Differential Equations Summary

1. Basic Differential Equations

1.1 First order equations

1.1.1 Basic form

Equations containing derivatives are **differential equations**. Differential equations of the **first order** (meaning only first derivatives can occur, but no second or higher derivatives) can be written as

$$\frac{dy}{dt} = y' = f(t, y). \tag{1.1.1}$$

Note that to keep equations short, we write $\frac{dy}{dt} = y'$. A function $y = \phi(t)$ is called a **solution** if it satisfies the above equation. No simple solution method exists that can solve all differential equations of this form.

1.1.2 Linear equations

However, for some forms there are methods to find solutions. For example, if the equation is **linear in** y, it can be written as

$$y' + p(t)y = g(t). \tag{1.1.2}$$

Note that sometimes differential equations have to be rewritten to bring them to the right form. To find a solution on a particular interval (α, β), $p(t)$ must be **continuous on** (α, β), that is, $p(t)$ exists for every t in the interval (α, β).

The **technique of integrating factor** can be applied to solve this form of differential equation. First find any integral of $p(t)$. Then define the **integrating factor** $\mu(t)$ as

$$\mu(t) = e^{\int p(t)\,dt}. \tag{1.1.3}$$

Now multiply equation 1.1.2 by $\mu(t)$. Using the chain rule, it can be rewritten as

$$\frac{d(\mu(t)y)}{dt} = \int \mu(t)\, g(t)\, dt. \tag{1.1.4}$$

The solution then becomes

$$y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\, g(s)\, ds + \frac{c}{\mu(t)}. \tag{1.1.5}$$

Of this equation, the part $\frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\, g(s)\, ds$ is called the **particular solution** and $\frac{c}{\mu(t)}$ is called the **general solution**. In differential equations the complete set of solutions is usually formed by the general solution, plus any linear combination of the particular solution(s).

1.1.3 Separable differential equations

A differential equation is called a **separable differential equation**, if it can be written as

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}. \tag{1.1.6}$$

We can rewrite it as

$$N(y)\,dy = M(x)\,dx. \tag{1.1.7}$$

The solution of this differential equation is acquired by simple integration of the equation

$$\int N(y)\,dy = \int M(x)\,dx. \tag{1.1.8}$$

1.2 Second order linear differential equations

1.2.1 Basic form

The basic form of a **second order differential equations** is

$$\frac{d^2y}{dt^2} = y'' = f(t, y, y'). \tag{1.2.1}$$

Such equations are hard to solve. So we will be looking at **second order linear differential equations**, which have the form

$$y'' + p(t)y' + q(t)y = g(t). \tag{1.2.2}$$

If a second order equation can be written in the form of the above equation, it is called **linear**, and otherwise **nonlinear**. For such differential equations solving methods exist. However, we do assume that the function $p(t)$, $q(t)$ and $g(t)$ are continuous functions.

A second order linear differential equation is said to be **homogeneous** if the term $g(t)$ in equation 1.2.2 is 0 for all t. Otherwise it is called **nonhomogeneous**.

1.2.2 Homogeneous equations with constant coefficients

Suppose our differential equation has the form

$$ay'' + by' + cy = 0, \tag{1.2.3}$$

with a, b and c constants. Let's define the **characteristic equation** to be

$$ar^2 + br + c = 0. \tag{1.2.4}$$

If we can find an r that satisfies the characteristic equation, then we know that $y = e^{rt}$ is a solution. In fact all linear combinations $y = ce^{rt}$ are solutions. So let's look at three specific cases.

- $b^2 - 4ac > 0$

  There are two real solutions r_1 and r_2 to equation 1.2.4. Both $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ and all linear combinations of them are solutions. So the general solution of the differential equation is:

  $$y = c_1 y_1 + c_2 y_2 = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{1.2.5}$$

- $b^2 - 4ac = 0$

  There is only one solution $r = -\frac{b}{2a}$ to the characteristic equation. We know that $y_1 = e^{rt}$ is a solution. However, also $y_2 = te^{rt}$ is a solution. So the general solution of the differential equation is:

  $$y = c_1 y_1 + c_2 y_2 = c_1 e^{rt} + c_2 te^{rt} \tag{1.2.6}$$

- $b^2 - 4ac < 0$

  There are no real solutions now, only complex ones. So if $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$, and also $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are solutions. Working out the complex numbers in all linear combinations of the two solutions gives as general solution:

$$y = c_1 y_1 + c_2 y_2 = e^{\alpha t} \left(c_1 \cos \beta t + c_2 \sin \beta t\right) \tag{1.2.7}$$

The solutions given by the methods above are all possible solutions of the differential equation.

1.2.3 Nonhomogeneous equations - Method of undetermined coefficients

Suppose our differential equation has the form

$$ay'' + by' + cy = g(t). \tag{1.2.8}$$

with a, b and c still constants. The function $g(t)$ here is called the **forcing function**. Suppose we find any particular solution $Y(t)$ that satisfies the above equation. We already know from the previous paragraph how to find the general solution set $c_1 y_1 + c_2 y_2$ for the homogeneous differential equation $ay'' + by' + c = 0$. If we add those two solutions up, we find all solutions for the above differential equation.

So the trick is to find a single $Y(t)$ that satisfies the differential equation. One way to do that is to use the **method of undetermined coefficients**. We make an initial assumption on the form of $Y(t)$ (called the **auxiliary equation**), with a couple of undetermined coefficients, and then try to find the coefficients. The downside of this method is that it only works on equations that contain terms at^n, $e^{\alpha} t$ and $\sin \beta t$, or combinations of those terms.

First take a look at $g(t)$. If it consists of multiple parts, separated by plus or minus signs (for example, $g(t) = t + \sin t - e^t$), then split the problem up in parts and find a particular solution $Y_i(t)$ for every part $g_i(t)$.

To find a particular solution for $g_i(t)$, use the auxiliary equation

$$t^s \left(\left(a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n\right) e^{\alpha t} \cos \beta t + \left(b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n\right) e^{\alpha t} \sin \beta t\right). \tag{1.2.9}$$

The variables α, β and n can be found in $g_i(t)$. (For example, for $g_i(t) = te^{2t}$ the auxiliary equation becomes $t^s \left(\left(a_0 + a_1 t\right) e^{2t}\right)$.) The variable s, however, is a different story. It's a matter of trial and error. Usually $s = 0$ works. If this doesn't work, try $s = 1$. If it still doesn't work (unlikely, but possible), try $s = 2$.

Now we have an auxiliary equation $Y_i(t)$ with undetermined coefficients $a_0, \ldots, a_n, b_0, \ldots b_n$. First find $Y_i'(t)$ and $Y_i''(t)$. Then write down the equation

$$aY_i''(t) + bY_i'(t) + cY_i(t) = g_i(t). \tag{1.2.10}$$

Use this equation to solve the undetermined coefficients and find the particular solution for $ay'' + by' + cy = g_i(t)$.

So having found all the particular solutions $Y_i(t)$ for $ay'' + by' + cy = g_i(t)$, add them all up to find the particular solution $Y(t) = Y_1(t) + \ldots + Y_n(t)$. Now add this up to the general solution $c_1 y_1 + c_2 y_2$ of the homogeneous equation $ay'' + by' + cy = 0$ to find the full solution set of the differential equation:

$$y = c_1 y_1 + c_2 y_2 + (Y_1(t) + \ldots + Y_n(t)). \tag{1.2.11}$$

1.2.4 Nonhomogeneous equations - Variation of parameters

The method **variation of parameters** is applied to differential equations of the form of equation 1.2.8 and goes as follows. First find the solution $y = c_1 y_1 + c_2 y_2$ of the differential equation $ay'' + by' + c = 0$.

Now replace c_1 by $u_1(t)$ and c_2 by $u_2(t)$ to get $y = u_1(t)y_1 + u_2(t)y_2$. Now it is possible to find y' and y''. Let's first (for no specific reason but that the outcome will be convenient) assume that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \tag{1.2.12}$$

Working everything out, we eventually find that

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \tag{1.2.13}$$

Now, let's define the **Wronskian determinant** (or simply **Wronskian**) $W(y_1, y_2)$ as

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t). \tag{1.2.14}$$

If we solve for u_1' and u_2' in equations 1.2.12 and 1.2.13, we find that

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \qquad \text{and} \qquad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}. \tag{1.2.15}$$

Solving this gives as a particular solution for the differential equation

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}ds, \tag{1.2.16}$$

for any convenient t_0 in the interval. So this makes the general solution for the differential equation:

$$y = c_1y_1 + c_2y_2 + Y(t). \tag{1.2.17}$$

1.3 Initial value problems and boundary value problems

1.3.1 Initial value problems

Sometimes, next to a differential equation, also certain conditions are given. For example, the values of y and y' at a given time t_0 are given:

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'. \tag{1.3.1}$$

Such conditions are called **initial conditions**. If these conditions are given, the problem is called an **initial value problem**. Finding the general solution for the differential equation isn't sufficient to solve the problem. You have to find the values c_1 and c_2 such that the initial conditions are met.

1.3.2 Boundary value problems

Sometimes not the initial conditions at a time t_0 are given, but the conditions are two different times are given:

$$y(\alpha) = y_0, \qquad y(\beta) = y_1. \tag{1.3.2}$$

Such conditions are called **boundary conditions**. If these conditions are given, the problem is called a **(two-point) boundary value problem**. Once more the values c_1 and c_2 should be found such that the boundary conditions are met, to solve the problem.

