Section 8.3

1. The ODE is linear, with $f(t, y) = 3 + t - y$. The Runge-Kutta algorithm requires the evaluations

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}\right) \\ k_{n3} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}\right) \\ k_{n4} &= f(t_n + h\,, y_n + hk_{n3}). \end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a). For $h = 0.1$:

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	1.19516	1.38127	1.55918	1.72968

(b). For $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.19516	1.38127	1.55918	1.72968

The exact solution of the IVP is $y(t) = 2 + t - e^{-t}$.

2. In this problem, $f(t, y) = 5t - 3\sqrt{y}$. At each time step, the Runge-Kutta algorithm requires the evaluations

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}\right) \\ k_{n3} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}\right) \\ k_{n4} &= f(t_n + h\,, y_n + hk_{n3}). \end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a). For $h = 0.1$:

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	1.62231	1.33362	1.12686	0.993839

(b). For $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.62230	1.33362	1.12685	0.993826

The exact solution of the IVP is given *implicitly* as

$$\frac{1}{\left(2\sqrt{y}+5t\right)^5\left(t-\sqrt{y}\right)^2} = \frac{\sqrt{2}}{512}\,.$$

3. The ODE is linear, with $f(t,y) = 2y - 3t$. The Runge-Kutta algorithm requires the evaluations

$$\begin{aligned}
k_{n1} &= f(t_n, y_n) \\
k_{n2} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}\right) \\
k_{n3} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}\right) \\
k_{n4} &= f(t_n + h\,, y_n + hk_{n3}).
\end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a). For $h = 0.1$:

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	1.20535	1.42295	1.65553	1.90638

(b). For $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.20535	1.42296	1.65553	1.90638

The exact solution of the IVP is $y(t) = e^{2t}/4 + 3t/2 + 3/4$.

5. In this problem, $f(t,y) = (y^2 + 2ty)/(3 + t^2)$. The Runge-Kutta algorithm

requires the evaluations

$$
\begin{aligned}
k_{n1} &= f(t_n, y_n) \\
k_{n2} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}\right) \\
k_{n3} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}\right) \\
k_{n4} &= f(t_n + h\,, y_n + hk_{n3}).
\end{aligned}
$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a). For $h = 0.1$:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
t_n	0.1	0.2	0.3	0.4
y_n	0.510170	0.524138	0.542105	0.564286

(b). For $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	0.520169	0.524138	0.542105	0.564286

The exact solution of the IVP is $y(t) = (3 + t^2)/(6 - t)$.

6. In this problem, $f(t, y) = (t^2 - y^2) sin\, y$. At each time step, the Runge-Kutta algorithm requires the evaluations

$$
\begin{aligned}
k_{n1} &= f(t_n, y_n) \\
k_{n2} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}\right) \\
k_{n3} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}\right) \\
k_{n4} &= f(t_n + h\,, y_n + hk_{n3}).
\end{aligned}
$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

(a). For $h = 0.1$:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924517	-0.864125	-0.816377	-0.779706

(b). For $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924517	-0.864125	-0.816377	-0.779706

7. (a). For $h = 0.1$:

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
t_n	0.5	1.0	1.5	2.0
y_n	2.96825	7.88889	20.8349	55.5957

(b). For $h = 0.05$:

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	2.96828	7.88904	20.8355	55.5980

The exact solution of the IVP is $y(t) = e^{2t} + t/2$.

8. See Prob. 2 . for the *exact* solution.

(a). For $h = 0.1$:

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
t_n	0.5	1.0	1.5	2.0
y_n	0.925725	1.28516	2.40860	4.10350

(b). For $h = 0.05$:

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.925711	1.28515	2.40860	4.10350

9(a). For $h = 0.1$:

	$n=5$	$n=10$	$n=15$	$n=20$
t_n	0.5	1.0	1.5	2.0
y_n	3.96219	5.10890	6.43139	7.92338

(b). For $h = 0.05$:

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	3.96219	5.10890	6.43139	7.92338

The exact solution is given *implicitly* as

$$ln\left[\frac{2}{y+t-1}\right] + 2\sqrt{t+y} - 2\,arctanh\,\sqrt{t+y} = t + 2\sqrt{3} - 2\,arctanh\,\sqrt{3}\,.$$

10. See Prob. 4 .

(a). For $h = 0.1$:

	$n=5$	$n=10$	$n=15$	$n=20$
t_n	0.5	1.0	1.5	2.0
y_n	1.61262	2.48091	3.74548	5.49587

(b). For $h = 0.05$:

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	1.61262	2.48091	3.74548	5.49587

12. See Prob. 5 . for the *exact* solution.

(a). For $h = 0.1$:

