{ij}]$ is a 3×3 matrix. It has thus 9 coefficients σ_{ij}. Once these parameters are known, the stress vector **t** for any unit normal vector **n** can be found using

$$\mathbf{t}(\mathbf{n}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{1.2.2}$$

The question now remains, how can we find the stress tensor? To do that, we have to first find the stress vector **t** for three (linearly independent) normal vectors **n**. (It is often convenient to choose the three unit normal vectors $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$.) We should then find the corresponding stress vectors $\mathbf{t_1}$, $\mathbf{t_2}$ and $\mathbf{t_3}$. Inserting all these data into equation (1.2.2) gives us 9 equations and 9 unknowns. The unknown coefficients can then be solved.

When solving for the coefficients, you can use a small trick. You can use that the stress tensor is symmetric. (This can be derived from balance of angular momentum.) So we have

$$\sigma_{12} = \sigma_{21}, \qquad \sigma_{13} = \sigma_{31} \qquad \text{and} \qquad \sigma_{23} = \sigma_{32}. \tag{1.2.3}$$

1.2.3 Stress tensor eigenvalues and eigenvectors

The stress tensor $[\sigma_{ij}]$ has three eigenvalues $\sigma^{(1)}$, $\sigma^{(2)}$ and $\sigma^{(3)}$. These eigenvalues are called the **principal stresses**. Because the stress tensor is symmetric, these eigenvalues must be real. We usually order them such that $\sigma^{(1)} \geq \sigma^{(2)} \geq \sigma^{(3)}$.

Of course, there are eigenvectors $\mathbf{n^{(1)}}$, $\mathbf{n^{(2)}}$ and $\mathbf{n^{(3)}}$ corresponding to these eigenvalues. Usually, these eigenvectors are normalized, such that their length $|\mathbf{n}|$ is one. These vectors are called the **principal stress directions**. It can be shown that they are mutually perpendicular. Because of this, they together form an orthogonal basis, called the **principal stress basis**.

There is something special about this basis. Previously, we have built our stress tensor $[\sigma_{ij}]$ with respect to our normal Cartesian basis $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$. If we, however, build it with respect to the principal stress basis, we find a very peculiar stress tensor, being

$$[\sigma_{ij}] = \begin{bmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{bmatrix}. \tag{1.2.4}$$

1.2.4 Relevance of principal stresses and their directions

You may wonder, what are these principal stresses and principal stress directions good for? Well, the principal stresses are used in many stress criterions. For example, there is the **tresca criterion**, demanding that

$$\max\left(|\sigma^{(1)} - \sigma^{(2)}|, |\sigma^{(1)} - \sigma^{(3)}|, |\sigma^{(2)} - \sigma^{(3)}|\right) \leq \sigma_y, \tag{1.2.5}$$

where the critical value σ_y is known as the **(initial) yield stress**. Similarly, there is the **Huber-von Mises-Hencky criterion**, demanding that

$$\sigma_m = \sqrt{\frac{\left(\sigma^{(1)} - \sigma^{(2)}\right)^2 + \left(\sigma^{(1)} - \sigma^{(3)}\right)^2 + \left(\sigma^{(2)} - \sigma^{(3)}\right)^2}{2}} \leq \sigma_y, \tag{1.2.6}$$

where σ_m is the **maximum distortion energy**.

The principal stress directions are also important. They are closely related to the directions and planes in which failure will initiate and propagate. This data is important when trying to optimize a structure.

2. Displacements and strains

After examining forces and stresses, we will now examine displacements and strains. How are they defined? And what can we do with them? We also examine their relation with stresses.

2.1 Definitions of the displacements and strains

2.1.1 Introduction of the strain tensor

Let's suppose we have some object Ω, we're deforming. Let's examine some point P. We call its initial position $\mathbf{x}$ and its final position $\mathbf{y}$. The **displacement** of P (its movement) then is

$$\mathbf{u} = \mathbf{y} - \mathbf{x}. \tag{2.1.1}$$

However, we are usually interested in the deformations of the material. The movement of some point P doesn't say much about that. To examine the deformations, we use **displacement gradients** $\partial u_i / \partial x_j$. In fact, the **strain tensor** is defined as

$$[\epsilon_{ij}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}. \tag{2.1.2}$$

Note that the strain tensor is symmetric.