2. Power Series Solutions

2.1 Power Series for Ordinary Points

2.1.1 Power series

A power series $f(x)$ is a series given by

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \lim_{m\to\infty}\sum_{n=0}^{m} a_n(x-x_0)^n. \tag{2.1.1}$$

It is said to converge if this limit exists. Otherwise it diverges. There is a **radius of convergence** ρ such that the series converges for $|x-x_0|<\rho$ and diverges for $|x-x_0|>\rho$. For $|x-x_0|=\rho$ the series may either converge or diverge. This needs to be tested separately.

The radius of convergence can often be found using the **ratio test**. Consider the limit of the absolute value of the ratio of two subsequent terms in the summation, being

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L. \tag{2.1.2}$$

The series converges if $|x-x_0|L<1$ and diverges if $|x-x_0|L>1$. In other words, the radius of convergence here is $\rho = 1/L$.

Series can also be tested for equality. If we have two series a and b such that

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n, \tag{2.1.3}$$

then all coefficients must be equal. So $a_n = b_n$ for every n.

Let $f^{(n)}(x)$ denote the n^{th} derivative of $f(x)$. If the coefficients a_n are such that

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \tag{2.1.4}$$

then the series is called a **Taylor series** for the function f about $x=x_0$.

2.1.2 Ordinary and singular points

Let's consider second order linear differential equations, where the coefficients are functions of the independent variable (which is usually x). The general form of the homogeneous equation is

$$P(x)y'' + Q(x)y' + R(x) = 0. \tag{2.1.5}$$

For simplicity we will assume that $P(x)$, $Q(x)$ and $R(x)$ are all polynomials. Let's suppose we want to solve this equation in the neighborhood of a point x_0. Such a point x_0 is called a **ordinary point** if $P(x_0)\neq 0$. If, however, $P(x_0)=0$, then the point is called a **singular point**.

2.1.3 Series solutions near an ordinary point

It's often hard to find a normal solution for equation 2.1.5. But let's suppose that we look for solutions of the form

$$y = a_0 + a_1(x-x_0) + \ldots + a_n(x-x_0)^n + \ldots = \sum_{n=0}^{\infty} a_n(x-x_0)^n. \tag{2.1.6}$$

We assume that this series converges in an interval $|x - x_0| < \rho$ for some $\rho > 0$. For example, if we want to solve the differential equation $y'' + y = 0$, we can first find that

$$y'' = 2a_2 + 6a_3(x - x_0) + \ldots + n(n-1)a_n(x - x_0)^{n-2} + \ldots = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x - x_0)^n. \quad (2.1.7)$$

The differential equation thus becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x - x_0)^n + a_n(x - x_0)^n = y'' + y = 0 = \sum_{n=0}^{\infty} 0(x - x_0)^n. \quad (2.1.8)$$

We now have an equation with two sums. The two sums are only equal if all the coefficients are equal. This results in

$$(n+2)(n+1)a_{n+2} + a_n = 0 \qquad \Rightarrow \qquad a_{n+2} = -\frac{a_n}{(n+2)(n+1)}. \quad (2.1.9)$$

This relation is a **recurrence relation**, expressing a coefficient as a function of its predecessors. For arbitrary coefficients a_0 and a_1 we can find all the coefficients, and thus find the solution to the differential equation.

2.1.4 Convergence of the solution

The solution found in the last paragraph converges around x_0. But what is the radius of convergence? It turns out that this depends on the roots of $P(x)$ (being the values x such that $P(x) = 0$). Let's consider all the roots of $P(x)$ and draw them in the complex plane. Now let's also draw x_0. The radius of convergence is the minimum distance between x_0 and any root of $P(x)$.

For example, if $P(x) = x^2 - 2x + 2 = (x-1)^2 + 1$, then the roots are $1 \pm i$. If also $x_0 = 0$, then the radius of convergence is simply $\sqrt{2}$.

2.2 Singular Points

2.2.1 Regular singular points

Let's define $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$. Normally we could rewrite the differential equation to

$$y'' + p(x)y' + q(x)y = 0. \quad (2.2.1)$$

For singular points this isn't possible since $P(x_0) = 0$. In this case using power series gives problems. That's why we need to find other ways to solve these problems. Solving this problem can be split up in two separate cases, depending on whether x_0 is a **regular singular point** or an **irregular singular point**. To determine this, we need to examine the limits

$$p_0 = \lim_{x \to x_0}(x - x_0)p(x) = \lim_{x \to x_0}(x - x_0)\frac{Q(x)}{P(x)}, \qquad \text{and} \qquad q_0 = \lim_{x \to x_0}(x - x_0)^2 = q(x) = \lim_{x \to x_0}(x - x_0)^2\frac{R(x)}{P(x)}. \quad (2.2.2)$$

If both these limits exist (they are finite), then the point x_0 is a regular singular point. If either of these limits (or both) do not exist, then x_0 is an irregular singular point.

2.2.2 Euler equation

A relatively simple differential equation with a regular singular point is the **Euler equation**, being

$$x^2y'' + \alpha xy' + \beta y = 0. \quad (2.2.3)$$

Let's assume a certain solution has the form $y = x^r$. The differential equation then becomes

$$x^2(x^r)'' + \alpha x(x^r)' + \beta x^r = x^r\left(r(r-1) + \alpha r + \beta\right) = 0. \tag{2.2.4}$$

So we need to solve $(r(r-1) + \alpha r + \beta)$ to find r. There are three given possibilities. If r has two real distinct roots r_1 and r_2, then the general solution is

$$y = c_1 x^{r_1} + c_2 x^{r_2}. \tag{2.2.5}$$

If the roots are real, but equal, then the general solution can be shown to be

$$y = (c_1 + c_2 \ln x)x^{r_1}. \tag{2.2.6}$$

If the roots are complex, such that $r = \lambda \pm \mu i$, then the general solution is

$$y = x^\lambda \left(c_1 \cos\left(\mu \ln x\right) + c_2 \sin\left(\mu \ln x\right)\right). \tag{2.2.7}$$

2.2.3 Negative x

The above solutions are correct for positive x. If $x < 0$ strange situations occur with possibly complex or undefined numbers. But if we define $\xi = -x$, then we find the same solutions (with ξ instead of x). So we can rewrite the equations of the last paragraph to

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}, \tag{2.2.8}$$

$$y = (c_1 + c_2 \ln|x|)|x|^{r_1}, \tag{2.2.9}$$

$$y = |x|^\lambda \left(c_1 \cos\left(\mu \ln|x|\right) + c_2 \sin\left(\mu \ln|x|\right)\right). \tag{2.2.10}$$

2.2.4 Series solutions near a singular point

Let's consider a regular singular point. We assume this point is $x_0 = 0$. (If $x_0 \neq 0$ simply make the change of variable $t = x - x_0$.) We can rewrite our differential equation to

$$y'' + p(x)y' + q(x)y = x^2y'' + x(xp(x))y' + x^2q(x)y = 0. \tag{2.2.11}$$

Note that for a regular singular point the parts $xp(x)$ and $x^2q(x)$ have a value as $x \to 0$. Let's now assume that a solution has the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}. \tag{2.2.12}$$

We only need to find the values of r and the coefficients a_n. If we calculate y' and y'' and put this back in the differential equation, we find that

$$F(r) = r(r-1) + p_0 r + q_0 = 0. \tag{2.2.13}$$

This equation is called the **indicial equation**. Its roots r_1 and r_2 (in which we suppose that $r_1 \geq r_2$) are called the **exponents of singularity**. We have now found r. The coefficients can be found using the **recurrence relation**

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k\left((r+k)p_{n-k} + q_{n-k}\right) = 0. \tag{2.2.14}$$

Note that the coefficients depend on the values of a_0 and r. a_0 is arbitrary, and for simplicity usually $a_0 = 1$ is chosen. r is not arbitrary though. To indicate which r has been used to calculate the coefficients, the coefficients are usually written as $a_n(r_1)$ or $a_n(r_2)$. Now that we have found the coefficients, we can write the solutions. Since we have two solutions r_1 and r_2, we have two solutions, being

$$y_1(x) = |x|^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right) \qquad \text{and} \qquad y_2(x) = |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right). \tag{2.2.15}$$

Note that we have taken the absolute value of x again, according to the trick of the previous paragraph. The general set of solutions now consists of all linear combination $c_1y_1 + c_2y_2$ of these solutions.

2.2.5 Equal roots

There are, however, a few catches to the method described in the last paragraph. If $r_1 = r_2$ we will find the same solution twice. We want two solutions to find the general solution set, and we only have one, being y_1. So we need another method to find a new solution. Let's assume that the new solution has the form

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n x^n. \tag{2.2.16}$$

All we need to do now is find the coefficients b_n. The procedure for this is simple. First calculate y_2' and y_2''. Then substitute these in the differential equation, and solve for the coefficients b_n.

2.2.6 Roots differing by an integer

Let's consider the term $F(r + n)$ of the recurrence relation. If this term is 0, it is impossible to find a_n. If $r = r_1$ we will always find a correct solution y_1. But if $r = r_2$ and $n = r_1 - r_2$ we find that $F(r + n) = F(r_2) = 0$. So there is a problem. Now let's assume the second solution has the form

$$y_2(x) = ay_1(x)\ln|x| + |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right). \tag{2.2.17}$$

Just like in the previous paragraph, the values of the constant a and the coefficients c_n can be found by substituting y_2, y_2' and y_2'' in the differential equation.

3. The Laplace Transform

3.1 Laplace transform definitions

3.1.1 Improper integrals

The **Laplace transform** involves an integral from zero to infinity, which is a so-called **improper integral**. Such an integral is defined as

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt. \tag{3.1.1}$$

Such an integral can **converge** to a certain value or **diverge**.

