Section 3.9

2. We have $sin(\alpha \pm \beta) = sin\,\alpha\, cos\,\beta \pm cos\,\alpha\, sin\,\beta$. Subtracting the two identities, we obtain $sin(\alpha+\beta) - sin(\alpha-\beta) = 2\,cos\,\alpha\, sin\,\beta$. Setting $\alpha+\beta = 7t$ and $\alpha-\beta = 6t$, $\alpha = 6.5t$ and $\beta = 0.5t$. Hence $sin\,7t - sin\,6t = 2\,sin\,\frac{t}{2}\,cos\,\frac{13t}{2}$.

3. Consider the trigonometric identity $cos(\alpha \pm \beta) = cos\,\alpha\, cos\,\beta \mp sin\,\alpha\, sin\,\beta$. Adding the two identities, we obtain $cos(\alpha-\beta) + cos(\alpha+\beta) = 2\,cos\,\alpha\, cos\,\beta$. Comparing the expressions, set $\alpha+\beta = 2\pi t$ and $\alpha-\beta = \pi t$. Hence $\alpha = 3\pi t/2$ and $\beta = \pi t/2$. Upon substitution, we have $cos(\pi t) + cos(2\pi t) = 2\,cos(3\pi t/2)\,cos(\pi t/2)$.

4. Adding the two identities $sin(\alpha \pm \beta) = sin\,\alpha\, cos\,\beta \pm cos\,\alpha\, sin\,\beta$, it follows that $sin(\alpha-\beta) + sin(\alpha+\beta) = 2sin\,\alpha\, cos\,\beta$. Setting $\alpha+\beta = 4t$ and $\alpha-\beta = 3t$, we have $\alpha = 7t/2$ and $\beta = t/2$. Hence $sin\,3t + sin\,4t = 2\,sin(7t/2)\,cos(t/2)$.

6. Using *mks* units, the spring constant is $k = 5(9.8)/0.1 = 490$ *N/m*, and the damping coefficient is $\gamma = 2/0.04 = 50$ *N-sec/m*. The equation of motion is

$$5u'' + 50u' + 490u = 10\,sin(t/2)\,.$$

The initial conditions are $u(0) = 0$ *m* and $u'(0) = 0.03$ *m/s*.

8(a). The homogeneous solution is $u_c(t) = Ae^{-5t}cos\sqrt{73}\,t + Be^{-5t}sin\sqrt{73}\,t$. Based on the method of *undetermined coefficients*, the particular solution is

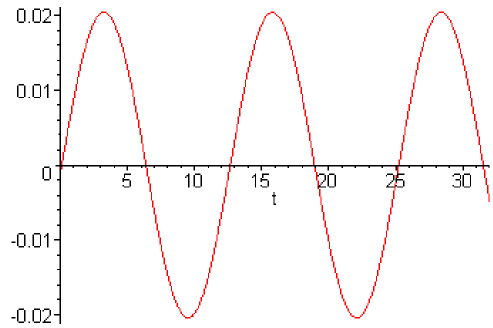
$$U(t) = \frac{1}{153281}[-160\,cos(t/2) + 3128\,sin(t/2)].$$

Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 160/153281$ and $B = 383443\sqrt{73}/1118951300$. Hence the response is

$$u(t) = \frac{1}{153281}\left[160\,e^{-5t}cos\,\sqrt{73}\,t + \frac{383443\sqrt{73}}{7300}e^{-5t}sin\,\sqrt{73}\,t\right] + U(t).$$

(b). $u_c(t)$ is the transient part and $U(t)$ is the steady state part of the response.

(c).



(d). Based on Eqs. (9) and (10), the amplitude of the forced response is given by $R = 2/\Delta$, in which

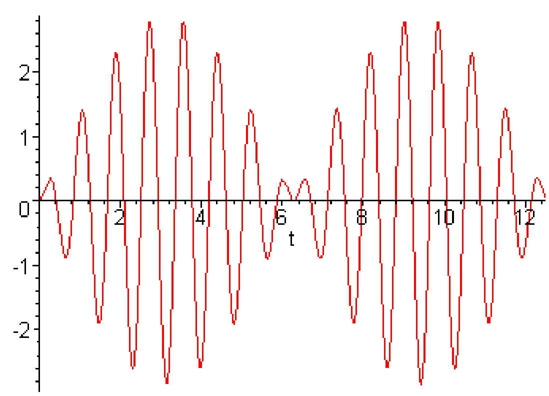
$$\Delta = \sqrt{25(98-\omega^2)^2 + 2500\,\omega^2}\,.$$

The maximum amplitude is attained when Δ is a *minimum*. Hence the amplitude is maximum at $\omega = 4\sqrt{3}$ *rad/s*.

9. The spring constant is $k = 12$ *lb/ft* and hence the equation of motion is

$$\frac{6}{32}u'' + 12u = 4\,cos\,7t\,,$$

that is, $u'' + 64u = \frac{64}{3}cos\,7t$. The initial conditions are $u(0) = 0$ *ft*, $u'(0) = 0$ *fps*. The general solution is $u(t) = Acos\,8t + Bsin\,8t + \frac{64}{45}cos\,7t$. Invoking the initial conditions, we have $u(t) = -\frac{64}{45}cos\,8t + \frac{64}{45}cos\,7t = \frac{128}{45}sin(t/2)sin(15t/2)$.



12. The equation of motion is

$$2u'' + u' + 3u = 3cos\,3t - 2sin\,3t\,.$$

Since the system is *damped*, the steady state response is equal to the particular solution. Using the method of *undetermined coefficients*, we obtain

$$u_{ss}(t) = \frac{1}{6}(sin\,3t - cos\,3t).$$

Further, we find that $R = \sqrt{2}/6$ and $\delta = arctan(-1) = 3\pi/4$. Hence we can write $u_{ss}(t) = \frac{\sqrt{2}}{6}cos(3t - 3\pi/4)$.