2.1.2 The meaning of the strain tensor

So, what is the use of the strain tensor? Well, it is closely related to the deformations in the object. To see how, we examine two points P and Q, originally being a very small distance l_0 apart. Their new relative distance is l. We call the unit vector in the direction of the line PQ the vector $\mathbf{n}$. The **relative elongation** ϵ of the distance PQ (also known as the normal strain) can then be approximated by

$$\epsilon(\mathbf{n}) = \frac{l - l_0}{l} \approx \sum_{i,j=1}^{3} \epsilon_{ij} n_i n_j = \mathbf{n}^T [\epsilon_{ij}] \mathbf{n}. \tag{2.1.3}$$

Using this, we can more closely examine the meaning of the strain tensor. If we examine the relative elongation in the direction of the Cartesian unit vector $\mathbf{e_1}$, then we find that $\epsilon(\mathbf{e_1}) = \epsilon_{11}$. Similarly $\epsilon(\mathbf{e_2}) = \epsilon_{22}$ and $\epsilon(\mathbf{e_3}) = \epsilon_{33}$. So the diagonal components simply indicate normal strain.

The next question is, what do the non-diagonal terms of the strain tensor mean? They indicate a change in angle of two lines that were previously perpendicular. (It's also known as the shear strain.) Let's examine two lines PQ and PR. PQ is in the direction of $\mathbf{e_1}$, while PR is in the direction of $\mathbf{e_2}$. Their relative angle is thus $\pi/2$. It can be shown that, after deformation, their relative angle is $\pi/2 - \epsilon_{12} - \epsilon_{21} = \pi/2 - 2\epsilon_{12}$.

2.1.3 Principal strains and their directions

Just like the stress tensor $[\sigma]$, also the strain tensor $[\epsilon]$ is symmetric. This means that its three eigenvalues $\epsilon^{(1)}$, $\epsilon^{(2)}$ and $\epsilon^{(3)}$, called the **principal strains**, are all real. The three corresponding eigenvectors $\mathbf{m^{(1)}}$, $\mathbf{m^{(2)}}$ and $\mathbf{m^{(3)}}$, called the **principal strain directions**, are mutually perpendicular. Together, they form the **principal strain basis**.

Previously, we have built our strain tensor $[\epsilon_{ij}]$ with respect to our normal Cartesian basis $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$. If we, however, build it with respect to the principal strain basis, we get the strain tensor

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon^{(1)} & 0 & 0 \\ 0 & \epsilon^{(2)} & 0 \\ 0 & 0 & \epsilon^{(3)} \end{bmatrix}. \tag{2.1.4}$$

All non-diagonal terms are zero. There is thus no shear strain. So we can conclude that, with respect to the principal strain basis, all perpendicular angles remain perpendicular.

You may wonder whether the principal stress directions and principal strain directions are the same. They usually are not. Only for **isotropic materials** (materials with the same properties in every direction) will these directions coincide.

2.1.4 Finding the displacement field from the strain tensor

Let's suppose we know the strain tensor $[\epsilon_{ij}]$ at every given position $\mathbf{x}$. Can we then find the displacements? Well, it turns out that we almost can do that. Only the so-called **rigid body modes**, being pure translation and rotation (without any deformation), can't be included. However, by using appropriate boundary conditions, we can get rid of these rigid body modes.

So how do we find the displacement field? Since this is a rather difficult process, we only consider the two-dimensional **plane strain** case. So $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$. The first step is to check whether the strains ϵ_{11}, ϵ_{12} and ϵ_{22} are **integrable**. To do this, we need to check the **compatibility equation**, being

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = 0. \tag{2.1.5}$$

If this equation holds, there is a solution.