3.1.2 Integral transforms

An **integral transform** is a relation of the form

$$F(s) = \int_\alpha^\beta K(s,t)\,f(t)\,dt, \tag{3.1.2}$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation. This relation transforms the function f into another function F, which is called the **transform** of f.

3.1.3 Laplace transform

One such integral transform is the **Laplace transform**, which is often useful for linear differential equations. In this transform, $K(s,t) = e^{-st}$, $\alpha = 0$ and $\beta = \infty$. So the Laplace transform, denoted by $L\{f(t)\}$ (even though the L is often written slightly different), is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st}\,f(t)\,dt. \tag{3.1.3}$$

Now suppose $|f(t)| \le Ke^{at}$ for $t \ge M$ for certain constants K, a and M, then the Laplace transformation exists for $s > a$. An overview of Laplace transforms can be seen in table 1.

Function $f(t) = L^{-1}\{F(s)\}$	**Laplace Transform** $F(s) = L\{f(t)\}$	**Range**	**Notes**
1	$\frac{1}{s}$	$s > 0$	
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$	n = positive integer
e^{at}	$\frac{1}{s-a}$	$s > a$	
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$	
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$	
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a\rvert$	
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a\rvert$	

Table 1: Laplace transforms of basic functions.

3.1.4 Linear operators

It can also be shown that the Laplace transform is a **linear operator**, meaning that for any constants c_1 and c_2 and functions $f_1(t)$ and $f_2(t)$,

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}. \tag{3.1.4}$$

Using this theorem and table 1, it is possible to transform many functions quite easily.

Define $L^{-1}\{F(s)\}$ as the **inverse transform** of $F(s)$, meaning that $f(t) = L^{-1}\{L\{f(t)\}\}$. Then also L^{-1} is a linear operator. So this gives

$$L^{-1}\{c_1 F_1(s) + c_2 F_2(s)\} = c_1 L^{-1}\{F_1(s)\} + c_2 L^{-1}\{F_2(s)\} \tag{3.1.5}$$

3.1.5 Laplace transform of derivatives

The Laplace transform of $f'(t)$ is related to the Laplace transform of $f(t)$ (if it exists), by the equation

$$L\{f'(t)\} = sL\{f(t)\} - f(0). \tag{3.1.6}$$

If $f^{(n)}$ is the n'th derivative of f, then also

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0). \tag{3.1.7}$$

3.2 Functions and operators

3.2.1 Unit step function

The **unit step function**, also called the **Heaviside function**, is denoted by $u_c(t)$. It is defined such that $u_c(t) = 0$ if $t < c$ and $u_c(t) = 1$ for $t \geq c$.

(In other words, in an equation like $u_c(t)f(t)$, the function u_c "activates" the function $f(t)$ only for $t \geq c$, meaning for values of t smaller than c, the function is just 0. To "deactivate" the function $f(t)$, the function $(1 - u_c(t))f(t)$ can be used.)

The Laplace transform of u_c, with range $s > 0$, is

$$L\{u_c(t)\} = \frac{e^{-cs}}{s}. \tag{3.2.1}$$

If $F(s) = L\{f(t)\}$ and $f(t) = L^{-1}\{F(s)\}$, then

$$L\{u_c(t)f(t-c)\} = e^{-cs}F(s) \qquad \Leftrightarrow \qquad u_c(t)f(t-c) = L^{-1}\{e^{-cs}F(s)\}. \tag{3.2.2}$$

Analogous, it can be shown that

$$L\{e^{ct}f(t)\} = F(s-c) \qquad \Leftrightarrow \qquad e^{ct}f(t) = L^{-1}\{F(s-c)\}. \tag{3.2.3}$$

3.2.2 Delta function

The **Dirac delta function** $\delta(t)$ (also called the **delta function** or the **unit impulse function**) is defined such that

$$\delta(t) = 0 \text{ for } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{3.2.4}$$

The Laplace transform of this function is

$$L\{\delta(t-t_0)\} = e^{-t_0 s}. \tag{3.2.5}$$

From this follows that $L\{\delta(t)\} = e^0 = 1$. And finally, the integral of the product of the delta function and any continuous function f is

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = f(t_0). \tag{3.2.6}$$

3.2.3 The convolution integral

If $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = L\{h(t)\}, \qquad s > a, \tag{3.2.7}$$

where $h(t)$ is

$$h(t) = \int_0^t f(t-\tau) g(\tau) d\tau = \int_0^t f(\tau) g(t-\tau) d\tau. \tag{3.2.8}$$

Note the difference between t and τ. The function h is known as the **convolution** of f and g and the integrals in the last equation are known as **convolution integrals**. It is conventional to write the above equation as $h(t) = (f * g)(t)$.

The $*$ is more or less similar to a multiplication. The following rules apply.

$$f * g = g * f, \tag{3.2.9}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2, \tag{3.2.10}$$

$$(f * g) * h = f * (g * h), \tag{3.2.11}$$

$$f * 0 = 0 * f. \tag{3.2.12}$$

However, in general $f * 1 \neq f$. Keep these rules in mind.

3.3 Solving differential equations

3.3.1 Solving a second order initial value problem

Suppose we have a differential equation of the form

$$ay'' + by' + cy = f(t), \tag{3.3.1}$$

with a, b and c constants. Taking the Laplace transform of both sides, and applying equations 3.1.4 and 3.1.7, gives

$$aL\{y''\} + bL\{y'\} + cL\{y\} = a(s^2 L\{y\} - sy(0) - y'(0)) + b(sL\{y\} - y(0)) + cL\{y\} = L\{f(t)\} = F(s). \tag{3.3.2}$$

Solving this for $L\{y\} = Y(s)$ gives

$$Y(s) = L\{y\} = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{3.3.3}$$

Now $L\{y\}$ is known. To find $f(t)$, we simply need to transform it back: $f(t) = L^{-1}\{L\{y\}\}$. But the inverse Laplace transform is not always easy to find. This problem is known as **the inversion problem for the Laplace transform**.

3.3.2 Inversion problem for the Laplace transform

To find $L^{-1}\{F(s)\}$ for some function $F(s)$, it's wise to split $F(s)$ up in pieces that occur in table 1, and use equation 3.1.5 to inversely transform all the pieces. However, this splitting up in pieces often isn't that easy. Especially when fractions are present, this can be difficult. That's why the following example shows a method in which fractions can be split up.

From the differential equation $y'' + y = \sin 2t$ follows

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 4)(s^2 + 1)}. \tag{3.3.4}$$

We want to split this fraction up in separate fractions, one with denominator $s^2 + 4$ and the other with denominator $s^2 + 1$, like

$$Y(s) = \frac{a}{s^2 + 4} + \frac{b}{s^2 + 1} = \frac{a(s^2 + 1) + b(s^2 + 4)}{(s^2 + 4)(s^2 + 1)}, \tag{3.3.5}$$

for certain a and b. From this we see that $a(s^2 + 1) + b(s^2 + 4) = 2s^3 + s^2 + 8s + 6$. But, if a and b are just constants, there are no third powers of s on the left side of the equation. So let's just suppose that $b = cs + d$. Now it's easy to see that $c = 2$. Working out the rest of the equation gives $a = -2/3$ and $d = 5/3$. So finally we have split up the fraction to

$$Y(s) = \frac{-2/3}{s^2 + 4} + \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1}. \tag{3.3.6}$$

Using table 1 we can find

$$y = -\frac{1}{3}\sin 2t + 2\cos t + \frac{5}{3}\sin t. \tag{3.3.7}$$

3.3.3 Discontinuous forcing functions

If the nonhomogeneous term of the differential equation, also called the **forcing term**, is discontinuous, solving the differential equation can be difficult. To illustrate how to solve such equations, we handle an example. To solve the differential equation

$$y'' + 4y = u_4(t)(t - 4) - u_8(t)(t - 8) \qquad y(0) = 0 \qquad y'(0) = 0, \tag{3.3.8}$$

we take the Laplace transform to find

$$Y(s) = \frac{e^{-4s} - e^{-8s}}{s^2(s^2 + 4)}. \tag{3.3.9}$$

It is now often wise to define $H = \frac{1}{s^2(s^2+4)}$, such that $Y(s) = (e^{-4s} - e^{-8s})H(s)$. If we define $h(t) = L^{-1}\{H(s)\}$, then taking the inverse Laplace transform, and using equation 3.2.2, gives

$$y(t) = u_4(t)h(t - 4) - u_8(t)h(t - 8). \tag{3.3.10}$$

We only need to find $h(t)$. Rewriting $H(s)$ differently gives

$$H(s) = \frac{1}{4}\frac{1}{s^2} - \frac{1}{8}\frac{2}{s^2 + 2^2} \qquad \Rightarrow \qquad h(t) = L^{-1}\{H(s)\} = \frac{1}{4}t - \frac{1}{8}\sin 2t, \tag{3.3.11}$$

which can be inserted in equation 3.3.10 to get the solution of our differential equation.

3.3.4 Using the convolution integral

Consider the differential equation

$$ay'' + by' + cy = g(t), \tag{3.3.12}$$

where a, b and c are constants. Let's define $L\{g(t)\} = G(s)$ and define $\Phi(s)$ and $\Psi(s)$ as

$$\Phi(s) = \frac{(as+b)y_0 + ay_0'}{as^2+bs+c}, \qquad \Psi(s) = \frac{G(s)}{as^2+bs+c}. \tag{3.3.13}$$

By taking the Laplace transform of the differential equation, we find

$$Y(s) = \Phi(s) + \Psi(s) \qquad \Leftrightarrow \qquad y(t) = \phi(t) + \psi(t), \tag{3.3.14}$$

where $\phi(t) = L^{-1}\{\Phi(s)\}$ and $\psi(t) = L^{-1}\{\Psi(s)\}$. It is convenient to write $\Psi(s)$ as

$$\Psi(s) = H(s)G(s), \tag{3.3.15}$$

where $H(s) = \frac{1}{as^2+bs+c}$. The function $H(s)$ is known as the **transfer function**. Using the convolution integral, we can solve for $\psi(t)$:

$$\psi(t) = L^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)d\tau, \tag{3.3.16}$$

where $h(t) = L^{-1}\{H(s)\}$. The function $h(t)$ is called the **impulse response** of the system.