	$n=5$	$n=10$	$n=15$	$n=20$
t_n	0.5	1.0	1.5	2.0
y_n	0.590909	0.800000	1.166667	1.75000

(b). For $h = 0.05$:

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	0.590909	0.800000	1.166667	1.75000

13. The ODE is linear, with $f(t, y) = 1 - t + 4y$. The Runge-Kutta algorithm requires

the evaluations

$$\begin{aligned}
k_{n1} &= f(t_n, y_n) \\
k_{n2} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n1}\right) \\
k_{n3} &= f\left(t_n + \frac{1}{2}h\,, y_n + \frac{1}{2}hk_{n2}\right) \\
k_{n4} &= f(t_n + h\,, y_n + hk_{n3}).
\end{aligned}$$

The next estimate is given as the weighted average

$$y_{n+1} = y_n + \frac{h}{6}(k_{n1} + 2\,k_{n2} + 2\,k_{n3} + k_{n4}).$$

The exact solution of the IVP is $y(t) = \frac{19}{16}e^{4t} + \frac{1}{4}t - \frac{3}{16}$.

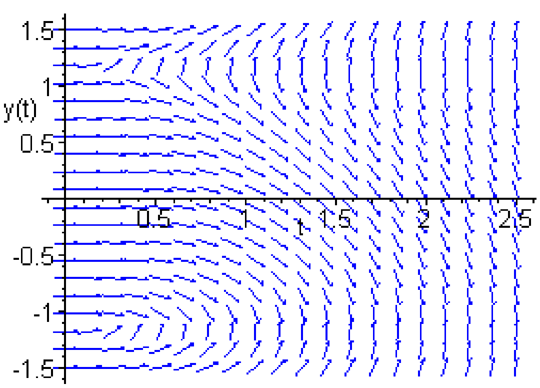
(a). For $h = 0.1$:

	$n=5$	$n=10$	$n=15$	$n=20$
t_n	0.5	1.0	1.5	2.0
y_n	8.7093175	64.858107	478.81928	3535.8667

(b). For $h = 0.05$:

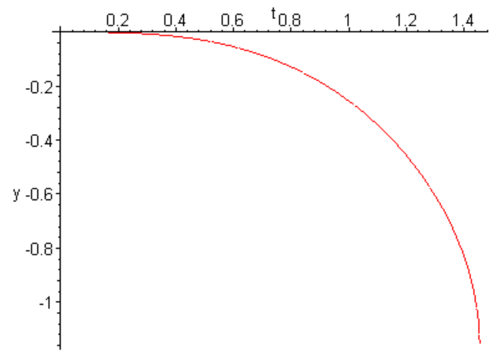
	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	8.7118060	64.894875	479.22674	3539.8804

$15(a)$.



(b). For the integral curve starting at $(0,0)$, the slope becomes *infinite* near $t_M \approx 1.5$. Note that the exact solution of the IVP is defined implicitly as

$$y^3 - 4y = t^3.$$



Using the classic Runge-Kutta algorithm, with $h = 0.01$, we obtain the values

	$n = 70$	$n = 80$	$n = 90$	$n = 95$
t_n	0.7	0.8	0.9	0.95
y_n	-0.08591	-0.12853	-0.18380	-0.21689

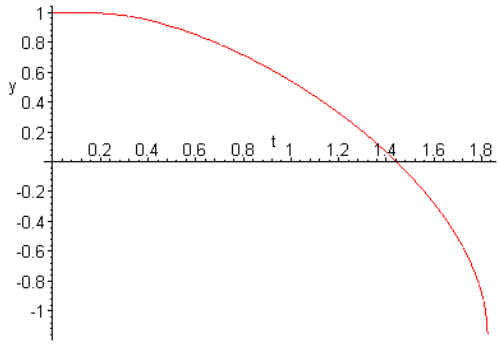
(c). Based on the direction field, the solution should *decrease* monotonically to the limiting value $y = -2/\sqrt{3}$. In the following table, the value of t_M corresponds to the approximate time in the iteration process that the calculated values begin to *increase*.

h	t_M
0.1	1.9
0.05	1.65
0.025	1.55
0.01	1.455

(d). Numerical values will continue to be generated, although they will *not* be associated with the integral curve starting at $(0,0)$. These values are approximations to nearby integral curves.

(e). We consider the solution associated with the initial condition $y(0) = 1$. The exact solution is given by

$$y^3 - 4y = t^3 - 3.$$



For the integral curve starting at $(0\,,1)$, the slope becomes *infinite* near $t_M \approx 2.0$. In the following table, the values of t_M corresponds to the approximate time in the iteration process that the calculated values begin to *increase*.

h	t_M
0.1	1.85
0.05	1.85
0.025	1.86
0.01	1.835