The next step is to use $\epsilon_{11} = \partial u_1/\partial x_1$ and $\epsilon_{22} = \partial u_2/\partial x_2$. In other words, we need to integrate ϵ_{11} and ϵ_{22} with respect to x_1 and x_2, respectively. This results in certain unknown functions. These functions can often be determined using $\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$ up to certain unknown constants. In the end, we should remain with a solution with a few unknown constants, indicating the rigid body modes.

2.1.5 The rotation tensor

Sometimes deformations aren't the only thing we're interested in. Rotations can also be important. To examine them, we use the **infinitesimal rotation tensor**, defined as

$$[\omega_{ij}] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}. \tag{2.1.6}$$

Note that this matrix is not symmetric. In fact, it is anti-symmetric $(\omega_{ij} = -\omega_{ji})$. The tensor above contains information about the **average rotation**. However, we'll not go into detail about this.

2.2 The constitutive relations

Constitutive relations relate loads with displacements. Or equivalently, they relate the stress tensor with the strain tensor. What are the relationships between those two tensors?

2.2.1 Linearly elastic solids

We want to find a relationship between the stress tensor and the strain tensor. There are, however, many types of materials. For some weird materials, the stress depends on the strain, the strain rate and the loading history.

Luckily, for most materials, the stress (approximately) only depends on the strain. And it does this in a linear way. Such materials are called **linear elastic solids**. For these materials, the stress tensor and the strain tensor can be related by a linear relation, such as

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl}\epsilon_{kl}. \tag{2.2.1}$$

The numbers C_{ijkl} are called the **elastic coefficients**. There are 81 of these components. Together, they form the **elasticity tensor** $[C_{ijkl}]$ (also denoted as $\mathbb{C}$). Usually, these components need to be determined experimentally. However, they are not all independent. So, we can apply some tricks.

2.2.2 Voigt's notation

We know that both the stress tensor and the strain tensor are symmetric. (So $\sigma_{ij} = \sigma_{ji}$ and $\epsilon_{ij} = \epsilon_{ji}$.) Because of this, we also must have $C_{ijkl} = C_{ijlk}$ and $C_{ijkl} = C_{jikl}$. (These relations are called **minor symmetries**.) So, instead of 81 independent coefficients, we now only have 36. And because of this, we can also write equation (2.2.1) as

$$\sigma_{ij} = C_{ij11}\epsilon_{11} + C_{ij22}\epsilon_{22} + C_{ij33}\epsilon_{33} + C_{ij23}(2\epsilon_{23}) + C_{ij13}(2\epsilon_{13}) + C_{ij12}(2\epsilon_{12}) \tag{2.2.2}$$

However, since there are only 36 independent coefficients, it's not useful to write 81 coefficients down every time. That's why **Voigt's notation** is often convenient. In Voigt's notation, the stress and strain tensors aren't written as 3×3 matrices, but as 6×1 vectors. Also, the elasticity tensor $\mathbb{C}$ is written as a 6×6 matrix. This gives us the following relation

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix} = \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{23}\\ 2\epsilon_{13}\\ 2\epsilon_{12}\end{bmatrix}. \tag{2.2.3}$$

It's important to note the order of the coefficients. (In other books this order may be different.) Also note the twos in the strain vector.

The elasticity tensor above has another interesting property. By examining elastic strain energy, it can be shown that $C_{ijkl} = C_{klij}$. (This is a so-called **major symmetry**.) This implies that the elasticity tensor above is symmetric. So there are only 21 independent coefficients left.

2.2.3 The compliance matrix

If we know the strain and the elasticity tensor, then we can find the stress. But, we can also do it the other way around. For this, we use the **compliance tensor** $\mathbb{S} = \mathbb{C}^{-1}$. This then gives us that

$$\bar{\epsilon} = \mathbb{S}\sigma. \tag{2.2.4}$$

By the way, with $\bar{\epsilon}$ we mean the new strain vector (with the added twos).