4. Systems of First Order Linear Equations

4.1 Introduction to First Order Equations

4.1.1 Relevance

This chapter is about first order linear equations. But why would we devote an entire chapter to it? Well, they can be very useful, as you can transform any differential equation to a set of first order equations. For example, the n^{th} order equation

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}) \tag{4.1.1}$$

can be transformed into a system of linear equations by setting

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = y'', \qquad \ldots, \qquad x_n = y^{(n-1)}. \tag{4.1.2}$$

This would give a system of first order differential equations, consisting of the equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= G(t, x_1, x_2, \ldots, x_n), \end{aligned} \tag{4.1.3}$$

where G is a function that depends on the original function F.

4.1.2 Definitions

The general form of a **system of first order differential equations** is

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= F_1(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= F_1(t, x_1, x_2, \ldots, x_n). \end{aligned} \tag{4.1.4}$$

The system is said to have a **solution** on the interval $I : \alpha < t < \beta$ if there exists a set of n functions

$$x_1 = \phi_1(t), \quad x_2 = \phi_2(t), \quad \ldots, \quad x_n = \phi_n(t), \tag{4.1.5}$$

that satisfies the corresponding system of equations on that interval.

If each of the functions $F_1, F_2, \ldots, F_n$ is a linear function of $x_1, x_2, \ldots, x_n$, then the system of equations is said to be **linear**. Otherwise it is **nonlinear**. The most general form of a system of n first order linear differential equations is therefore

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_1' &= p_{21}(t)x_1 + \ldots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_1' &= p_{n1}(t)x_1 + \ldots + p_{nn}(t)x_n + g_n(t). \end{aligned} \tag{4.1.6}$$

If also each of the functions $g_1(t), \ldots, g_n(t)$ is zero for all t in the interval I, then the system is said to be **homogeneous**. Otherwise it is **nonhomogeneous**.

We can rewrite the general form to a much simpler form, involving matrices. This would give

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t), \tag{4.1.7}$$

where $P(t)$ is the matrix formed by all the functions $p_{ij}(t)$.

4.1.3 Homogeneous systems

In homogeneous systems $\mathbf{g}(t) = 0$. Such systems can therefore be written like

$$\mathbf{x}' = P(t)\mathbf{x}. \tag{4.1.8}$$

Let's suppose we're dealing with a system of the n^{th} order. Also suppose we have n solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to this system. Now any linear combination $c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n$ of these vectors is also a solution. In fact, we can put the n solutions we had in a matrix $X(t)$, being

$$X(t) = \Big[\mathbf{x}_1(t) \ \ldots \ \mathbf{x}_n(t)\Big]. \tag{4.1.9}$$

Now every vector $\phi(t)$ satisfying

$$\phi(t) = c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n = X(t)\mathbf{c} \tag{4.1.10}$$

is a solution to our system of first order differential equations. If the linear combinations of the set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ contain all solutions to the system, then this set is called a **general solution set**.

Any general solution set that is linearly independent at the interval I is said to be a **fundamental set of solutions** for this interval. For such a set, every solution ϕ can be expressed as $\phi = X\mathbf{c}$ in exactly one way.

Let's define the **Wronskian** of the n solutions (denoted by $W[\mathbf{x}_1, \ldots, \mathbf{x}_n]$) as

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det X(t). \tag{4.1.11}$$

If $W[\mathbf{x}_1, \ldots, \mathbf{x}_n] \neq 0$ on a certain interval $I : \alpha < t < \beta$, then the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent on I and thus form a fundamental set of solutions.

4.1.4 Fundamental matrices

If the set $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a fundamental set of solutions, then the matrix $X(t)$ with columns $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is called the **fundamental matrix**. Since this is an important matrix, it is written with a different sign, being $\Psi(t)$. Any solution $\mathbf{x}$ to the system of differential equations can now be written as

$$\mathbf{x}(t) = \Psi(t)\mathbf{c}, \tag{4.1.12}$$

for some constant vector $\mathbf{c}$. Now let's suppose we need to solve an initial value problem. An **initial vector** $\mathbf{x}_0 = \mathbf{x}(t_0)$ is given. We have already found $\Psi(t)$ with the methods described above. We just have to find $\mathbf{c}$ such that

$$\mathbf{x}_0 = \Psi(t_0)\mathbf{c} \qquad \Rightarrow \qquad \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}_0. \tag{4.1.13}$$

The solution can now be found using

$$\mathbf{x} = \Psi(t)\mathbf{c} = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}_0. \tag{4.1.14}$$

If the matrix $\Psi(t)$ satisfies the condition $\Psi(t_0) = I$, where I is the identity matrix, then it is a **special fundamental matrix**. Such a matrix is denoted by $\Phi(t)$. Using this fact, the solution reduces to

$$\mathbf{x} = \Phi(t)\mathbf{x}_0. \tag{4.1.15}$$

4.2 Homogeneous Systems with Constant Coefficients

4.2.1 Relevance of eigenvalues

Let's turn our attention to **homogeneous systems with constant coefficients**, meaning that the matrix P does not depend on t. Since the matrix is now constant, we use a different sign for it, being A. A system of equations can now be expressed as $\mathbf{x}' = A\mathbf{x}$. Let's suppose every solution $\mathbf{x}(t)$ to this system of equations can be written as

$$\mathbf{x}(t) = \xi e^{rt}, \tag{4.2.1}$$

for some constant vector ξ. Using this, we can rewrite the system of equations to

$$r\xi e^{rt} = A\xi e^{rt} \qquad \Rightarrow \qquad (A - rI)\xi = \mathbf{0}. \tag{4.2.2}$$

There are nontrivial solutions (meaning $\xi \neq \mathbf{0}$) to this equation if $\det(A - rI) = 0$. This is a familiar equation in Linear Algebra. It is only true if r is an eigenvalue of A. The solution set of $\mathbf{x}' = A\mathbf{x}$ therefore depends on those eigenvalues. In the following paragraphs, a closer look is given to the eigenvalues.

4.2.2 Real and different eigenvalues

If A has n eigenvalues (with n being the size of the square matrix A) that are all real and different from each other, the solutions are relatively easy to find. Let's call the eigenvalues $r_1, \ldots, r_n$ and the corresponding eigenvectors $\xi_1, \ldots, \xi_n$. The corresponding solutions are

$$\mathbf{x}_1(t) = \xi_1 e^{r_1 t}, \ldots, \mathbf{x}_n(t) = \xi_n e^{r_n t}. \tag{4.2.3}$$

These solutions also form a fundamental solution set. This can be shown by looking at the Wronskian of the set, being

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \left|\xi_1 e^{r_1 t} \ \ldots \ \xi_n e^{r_n t}\right| = e^{(r_1 + \ldots + r_n)t} \left|\xi_1 \ \ldots \ \xi_n\right|. \tag{4.2.4}$$

The second part of this equation was derived using determinant calculation rules. Since the eigenvectors are linearly independent, the determinant of the matrix on the right side is nonzero. Also the exponential in this equation is nonzero. Therefore the Wronskian is nonzero, proving that the set of solutions $\xi_1, \ldots, \xi_n$ forms a fundamental set of solutions.

4.2.3 Complex eigenvalues

The solutions of $\det(A - rI) = 0$ are not always real. Sometimes the eigenvalues are complex. Let's assume A contains no complex numbers. In that case complex eigenvalues always come in pairs. In fact, if $r_1 = \lambda + \mu i$ is an eigenvalue, then its complex conjugate $r_2 = \lambda - \mu i$ is also an eigenvalue. The corresponding eigenvectors $\xi_1 = \mathbf{a} + \mathbf{b}i$ and $\xi_2 = \mathbf{a} - \mathbf{b}i$ are also complex conjugates.

Using these data, we can rewrite $\mathbf{x}_1 = \xi_1 e^{rt}$ to

$$\mathbf{x}_1(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)i = \mathbf{u}(t) + \mathbf{v}(t)i. \tag{4.2.5}$$

Equivalently $\mathbf{x}_2(t) = \mathbf{u}(t) - \mathbf{v}(t)i$. Now we have two solutions to the system of equations. But these solutions are complex, and we are looking for real solutions.

However, it turns out that $\mathbf{u}$ and $\mathbf{v}$ are also linearly independent solutions of the system of equations. Therefore the solutions belonging to the two complex eigenvalues r_1 and r_2 are

$$\begin{aligned} \mathbf{u}(t) &= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t), \\ \mathbf{v}(t) &= e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t). \end{aligned} \tag{4.2.6}$$

4.2.4 Repeated eigenvalues

It may occur that a matrix A with size $n \times n$ has a repeated eigenvalue r (the multiplicity of r is greater than one). If the amount of eigenvectors corresponding to r is equal to the multiplicity of r, then there is no problem. The system can be solved using the methods from paragraph 4.2.2. If, however, an eigenvalue r doesn't have enough eigenvectors, we don't get n solutions for the entire system. So more tricks are needed to find more solutions.