2.2.4 Material symmetries

We still have 21 independent properties. But we can often reduce that number, due to **geometrical symmetries** in the material. Some materials have no such intrinsic symmetries. They are called **triclinic materials** and need to be described by 21 independent variables.

Some materials have a plane of symmetry. A material has such a **plane of summetry** if, after a reflection about that plane, is indistinguishable from the original material.

A material that has only one such plane of symmetry is called **monoclinic**. For such a material, eight of the coefficients will be zero. So there are 13 remaining independent coefficients. A material that has three mutually perpendicular planes of symmetry is called **orthotropic**. Such a material has 9 independent coefficients.

A material can also be **transversely isotropic**. In this case the material has three planes of symmetry, with an angle of 60° between them. (Like in a honeycomb structure.) In this case, there are only 5 independent coefficients.

2.2.5 Isotropic materials

For some materials every plane is a plane of symmetry. Such materials are called **isotropic materials**. Such materials have only two independent properties. We can write the elasticity tensor for these materials as

$$[C_{IJ}] = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{2.2.5}$$

The parameters λ and μ are called the **Lamé coefficients**. μ is also known as the **shear modulus**. (λ has no clear physical meaning.) From the above matrix, we can also directly find that

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\sum_{k=1}^{3}\epsilon_{kk}. \tag{2.2.6}$$

By the way, the symbol δ_{ij} is the **Kronecker delta** and is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.2.7}$$

We can also invert the above relation to express the strain ε as a function of the stress σ. We then get

$$\epsilon_{ij} = \frac{1}{E}\left((1+\nu)\sigma_{ij} - \nu\delta_{ij}\sum_{k=1}^{3}\sigma_{kk}\right). \tag{2.2.8}$$

Here E is the **elastic modulus** and ν is **Poisson's ratio**.

3. Application of the consitutive models

We now know how stresses and strains relate to each other. It's time to find out how we can use this to solve problems. First we examine static problems. We then move on to dynamic problems.

3.1 Static problems

3.1.1 Conditions and equations

When solving problems, the stress field should obey certain conditions. First let's take a look at what conditions there are. From the first chapter of this summary, we can recall the balance of linear momentum. It stated that

$$\int_{\partial\Omega} \mathbf{t}\, dA + \int_{\Omega} \mathbf{b}\, dV = \int_{\Omega} \rho \mathbf{a}\, dV. \tag{3.1.1}$$

For static problems, the acceleration is zero. If we rewrite the above equation, and split it up in components, we can then find that

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0. \tag{3.1.2}$$

This is the **balance of linear momentum** for static problems. It is our first condition. There are also the so-called **compatibility conditions**. They demand that

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} = \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k}. \tag{3.1.3}$$

Note that there are 81 different compatibility conditions, for every combination of i, j, k and l. There are often also boundary conditions. Sometimes the displacement in some direction u_i is set. At other times, the boundary traction $\hat{t}_i$ is set. In this case, you can use the stress tensor to find a relation for $\hat{t}_i$.

So our task is to find a stress field which satisfies all the conditions. With that, we can then find the displacement field. For that, we use the constitutive relations

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl} \epsilon_{kl}. \tag{3.1.4}$$

and the strain-displacement relations

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{3.1.5}$$

There is, however, one small problem. An analytical solution only exists for a few simple problems. Therefore the above equations are often used in numerical methods. Nevertheless, we will examine some analytical solutions now.

3.1.2 Plane stress case

The first case we examine is the **plane stress case**. Stress occurs only in a plane. Therefore $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. The stress also only depends on the position on the plane. Thus $\sigma_{ij} = \sigma_{ij}(x_1, x_2)$. (Note that in general $\varepsilon_{33} \neq 0$.) We also assume that the material is **isotropic** (it has the same properties in every direction) and **homogeneous** (the material has the same properties at every point in the structure). Also, there are no body forces. (Thus $b_1 = b_2 = 0$.)

To find the stress distribution, it is handy to use a so-called **Airy stress function** ψ. We define ψ such that

$$\sigma_{11} = \frac{\partial^2 \psi}{\partial x_2^2}, \qquad \sigma_2 = \frac{\partial^2 \psi}{\partial x_1^2} \qquad \text{and} \qquad \sigma_{12} = -\frac{\partial^2 \psi}{\partial x_1 \partial x_2}, \tag{3.1.6}$$

if such a function exists. This has several advantages. We can see that the balance of linear momentum is now automatically satisfied. But what about the 81 compatibility equations? Well, it turns out that there are only 6 independent compatibility equations. And of these 6, only 1 actually matters. (The others are not important or automatically satisfied.) This equation demands that

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial \epsilon_{12}}{\partial x_1 \partial x_2}. \tag{3.1.7}$$

We can now use the relations between stress and strain (the constitutive relations). This turns the above equation into a single compatibility equation for the stress function, being

$$\frac{\partial^4 \psi}{\partial x_1^4} + 2\frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \psi}{\partial x_2^4} = 0. \tag{3.1.8}$$

All we have to do is find a stress function ψ which satisfies this compatibility equation, and any given boundary conditions. Once we have done that, we have solved our problem.

3.1.3 Plane strain case

We now examine the **plain strain case**. Now we the strain occurs only in a plane. So $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$. (But not $\sigma_{33} = 0$.) Also, $\epsilon_{ij} = \epsilon_{ij}(x_1, x_2)$. We again assume that the material is isotropic and homogeneous, and that there are no body forces.

We define the stress function the same as in the plane stress case. So,

$$\sigma_{11} = \frac{\partial^2 \psi}{\partial x_2^2}, \qquad \sigma_2 = \frac{\partial^2 \psi}{\partial x_1^2} \qquad \text{and} \qquad \sigma_{12} = -\frac{\partial^2 \psi}{\partial x_1 \partial x_2}, \tag{3.1.9}$$

After examining compatibility equations, we find that the only remaining equation again is

$$\frac{\partial^4 \psi}{\partial x_1^4} + 2\frac{\partial^4 \psi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \psi}{\partial x_1^4} = 0. \tag{3.1.10}$$

So the plane stress and the plane strain case work quite the same.

3.1.4 Finding a stress function

So how do we find an appropriate stress function ψ? To do this, we simply assume a form for ψ. Usually, an exponential form would do well. Therefore we assume that

$$\psi = \sum_m \sum_n c_{mn} x_1^n x_2^m. \tag{3.1.11}$$

Terms with $m + n \leq 1$ will drop out of all the compatibility equations. We therefore don't consider them. Also, terms with $m + n \geq 5$ usually aren't necessary to get a good solution. This prevents us a bit from getting an incredibly huge polynomial.

After having a form for ψ, we insert it into the compatibility equation. This gives us some relations for the unknown coefficients c_{mn}. We also try to match ϕ with the boundary conditions. This gives us even more relations for the unknown coefficients. In the end, all the coefficients should be solved for.

It's not always possible to let ψ match exactly with the boundary conditions. In this case, **Saint Venant's principle** should often be used. This principle states that, relatively far away from the boundary, the introduced loads have spread out. This can be used to let ψ approximately match the boundary conditions.

3.2 Dynamic Problems

3.2.1 The wave equation

Let's examine the **half-space**. This is a space such that there is material for every $x_1 > 0$. The boundary of the half-space is thus simply the $\mathbf{e_2}, \mathbf{e_3}$ plane. We load this half-space on its boundary by a uniform time-varying compressive load $\hat{\mathbf{p}}(t)$ in $\mathbf{e_1}$-direction. Now let's ask ourselves, what happens?