Let's first look at the case where 1 eigenvector is missing. Of course the eigenvalue r always has at least one corresponding eigenvector ξ. So $\mathbf{x}_1 = \xi e^{rt}$ is already a solution. A second solution will then be of the form

$$\mathbf{x}_2 = \xi t e^{rt} + \eta e^{rt}, \tag{4.2.7}$$

where η is a constant vector called the **generalized eigenvector**. It can be found using

$$(A - rI)\eta = \xi. \tag{4.2.8}$$

This equation is always inconsistent, and therefore has infinitely many solutions. One of the components of η can therefore be taken as a constant. Which component is assumed to be a constant does not effect the final solution. Once η is determined, the missing solution can be found and the system of equations can be solved.

If two eigenvectors corresponding to one eigenvalue r are missing, things get slightly more complicated. The first two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ can be found using the method described above. The third solution can be found using

$$\mathbf{x}_3 = \xi \frac{1}{2} t^2 e^{rt} + \eta t e^{rt} + \varsigma e^{rt}, \tag{4.2.9}$$

where ς is a constant vector, which can be found using

$$(A - rI)\varsigma = \eta. \tag{4.2.10}$$

If more eigenvectors are missing, this method can be expanded, analog to the method shown above.

4.3 Nonhomogeneous Linear Systems

4.3.1 Basics of nonhomogeneous systems

Let's now consider the system of differential equations given by

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t). \tag{4.3.1}$$

The general solution has the form of

$$\mathbf{x} = c_1\mathbf{x}_1(t) + \ldots + c_n\mathbf{x}_n(t) + \mathbf{v}(t). \tag{4.3.2}$$

The first part of this solution is the general solution to the homogeneous system. It is already known how to find this. The vector $\mathbf{v}(t)$ is any specific solution to the nonhomogeneous system as given by equation 4.3.1. How to find that is something we'll be looking at now.

4.3.2 Methods of finding a solution

There are several methods of solving such a system. The first method is called **Diagonalization**, which uses matrix inverses. As finding the inverse of a matrix can be a tedious process, this method is often not preferred.

A second way of finding a solution is using the Laplace transform. This especially comes in handy when $\mathbf{g}(t)$ contains a unit step function or a unit impulse function. The method itself isn't very difficult. Let's consider the system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t), \tag{4.3.3}$$

where A is some constant matrix. Taking the Laplace transform gives

$$sX(s) - \mathbf{x}(0) = AX(s) + \mathbf{G}(s), \tag{4.3.4}$$

where $X(s) = L\{\mathbf{x}\}$ is the Laplace transform of $\mathbf{x}$. This equation should then be solved for $X(s)$, which should then be transformed back to the solution $\mathbf{x}$.

The other two methods we will discuss are the **method of undetermined coefficients** and the **method of variation of parameters**.

4.3.3 Method of undetermined coefficients

If the components of the function $\mathbf{g}(t)$ are polynomial, exponential or sinusoidal functions (or sums or products of these), then the method of undetermined coefficients can be used.

First assume a general form of the specific solution $\mathbf{v}$, with several undetermined coefficients. Then insert this solution into equation 4.3.1. After this, you should try to solve for the undetermined coefficients.

If the undetermined coefficients can not be solved, then you might try a different form of a specific solution. If any part of this form is already present in the general solution of the homogeneous system, it is often worth while multiplying this part by a factor t.

4.3.4 Method of variation of parameters

In homogeneous systems, the general solution can be found using $\mathbf{x} = \Psi(t)\mathbf{c}$, where Ψ is the fundamental matrix of the system and $\mathbf{c}$ is some constant vector. For nonhomogeneous systems this is not possible. However, when using the method of variation of parameters, we assume that the general solution can be written like

$$\mathbf{x} = \Psi(t)\mathbf{u}(t), \tag{4.3.5}$$

where $\mathbf{u}(t)$ is some vector function of t. It can then be shown that

$$\Psi(t)\mathbf{u}'(t) = \mathbf{g}(t), \tag{4.3.6}$$

or equivalently,

$$\mathbf{u}(t) = \int \Psi^{-1}(t)\mathbf{g}(t)dt + \mathbf{c}, \tag{4.3.7}$$

where $\mathbf{c}$ is an arbitrary constant vector. The general solution can now be written as

$$\mathbf{x} = \Psi(t)\mathbf{u} = \Psi(t)\mathbf{c} + \Psi(t)\int \Psi^{-1}(t)\mathbf{g}(t)dt. \tag{4.3.8}$$

Note that the part $\Psi(t)\mathbf{c}$ is the solution to the homogeneous system.

The method of variation hasn't got many constraints and is therefore the most general method of solving systems of nonhomogeneous linear differential equations.

5. Phase Portraits and Stability

5.1 Phase Portraits for Linear Systems

5.1.1 Phase Portraits

Many differential equations can't be solved analytically. If we have a system, described by a differential equation, we still want to get an idea of how that system behaves. Let's consider the system

$$\mathbf{x}' = A\mathbf{x}. \tag{5.1.1}$$

If at some given $\mathbf{x}$ the value of $\mathbf{x}' = 0$, the system doesn't change. In that case the vector $\mathbf{x}$ is an **equilibrium solution**, also called a **critical point**. These points are often of special importance. However, for a consistent matrix A ($\det A = 0$) only the point $\mathbf{0}$ is a critical point. In the rest of this chapter we assume A is consistent.

Let's suppose we have found a vector function $\mathbf{x}(t)$ that satisfies equation 5.1.1. In case A is a 2×2 matrix, such a function can be viewed as a parametric representation for a curve in the x_1x_2-plane. Such a curve is called a **trajectory**, the x_1x_2-plane is called the **phase plane** and a representative set of trajectories is called a **phase portrait**.

Phase portraits can have many shapes. To get a general idea of them, we examine phase portraits of first-order linear differential equations, which we have already studied in detail. In the following paragraphs we will only be looking at a 2×2 matrix A.

5.1.2 Real unequal eigenvalues of the same sign

If the matrix A has two real unequal eigenvalues of the same sign, then the solution of system 5.1.1 is

$$\mathbf{x} = c_1\xi_1 e^{r_1 t} + c_2\xi_2 e^{r_2 t}. \tag{5.1.2}$$

If r_1 and r_2 are both negative, then as $t \to \infty$, $\mathbf{x} \to \mathbf{0}$. In this case the point $\mathbf{0}$ is called a **nodal sink**. All trajectories go to this sink.

If, however, r_1 and r_2 are both positive, then as $t \to \infty$, $\mathbf{x}$ diverges away from $\mathbf{0}$. Now the point $\mathbf{0}$ is called a **nodal source**. All trajectories go away from this source.

Another thing can be noted for these kinds of solutions. If $r_1 > r_2 > 0$ or $r_1 < r_2 < 0$, then ξ_1 has the most influence on the trajectory of $\mathbf{x}$. Therefore the trajectory will be mostly tangent to ξ_1.

5.1.3 Real eigenvalues of opposite sign

If the matrix A has two eigenvalues of opposite sign, then the solution still has the form of equation 5.1.2. However, there won't be a sink or a source, but a **saddle point**. Let's suppose $r_1 > 0 > r_2$. As $t \to \infty$ the part of the solution $\xi_2 e^{r_2 t}$ disappears and $\mathbf{x}$ will be (approximately) a multiple of ξ_1. If, however, $c_1 = 0$ (which is the case if $\mathbf{x}_0$ is a multiple of ξ_2), then the trajectory of $\mathbf{x}$ does converge to $\mathbf{0}$.

5.1.4 Equal eigenvalues with independent eigenvectors

If A has two equal eigenvalues (so an eigenvalue with multiplicity 2) with independent eigenvectors, the solution will still be of the form of equation 5.1.2. In this case $r_1 = r_2 = r$. If $r < 0$, then all trajectories will directly converge to $\mathbf{0}$ in a straight line. If $r > 0$ all trajectories will diverge away from $\mathbf{0}$ in a straight line. As the phase portrait therefore looks like a star, the point $\mathbf{0}$ is called a **star point**. It's also called a **proper node**.

5.1.5 Equal eigenvalues with a missing eigenvector

If A has only one eigenvalue with one eigenvector, then the solution will be of the form

$$\mathbf{x} = c_1\xi e^{rt} + c_2(\xi t e^{rt} + \eta e^{rt}). \tag{5.1.3}$$

This can also be written as

$$\mathbf{x} = ((c_1\xi + c_2\eta) + c_2\xi t)\, e^{rt} = \mathbf{y}e^{rt}. \tag{5.1.4}$$

Here the vector $\mathbf{y}$ largely determines the direction of the vector, while e^{rt} determines the magnitude. As $t \to \infty$ the part $c_2\xi t$ will increase, so the direction of $\mathbf{y}$ will be in the direction of ξ. It is also interesting to note that at $t = 0$ always $\mathbf{x} = c_1\xi + c_2\eta$.

The trajectories will always converge to $\mathbf{0}$ if $r < 0$ and diverge from it if $r > 0$. This critical point is in this case called an **improper** or **degenerate node**.

5.1.6 Complex eigenvalues

Let's suppose A has only complex eigenvalues $\lambda \pm \mu i$ (with $\lambda \neq 0$ and $\mu > 0$). The system is typified by

$$\mathbf{x}' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \mathbf{x}. \tag{5.1.5}$$

We can transfer this system to polar coordinates, such that $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \tan^{-1} x_2/x_1$. Solving the system will give

$$r = ce^{\lambda t} \qquad \text{and} \qquad \theta = -\mu t + \theta_0. \tag{5.1.6}$$

As t increases, the trajectory will spiral around the origin, which is thus called a **spiral point**. If $r < 0$ it will spiral inward, so then the origin is a **spiral sink**. If $r > 0$ it will spiral outward, so then the origin is a **spiral source**.