Now let's examine linear momentum in the $\mathbf{e_1}$-direction. We assume that there are no body forces ($\mathbf{b} = \mathbf{0}$). We then see that

$$\frac{\partial \sigma_{11}}{\partial x_1} = \rho \frac{\partial^2 u_1}{\partial t^2}. \tag{3.2.1}$$

Due to symmetry, there is only displacement in $\mathbf{e_1}$-direction. So $\epsilon_{22} = \epsilon_{33} = \epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$ and $\epsilon_{11} = \partial u_1 / \partial x_1$. We also have $\sigma_{11} = (\lambda + 2\mu) \partial u_1 / \partial x_1$. This turns the above equation into

$$c_p^2 \frac{\partial^2 u_1}{\partial x_1^2} = \frac{\partial^2 u_1}{\partial^2 t}, \qquad \text{where} \qquad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{3.2.2}$$

c_p is called the **longitudinal (pressure) wave speed**. We can now see that the above equation is the **wave equation**, known from partial differential equations. Of course, a PDE should have initial conditions and boundary conditions. The **initial conditions** are often assumed to be

$$u_1(x, 0) = 0 \qquad \text{and} \qquad \frac{\partial u_1}{\partial t}(x_1, 0) = 0. \tag{3.2.3}$$

There is only one **boundary condition**. It is set at $x_1 = 0$ and is given by

$$\sigma_{11}(0, t) = (\lambda + 2\mu) \frac{\partial u_1}{\partial x_1}(0, t) = -\hat{\mathbf{p}}(t). \tag{3.2.4}$$

3.2.2 The solution of the wave equation

It's time to solve the wave equation. The general solution of the wave equation is given by

$$u_1(x_1, t) = f\left(t - \frac{x_1}{c_p}\right) + g\left(t + \frac{x_1}{c_p}\right). \tag{3.2.5}$$

f and g are functions that need to be chosen such that the initial and boundary conditions are satisfied. f denotes a wave that travels in the positive $\mathbf{e_1}$-direction. Similarly, g denotes a wave that travels in the negative $\mathbf{e_1}$-direction.

If we also include the initial conditions and boundary conditions, we can derive relations for f and g. In fact, these two functions are given by

$$f(\eta) = \begin{cases} \frac{1}{\rho c_p} \int_0^\eta \hat{\mathbf{p}}(\tau)\, d\tau & \text{for } \eta \geq 0 \\ 0 & \text{for } \eta \leq 0 \end{cases} \qquad \text{and} \qquad g(\xi) = 0 \text{ for } \xi \geq 0. \tag{3.2.6}$$

Combining this with the general solution, we can find that

$$u_1(x_1, t) = \begin{cases} \frac{1}{\rho c_p} \int_0^{t - x_1/c_p} \hat{\mathbf{p}}(\tau)\, d\tau & \text{for } t \geq x_1/x_p, \\ 0 & \text{for } t \leq x_1/c_p. \end{cases} \tag{3.2.7}$$

So the displacement field is now known. The stress distribution can also be solved for. We then find that

$$\sigma_{11}(x_1, t) = \begin{cases} -\hat{\mathbf{p}}(t - x_1/c_p) & \text{for } t \geq x_1/x_p, \\ 0 & \text{for } t \leq x_1/c_p. \end{cases} \tag{3.2.8}$$

We can see something quite interesting from this equation. When a force is introduced into the half-space, it travels through the half-space with velocity c_p. That's interesting to know.

3.2.3 Multiple layers with different properties

What happens if we have two layers A and B, having different material properties? The two layers each have different wave velocities, being

$$c_p^A = \sqrt{\frac{\lambda_A + 2\mu_A}{\rho_A}} \qquad \text{and} \qquad c_p^B = \sqrt{\frac{\lambda_B + 2\mu_B}{\rho_B}}. \tag{3.2.9}$$

Let's suppayer A starts at $x_1 = 0$. It ends at $x_1 = d$, which is also where the other layer starts. For times $t < d/c_p^A$, layer B will not notice any of the waves coming from the applied load. However, for $t \geq d/c_p^A$, there will be an **incident pulse** f_i acting on layer B. At the boundary between these layers, part of this pulse will be reflected. This is the **reflected pulse** f_r. Another part will be transmitted into layer B. This is the **transmitted pulse** f_t. So, for $t \geq d/c_p^A$, we have

$$u_1(x_1,t) = f_i\left(t - \frac{x_1}{c_p^A}\right) + f_r\left(1 + \frac{x_1}{c_p^A}\right) \text{ for } x_1 < d \quad \text{and} \quad u_1(x_1,t) = f_t\left(t - \frac{x_1}{c_p^B}\right) \text{ for } x_1 > d. \tag{3.2.10}$$