Let's now look at the same situation, except we assume that $\lambda = 0$. In this case r is constant. So the trajectories are circles, with center at the origin. The origin is therefore called a **center**.

5.1.7 Intermediate summary

Eigenvalues	**Type of Critical Point**	**Stability**
$r_1 > r_2 > 0$	Nodal Source (Node)	Unstable
$r_1 < r_2 < 0$	Nodal Sink (Node)	Asymptotically Stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$, independent eigenvectors	Proper node/Star point	Unstable
$r_1 = r_2 < 0$, independent eigenvectors	Proper node/Star point	Asymptotically Stable
$r_1 = r_2 > 0$, missing eigenvector	Improper node	Unstable
$r_1 = r_2 < 0$, missing eigenvector	Improper node	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda > 0$	Spiral point	Unstable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda < 0$	Spiral point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda = 0$	Center	Stable

Table 2: Overview of behavior of linear systems.

All that we have discussed in this part can be summarized in a table. This is done in table 2. In this table is also a column concerning stability. This topic will be discussed in the next part.

5.2 Stability

5.2.1 Autonomous systems

Previously we have looked at systems of linear first order differential equations. Linear meant that only x_1, x_2 and such appeared in the equation, and not something like x_1^2 or $\ln x_1$. First order meant that only x' and not x'' or x''' appeared.

Now let's widen our view a bit more. Let's also consider systems of nonlinear first order differential equation. But we won't consider all nonlinear systems. We only consider systems that can be written as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{5.2.1}$$

Here the function $\mathbf{f}(\mathbf{x})$ does not depend on t. So given any position vector $\mathbf{x}$, the velocity $\mathbf{x}'$ will always be the same. In other words, the phase portrait of the system is constant in time. Such a system is said to be **autonomous**. An example of such a system is the linear system $\mathbf{x}' = A\mathbf{x}'$ (with A a constant matrix).

5.2.2 Stability Definitions

A point for which $\mathbf{x}' = 0$ is called a **critical point**. Now imagine a circle with radius ϵ around a critical point $\mathbf{x}_{cr}$. Also imagine a second smaller circle with radius δ. Let's take a point $\mathbf{x}_0$ in the δ-circle. If the trajectory of that point leaves the ϵ-circle, then the critical point is called **unstable**. If, however, the trajectory of every starting point $\mathbf{x}_0$ in the δ-circle remains entirely within the ϵ-circle, the critical point is called **stable**.

If a point is stable, it can also be **asymptotically stable**. This is the case if also

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_{cr}, \tag{5.2.2}$$

meaning that the trajectory of the starting point $\mathbf{x}_0$ goes to $\mathbf{x}_{cr}$. If a trajectory forms, for example, a circle around the critical point, then it is stable but not asymptotically stable.

For asymptotically stable points, certain trajectories approach the origin. If all trajectories approach the origin, then the critical point is said to be **globally asymptotically stable**. Linear systems with $\det A = 0$ always have only 1 critical point $\mathbf{x}_{cr} = \mathbf{0}$. If $\mathbf{0}$ is then stable, it is also globally asymptotically stable.

5.2.3 Almost linear systems

Let's now consider an **isolated** critical point $\mathbf{x}_{cr}$. A critical point is isolated if there are no other critical points very close next to it. For simplicity, let's assume $\mathbf{x}_{cr} = \mathbf{0}$.

An autonomous nonlinear system can be written like

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x}). \tag{5.2.3}$$

If $\mathbf{g}(\mathbf{x})$ is small, then this system is close to the linear system $\mathbf{x}' = A\mathbf{x}$. More precisely, the system is said to be an **almost linear system** if $\mathbf{g}$ has continuous partial derivatives and

$$\frac{|\mathbf{g}(\mathbf{x})|}{|\mathbf{x}|} \to 0 \qquad \text{as} \qquad \mathbf{x} \to \mathbf{0}. \tag{5.2.4}$$

If we define $r = |\mathbf{x}|$, then this can be written in scalar form as

$$\frac{g_1(\mathbf{x})}{r} \to 0, \ldots, \frac{g_2(\mathbf{x})}{r} \to 0 \qquad \text{as} \qquad r \to 0. \tag{5.2.5}$$

It can be shown that if $\mathbf{g}(\mathbf{x})$ is twice differentiable, then the system is always an almost linear system.

Previously we have treated stability for linear systems. An overview was shown in table 2. The stability for an almost linear system is shown in table 3. It is important to note the difference. For most eigenvalues the stability and the type of critical point stay the same. There are a few exceptions.

Let's consider the case when $r_1 = \lambda + \mu i$ and $r_2 = \lambda - \mu i$ with $\lambda = 0$. If small deviations occur, it is likely that $\lambda \neq 0$. So the critical point has become a spiral point. The other difference occurs when $r_1 = r_2$. But now there are several more types to which the critical point can change.

Eigenvalues of linear system	**Type of Critical Point**	**Stability**
$r_1 > r_2 > 0$	Nodal Source (Node)	Unstable
$r_1 < r_2 < 0$	Nodal Sink (Node)	Asymptotically Stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$, independent eigenvectors	Node or Spiral Point	Unstable
$r_1 = r_2 < 0$, independent eigenvectors	Node or Spiral Point	Asymptotically Stable
$r_1 = r_2 > 0$, missing eigenvector	Node or Spiral Point	Unstable
$r_1 = r_2 < 0$, missing eigenvector	Node or Spiral Point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda > 0$	Spiral point	Unstable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda < 0$	Spiral point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda = 0$	Center or Spiral Point	Indeterminate

Table 3: Overview of behavior of almost linear systems.

5.2.4 Periodic Solutions

It may occur that autonomous systems $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ have periodic solutions. Such solutions satisfy

$$\mathbf{x}(t+T) = \mathbf{x}(t), \tag{5.2.6}$$

where $T > 0$ is called the period. This period is usually the smallest value such that the above relation is satisfied. The corresponding trajectories form closed curves. If other non-closed curves spiral towards this curve, then it is called a **limit cycle**.

If all trajectories that start near the limit cycle (both inside and outside) spiral towards it, then it is called **asymptotically stable**. If all trajectories spiral outward, then the limit cycle is called **unstable**. If trajectories on one side spiral inward and on the other side spiral outward, it is called **semistable**. It may also occur that other trajectories neither spiral to nor away from a limit cycle. In that case the limit cycle is called **stable**.

It is usually difficult to determine whether limit cycles exist in a system. However, there are a few rules that may help. A closed trajectory always encloses at least one critical point. If it encloses only one critical point, then that critical point can not be a saddle point.

We can also consider the value

$$\frac{df_1(\mathbf{x})}{dx_1} + \frac{df_2(\mathbf{x})}{dx_2} + \ldots + \frac{df_n(\mathbf{x})}{dx_n}. \tag{5.2.7}$$

If this has the same sign throughout a simply connected region D (meaning that D has no holes), then there is no closed trajectory lying entirely in D.

Suppose a region R contains no critical points. If a certain trajectory lies entirely in R, then this trajectory either is a closed trajectory or spirals towards one. In either case, there is a closed trajectory present. This last rule is called the **Poincaré-Bendixson Theorem**.

6. Fourier Series

6.1 Function types

6.1.1 Periodic functions

In the last chapter we have already mentioned periodic functions, but we will briefly repeat that here. A function f is **periodic** is

$$f(x+T) = f(x) \tag{6.1.1}$$

for every x. Here $T > 0$ is the period. The smallest value of T is called the **fundamental period** of f.

If f and g are two functions with equal period T, then their product fg and any linear combination $c_1 f + c_2 g$ also have period T.

6.1.2 Orthogonal functions

The inner product of two functions u and v on the interval $I : \alpha \leq x \leq \beta$ is defined as

$$(u, v) = \int_\alpha^\beta u(x)v(x)dx. \tag{6.1.2}$$

The function u and v are said to be **orthogonal** on I if $(u, v) = 0$. A set of functions is said to be **mutually orthogonal** if each distinct pair of functions is orthogonal.

Now consider the functions

$$u_m(x) = \cos\frac{m\pi x}{L} \qquad \text{and} \qquad v_n(x) = \sin\frac{n\pi x}{L}. \tag{6.1.3}$$

It can now be shown that $(u_m, v_n) = 0$ for every m, n on an interval $-L \leq x \leq L$. Also, if $m \neq n$, then $(u_m, u_n) = (v_m, v_n) = 0$ on the same interval. On the contrary, if $m = n$, then $(u_m, u_n) = (v_m, v_n) = L$ (also on $-L \leq x \leq L$).

6.1.3 Even and odd functions

A function f is said to be an **even function** if

$$f(-x) = f(x) \tag{6.1.4}$$

for every x. An example is $f(x) = x^2$. Let's take a graph of a function and mirror it along the y-axis. If we get back the same graph is we put in, then it is an even function.

A function is even if Similarly, a function f is said to be an **odd function** is

$$f(-x) = -f(x) \tag{6.1.5}$$

for every x. So odd functions always have $f(0) = 0$. An example is $f(x) = x$ or $f(x) = x^3$. Let's take a graph of a function and rotate it 180° about the origin. If we get back the same graph as we put in, then it is an odd function.

If f and g are even functions and p and q are odd functions, then

- $c_1 f + c_2 g$ and fg are even.
- $c_1 p + c_2 q$ is odd. However pq is even.
- fp is odd. $f + p$ is neither even nor odd.
- $\int_{-L}^{L} f(x)dx = 2\int_{-L}^{0} f(x)dx = 2\int_{0}^{L} f(x)dx$.
- $\int_{-L}^{L} p(x)dx = 0$.

6.1.4 Eigenfunctions

The difference between initial value problems and boundary value problems was previously discussed. Initial value problems concerned differential equations where y and y' were given at a certain point, while boundary problems have y given at two different points. While there usually is a unique solution to initial value problems, there is often not a unique solution to boundary problems. For boundary value problems here are either 0, 1 or infinitely many solutions.

Let's take a look at a boundary value problem concerning a homogeneous differential equations with a certain unknown constant. For example, let's consider

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0. \tag{6.1.6}$$

Here λ is an unknown constant. The above differential equation has solution $y = 0$ for all λ. This is the **trivial solution** in which we are not interested.

Instead, it turns out that for some values of λ there are infinitely many solutions. These values of λ for which nontrivial solutions occur are called **eigenvalues**. The nontrivial solutions are called **eigenfunctions**. For the above example, the eigenvalues turn out to be

$$\lambda_1 = 1, \qquad \lambda_2 = 4, \qquad \lambda_3 = 9, \qquad \ldots, \qquad \lambda_n = n^2, \tag{6.1.7}$$

where the corresponding eigenfunctions are

$$y_1(x) = \sin x, \qquad y_2(x) = \sin 2x, \qquad y_3(x) = \sin 3x, \qquad \ldots, \qquad y_(x) = \sin nx. \tag{6.1.8}$$

Just like in linear algebra, any linear combination of solutions (eigenfunctions) is also a solution to the differential equation.

6.2 Fourier Series

6.2.1 Introduction to Fourier series

Let's suppose we have a continuous periodic function f with period $T = 2L$. In that case, it can be expressed as a **Fourier series**, being an infinite sum of sines and cosines that converges to $f(x)$. This goes according to

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{6.2.1}$$

Here the coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ need to be determined. It can be shown that

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \qquad \text{and} \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx. \tag{6.2.2}$$

If f is not a periodic function, it can not entirely be expressed as a Fourier series. However, the part of f on interval $-L \leq x \leq L$ can be expressed as a Fourier series, according to the above procedure.

6.2.2 Discontinuous functions

If the periodic function f is not a continuous function but a piecewise continuous function, it is still possible to express the function using a Fourier series. However, at positions of discontinuity (where the graph makes a "jump") the Fourier series never really converges to $f(x)$. This behavior is known as the **Gibbs phenomenon**.

Another interesting phenomenon always occurs. If the value of f at a certain point x jumps from y_1 to y_2, then the Fourier series at point x always returns a value of $\frac{y_1+y_2}{2}$.

For functions that are not even piecewise continuous (such as for example $\frac{1}{x}$), Fourier series often do not converge. Therefore Fourier series are hardly ever applicable for such functions.

6.2.3 Sine and cosine series

Let's suppose we have an even function f and want to find the corresponding Fourier series. When we are trying to find the b-coefficients, we will be integrating over $f(x)\sin\frac{n\pi x}{L}$. Since $\sin(x)$ is an odd function, this product is also odd. We know that an integral from $-L$ to L over an odd function will give 0 as result. Therefore $b_n = 0$ for every n.

Since b_n is always zero in the Fourier series of even functions, all the terms with sines disappear. Such a series thus only consists of cosines and is therefore called a **Fourier cosine series**.

Now let's suppose f is odd. If we make use of the fact that $\cos(x)$ is an even function, we will find that $a_n = 0$ for every n. Therefore the Fourier series for an odd function consists only of sines. and is thus called a **Fourier sine series**.

6.3 Heat Conduction Problems

6.3.1 Heat conduction in a rod

Let's consider a thin rod, ranging from $x = 0$ to $x = L$ of which the sides are insulated. Heat can only enter via the two edges. The temperature u is now only a function of x and t. To solve this problem, we need to use the **heat conduction equation**

$$\alpha^2 u_{xx} = u_t, \tag{6.3.1}$$

where α^2 is the **thermal diffusivity** (a material property). There are several boundary values for this problem. First there is the initial state of the rod $u(x, 0)$. This is simply equal to some known function $f(x)$, so

$$u(x, 0) = f(x). \tag{6.3.2}$$

6.3.2 Rod with open ends at $u = 0$

If heat can pass in/out of the rod at the edges, then the edges will always have constant temperature. For simplicity's sake we will assume that this temperature is 0 for both edges. Later we will consider the case in which this is not true. So the other boundary conditions are

$$u(0, t) = 0, \qquad u(L, t) = 0. \tag{6.3.3}$$

This differential equation is hard to solve. So to solve it, we make an assumption. We assume that the function $u(x, t)$ can be written as

$$u(x, t) = X(x)T(t). \tag{6.3.4}$$

So we assume it is a product of a function of x and a function of t. Using this assumption we can separate the problem in two differential equations

$$X'' + \lambda X = 0, \qquad T' + \alpha^2 \lambda T = 0, \tag{6.3.5}$$

where λ is an unknown separation constant. Now let's look at the first equation and combine it with the second boundary equation. Ignoring the trivial solution $X = 0$, we will find that the solutions are the eigenfunctions

$$X_n(x) = \sin \frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots, \tag{6.3.6}$$

associated with the eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$. Inserting these values in the second differential equation gives

$$T_n(x) = e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}}. \tag{6.3.7}$$

We can now find a solution $u_n = X_n T_n$. The general solution is then any linear combination of the specific solutions, so

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n X_n(t) T_n(t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin \frac{n\pi x}{L}. \tag{6.3.8}$$

But we haven't satisfied the first boundary conditions yet. Using $u(x,0) = f(x)$ we can find the coefficients c_n. The procedure for this is identical to finding a sine series for $f(x)$. From this follows that

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx. \tag{6.3.9}$$

6.3.3 Rod with open ends not at $u = 0$

But what if the edges don't have $u = 0$? Let's suppose $u(0,t) = T_1$ and $u(L,t) = T_2$. Now the problem is not homogeneous anymore. So we will first make it homogeneous. We can see that

$$g(x) = T_1 + (T_2 - T_1)\frac{x}{L} \tag{6.3.10}$$

is a solution. In fact, this is the limit solution as $t \to \infty$. If we now not use the initial condition $u(x,0) = f(x)$, but instead use $u(x,0) = f(x) - g(x)$, then we once more have a homogeneous problem. Then the coefficients can be found using

$$c_n = \frac{2}{L}\int_{0}^{L} (f(x) - g(x)) \sin \frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{L} \left(f(x) - T_1 - (T_2 - T_1)\frac{x}{L}\right) \sin \frac{n\pi x}{L} dx. \tag{6.3.11}$$

Note that this is equal to what we saw last chapter, except we replaced $f(x)$ by $f(x) - g(x)$. The corresponding solution then becomes

$$u(x,t) = g(x) + \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin \frac{n\pi x}{L} = T_1 + (T_2 - T_1)\frac{x}{L} + \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin \frac{n\pi x}{L}. \tag{6.3.12}$$

This is also equal to the solution of the last paragraph, except that we put the part $g(x)$ in front of it.

6.3.4 Rod with insulated ends

What happens if the ends of the rod are insulated? In that case they are no longer a constant temperature. Instead, in that case $X'(0) = 0$ and $X'(L) = 0$. The solution process is more or less similar to that of a rod without insulated ends. But instead of finding a sine series, the result now is a cosine series, given by

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{\frac{-n^2\pi^2\alpha^2 t}{L^2}} \cos \frac{n\pi x}{L}. \tag{6.3.13}$$

The coefficients are given by the equation

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{6.3.14}$$

A funny thing to note is that as $t \to \infty$, the temperature in the entire bar becomes equal to $c_0/2$. It can be shown that this just happens to be the average temperature of the bar.

6.4 The Wave Equation

6.4.1 Vibrations of an elastic string

Let's examine an elastic string, connected at $x = 0$ and $x = L$. Every point x at a time t has a deflection $u(x, t)$. If the string is given an initial deflection, it will vibrate. If damping effects are neglected, the governing equation is

$$a^2 u_{xx} = u_{tt}, \tag{6.4.1}$$

where a^2 is a constant. This equation is called the **wave equation**. One of the boundary conditions of this problem is rather trivial. As the ends of the string are fixed, we know that

$$u(0, t) = 0, \qquad u(L, t) = 0. \tag{6.4.2}$$

To solve the problem, we also need to know the initial position $u(x, 0)$. But this won't suffice to solve the problem. Also the initial velocity $u_t(x, 0)$ needs to be known. These boundary conditions can be expressed as

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x). \tag{6.4.3}$$

6.4.2 String with initial displacement

Suppose the string has been given an initial displacement, but no initial velocity. So $u_t(x, 0) = 0$. This implies that $T'(0) = 0$. Solving the wave equation is very similar to solving the heat conduction equation. The solution for $X(x)$ will be exactly the same. The solution for $T(t)$ will be

$$T_n(t) = \cos\frac{n\pi at}{L}. \tag{6.4.4}$$

The final solution will then have the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}. \tag{6.4.5}$$

The constants c_n can be found using

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{6.4.6}$$

For a fixed value of n the expression $\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$ is periodic with period $T = \frac{2L}{na}$ or equivalently having the frequency $\frac{na}{2L}$. This frequency is called the **natural frequency** of the string - being the frequency at which it will freely vibrate.

While vibrating, certain displacement patterns appear. Each displacement pattern is called a **natural mode of vibration** and is periodic in space. The corresponding spacial period $\frac{2L}{n}$ is called the **wavelength** of the mode.