The question remains, what are these functions f_r and f_t? To find them, we have to use conditions. We know that the displacement at the boundary must remain the same for both layers. Due to Newton's third law, also the stress must remain continuous. So the conditions at the boundary are

$$u_1(d^-,t) = u_1(d^+,t) \qquad \text{and} \qquad \sigma_{11}(d^-,t) = \sigma_{11}(d^+,t). \tag{3.2.11}$$

By using this, we can find that

$$f_r = \frac{\rho_B c_p^B - \rho_A c_p^A}{\rho_B c_p^B + \rho_A c_p^A} f_i \qquad \text{and} \qquad f_t = \frac{2\rho_A c_p^A}{\rho_B c_p^B + \rho_A c_p^A} f_i. \tag{3.2.12}$$

It is often convenient to define the **ratio of longitudinal acoustic impedances** γ_p as

$$\gamma_p = \frac{\rho_B c_p^B}{\rho_A c_p^A}. \tag{3.2.13}$$

In this case, the above equations turn into

$$f_r = \frac{1-\gamma_p}{1+\gamma_p} f_i \qquad \text{and} \qquad f_t = \frac{2}{1+\gamma_p} f_i. \tag{3.2.14}$$

We can find similar relations for the stress propagation. These relations are

$$\sigma_{11}^{(r)} = \frac{\gamma_p - 1}{\gamma_p + 1}\sigma_{11}^{(i)} \qquad \text{and} \qquad \sigma_{11}^{(t)} = \frac{2\gamma_p}{\gamma_p + 1}\sigma_{11}^{(i)}. \tag{3.2.15}$$

3.2.4 Shear stress propagation

We have seen how normal stress propagates in a half-space. But what about shear stress? Let's assume a shear load $\hat{\mathbf{s}}(t)$ is applied on the half-space boundary, in the $\mathbf{e_2}$-direction. What happens?

This time we have $\epsilon_{12} = \frac{1}{2}\frac{\partial u_2}{\partial x_1}$, while $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0$. We also have $\sigma_{12} = 2\mu\epsilon_{12} = \mu\frac{\partial u_2}{\partial x_1}$. This time, we can derive from balance of linear momentum that

$$c_s^2 \frac{\partial^2 u_2}{\partial x_1^2} = \frac{\partial^2 u_2}{\partial t^2}, \qquad \text{where} \qquad c_s = \sqrt{\frac{\mu}{\rho}}. \tag{3.2.16}$$

The quantity c_s is called the **transverse (shear) wave speed**. We again have a wave equation. The **initial conditions** now are

$$u_2(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u_2}{\partial t}(x_1,0) = 0. \tag{3.2.17}$$

The boundary condition is again set at $x_1 = 0$. It is now given by

$$\sigma_{12}(0,t) = \mu \frac{\partial u_2}{\partial x_1}(0,t) = \hat{\mathbf{s}}(t). \tag{3.2.18}$$

We can solve the wave equation for u_2. We then find that

$$u_2(x_1,t) = \begin{cases} \frac{1}{\rho c_s} \int_0^{t-x_1/c_s} \hat{\mathbf{s}}(\tau)\, d\tau & \text{for } t \geq x_1/x_s, \\ 0 & \text{for } t \leq x_1/c_s. \end{cases} \tag{3.2.19}$$

Similarly, we can find a relation for the shear stress distribution. We now find that

$$\sigma_{12}(x_1,t) = \begin{cases} \hat{\mathbf{s}}(t - x_1/c_s) & \text{for } t \geq x_1/x_s, \\ 0 & \text{for } t \leq x_1/c_s. \end{cases} \tag{3.2.20}$$