6.4.3 String with initial velocity

Now let's examine a string without initial displacement, but with initial velocity. So this time $u(x,0) = 0$, implying that $T(0) = 0$. Now we will find that

$$T_n(t) = \sin \frac{n\pi at}{L}. \tag{6.4.7}$$

Working out the results will give

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L}. \tag{6.4.8}$$

To find the coefficients c_n, we have to do a little bit more work than previously, as now we haven't been given an initial value but an initial velocity. Differentiating $u(x,t)$ and solving for c_n using Fourier series will give

$$c_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} dx. \tag{6.4.9}$$

6.4.4 String with both initial position and initial velocity

The two cases above have a disadvantage. To use the first case, we have to have a string of which all points have no velocity at $t = 0$. For the second all points on the string have to have zero displacement at that moment. This doesn't always occur.

If $f(x) \neq 0$ and $g(x) \neq 0$ it is possible to solve the problem using separation of variables, as we have previously performed. This is difficult though, and there is an easier way to solve the problem.

Let $v(x,t)$ be the solution for the vibrating string with no initial velocity ($g(x) = 0$). Also let $w(x,t)$ be the solution for the string with no initial displacement ($f(x) = 0$). If we add the two solutions up, we get

$$u(x,t) = v(x,t) + w(x,t). \tag{6.4.10}$$

It can now be shown that this solution satisfies all the boundary conditions. So if you have a string with both initial displacement and initial velocity, simply split the problem up and then add up the results.

6.5 Problem Variations

6.5.1 Expanding to multiple dimensions

The heat conduction problem and the wave problem described in the previous parts are only one-dimensional. They can be made two-dimensional or three-dimensional rather easily. We can replace the term u_{xx} by $u_{xx} + u_{yy}$ for a two-dimensional case or $u_{xx} + u_{yy} + u_{zz}$ for a three-dimensional case. This would make the heat conduction equation

$$\alpha^2 \left(u_{xx} + u_{yy} + u_{zz}\right) = u_t. \tag{6.5.1}$$

The wave equation would then be

$$a^2 \left(u_{xx} + u_{yy} + u_{zz}\right) = u_{tt}. \tag{6.5.2}$$

6.5.2 Steady-State problems

In for example heat conduction problems, the variable u usually converges to a constant value in time. But to what value does it occur? It stops changing if $u_t = 0$ or equivalently

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{6.5.3}$$

This equation is called **Laplace's Equation** for three dimensions. But can we solve it for a three-dimensional problem? What do we need to know before we can solve it?

In a one-dimensional problem we needed to know either the value of u or u_t at the edges of the rod. This can be expanded to three dimensions. To solve Laplace's equation in three dimensions, we need to know the value of u or u_t along the entire boundary of the three-dimensional space.

If u is given, the problem is slightly different than if u_t is given. It therefore also has a different name. If u is known along the edges, then the problem is called a **Dirichlet problem**. However, if we have been given u_t, then the problem is called a **Neumann problem**.

Both types of problems can be solved using the techniques demonstrated in this chapter. However, the equations for the solution and the corresponding coefficients need to be derived once more. As there are very many types of these problems, it is not possible to give the solution for every single type.

7. Sturm-Liouville Problems

7.1 Homogeneous Problems

7.1.1 Sturm-Liouville problems

In this chapter, we will be examining differential equations of the form

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0, \tag{7.1.1}$$

where $p(x)$, $q(x)$ and $r(x)$ are given functions. y is a function of x and y' denotes the derivative with respect to x. Let's define the differential operator L to be

$$L[y] = -(p(x)y')' + q(x)y. \tag{7.1.2}$$

We can now rewrite the differential equation to

$$L[y] = \lambda r(x)y. \tag{7.1.3}$$

By using $L = 1$ we can also rewrite the boundary conditions to

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \qquad \beta_1 y(1) + \beta_2 y'(1) = 0. \tag{7.1.4}$$

Such types of problems are called **Sturm-Liouville problems**.

7.1.2 Lagrange's identity

Lagrange's identity is

$$\int_0^1 (L[u]v - uL[v])\,dx = \left[-p(x)\left(u'(x)v(x) - u(x)v'(x)\right)\right]_0^1. \tag{7.1.5}$$

By using the boundary conditions of equation 7.1.4 we can show that the right side of this identity is 0, and thus also

$$\int_0^1 (L[u]v - uL[v])\,dx = 0. \tag{7.1.6}$$

Using the inner product, defined in the previous chapter, we can also write this as $(L[u], v) - (u, L[v]) = 0$.

7.1.3 Sturm-Liouville problem properties

Several things are known about Sturm-Liouville problems. It can be shown that all eigenvalues λ (for which there are nontrivial solutions to the problem) are real. In fact, if we find two eigenvalues λ_1 and λ_2 (with $\lambda_1 \neq \lambda_2$) and corresponding eigenfunctions ϕ_1 and ϕ_2, then

$$\int_0^1 r(x)\phi_1(x)\phi_2(x)dx = 0. \tag{7.1.7}$$

Also all eigenvalues are **simple**, meaning that each eigenvalue has only one eigenfunction (if you don't consider multiples of that eigenfunction). Furthermore, the eigenvalues can be ordered according to increasing magnitude, such that $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, where $\lambda_n \to \infty$ as $n \to \infty$.

7.1.4 Orthogonality

Equation 7.1.7 expresses the property of **orthogonality** of the eigenfunctions with respect to the weight function $r(x)$. The eigenfunctions are said to form an **orthogonal set** with respect to $r(x)$.

Every eigenvalue has one corresponding eigenfunction. However, every multiple of this eigenfunction is actually also an eigenfunction. So we can choose our eigenfunctions such that

$$\int_0^1 r(x)\phi_n^2(x)dx = 1. \tag{7.1.8}$$

Eigenfunctions satisfying this condition are said to be **normalized**. Normalized eigenfunctions are said to form an **orthonormal set** with respect to $r(x)$.

7.1.5 Expressing a function as a sum of eigenfunctions

Suppose we have found all the normalized eigenfunctions ϕ_n of a Sturm-Liouville problem. Can we now express a given function $f(x)$ as a sum of these eigenfunctions? If so, then a solution $f(x)$ could be written as

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x). \tag{7.1.9}$$

The only trick is to find the coefficients. To find any coefficient c_m, we can multiply the above equation by $r(x)\phi_m(x)$ and then integrate from 0 to 1, like

$$\int_0^1 = r(x)\phi_m(x)f(x)dx = \sum_{n=1}^{\infty} c_n \int_0^1 r(x)\phi_m(x)\phi_n(x)dx = c_m, \tag{7.1.10}$$

where we have used equation 7.1.7 in the last step. It follows that

$$c_m = \int_0^1 r(x)\phi_m(x)f(x)dx = (f(x), r(x)\phi_m(x)). \tag{7.1.11}$$

7.2 Nonhomogeneous Problems

7.2.1 Nonhomogeneous Sturm-Liouville problems

We have spend enough time on homogeneous problems. Now let's turn our attention to the nonhomogeneous problems. These problems have the form

$$L[y] = -\left(p(x)y'\right)' + q(x)y = \mu r(x)y + f(x), \tag{7.2.1}$$

where μ is a given constant. Note the extra term $f(x)$. Let the boundary conditions once more be

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \qquad \beta_1 y(1) + \beta_2 y'(1) = 0. \tag{7.2.2}$$

To solve this problem, we first look at the homogeneous problem $L[y] = \lambda r(x)y$ with eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenfunction $\phi_1, \phi_2, \ldots$. We will assume that the solution $y = \phi(x)$ can be written as

$$\phi(x) = \sum_{n=1}^{\infty} b_n\phi_n(x). \tag{7.2.3}$$

However, this time we can not find the coefficients b_n in the way we are used to. Instead, we can find them with a small detour. First define the coefficients c_n as

$$c_n = \int_0^1 f(x)\phi_n(x)dx. \tag{7.2.4}$$

The coefficients b_n can then be found using

$$b_n = \frac{c_n}{\lambda_n - \mu}. \tag{7.2.5}$$

If $\lambda_n \neq \mu$ for all n, then the solution will simply be equal to

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu}\phi_n(x). \tag{7.2.6}$$

However, if $\lambda_n = \mu$ for some n, then there is a problem. If $c_n \neq 0$ (for the same n), then we are dividing by zero. It can then be shown that the nonhomogeneous problem simply doesn't have a solution. However, if also $c_n = 0$, then b_n remains arbitrary. In this case there are infinitely many solutions to the nonhomogeneous problem.

7.2.2 Nonhomogeneous heat conduction problems

The generalized heat conduction equation is given by

$$r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t), \tag{7.2.7}$$

with two boundary conditions and one initial condition, being

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(1,t) + h_2 u(1,t) = 0, \qquad \text{and} \qquad u(x,0) = f(x). \tag{7.2.8}$$

We assume any solution will have the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x), \tag{7.2.9}$$

where $\phi_n(x)$ are the eigenfunctions of the problem. To find the coefficients b_n we need to do several steps. First we need to find two intermediate coefficients B_n and $\gamma_n(t)$, given by

$$B_n = \int_0^1 r(x)f(x)\phi_n(x)dx, \tag{7.2.10}$$

$$\gamma_n(t) = \int_0^1 F(x,t)\phi_n(x)dx. \tag{7.2.11}$$

Now the coefficient b_n can be calculated using

$$b_n(t) = B_n + e^{-\lambda_n t}\int_0^t e^{\lambda_n s}\gamma_n(s)ds. \tag{7.2.12}$$

All the necessary coefficients are now known. The solution can be found by using the sum

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x). \tag{7.2.13}$$