13. The amplitude of the steady-state response is given by

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2}}\,.$$

Since F_0 is constant, the amplitude is *maximum* when the denominator of R is *minimum*. Let $z = \omega^2$, and consider the function $f(z) = m^2(\omega_0^2 - z)^2 + \gamma^2 z$. Note that $f(z)$ is a quadratic, with *minimum* at $z = \omega_0^2 - \gamma^2/2m^2$. Hence the amplitude R attains a maximum at $\omega_{max}^2 = \omega_0^2 - \gamma^2/2m^2$. Furthermore, since $\omega_0^2 = k/m$, and therefore

$$\omega_{max}^2 = \omega_0^2\left[1 - \frac{\gamma^2}{2km}\right].$$

Substituting $\omega^2 = \omega_{max}^2$ into the expression for the amplitude,

$$\begin{aligned} R &= \frac{F_0}{\sqrt{\gamma^4/4m^2 + \gamma^2\,(\omega_0^2 - \gamma^2/2m^2)}} \\ &= \frac{F_0}{\sqrt{\omega_0^2\gamma^2 - \gamma^4/4m^2}} \\ &= \frac{F_0}{\gamma\omega_0\sqrt{1 - \gamma^2/4mk}}\,. \end{aligned}$$

14(a). The forced response is $u_{ss}(t) = Acos\,\omega t + Bsin\,\omega t$. The constants are obtain by the method of *undetermined coefficients*. That is, comparing the coefficients of $cos\,\omega t$ and $sin\,\omega t$, we find that

$$-m\omega^2 A + \gamma\omega B + kA = F_0\,, \text{ and } -m\omega^2 B - \gamma\omega A + kB = 0\,.$$

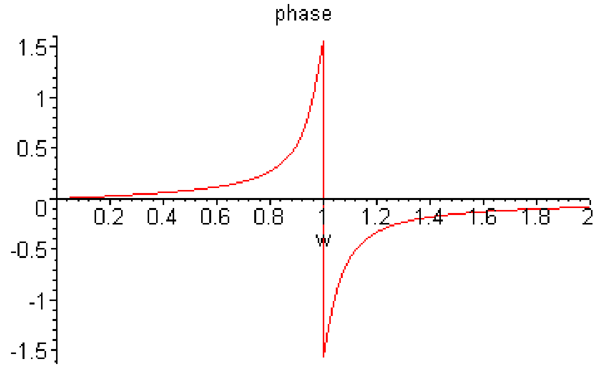
Solving this system results in

$$A = m\big(\omega_0^2 - \omega^2\big)/\Delta \quad \text{and} \quad B = \gamma\omega/\Delta\,,$$

in which $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2}$. It follows that

$$tan\,\delta = B/A = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}\,.$$

(b). Here $m = 1$, $\gamma = 0.125$, $\omega_0 = 1$. Hence $tan\,\delta = 0.125\omega/(1 - \omega^2)$.

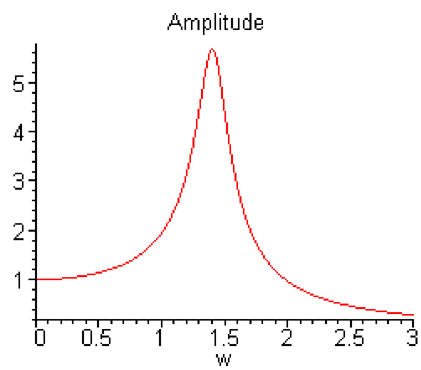


$17(a)$. Here $m = 1$, $\gamma = 0.25$, $\omega_0^2 = 2$, $F_0 = 2$. Hence $u_{ss}(t) = \frac{2}{\Delta}cos(\omega t - \delta)$, where $\Delta = \sqrt{(2-\omega^2)^2 + \omega^2/16} = \frac{1}{4}\sqrt{64 - 63\omega^2 + 16\,\omega^4}$, and $tan\,\delta = \frac{\omega}{4(2-\omega^2)}$.

(b). The amplitude is

$$R = \frac{8}{\sqrt{64 - 63\omega^2 + 16\,\omega^4}}\,.$$

(c).



(d). *See Prob. 13*. The amplitude is maximum when the denominator of R is minimum. That is, when $\omega = \omega_{max} = 3\sqrt{14}/8 \approx 1.4031$. Hence $R(\omega = \omega_{max}) = 64/\sqrt{127}$.

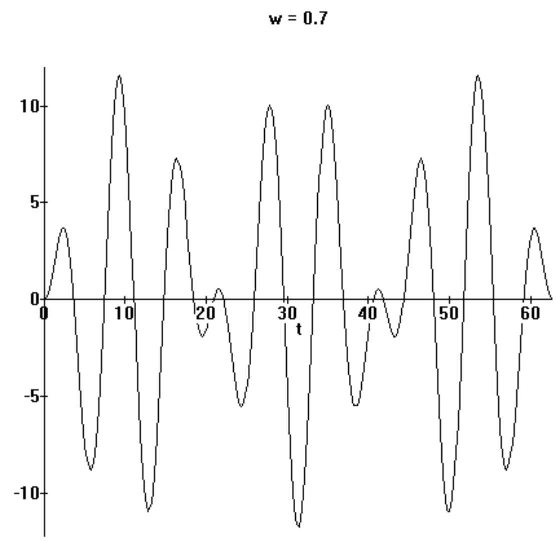
$18(a)$. The homogeneous solution is $u_c(t) = Acos\,t + Bsin\,t$. Based on the method of *undetermined coefficients*, the particular solution is

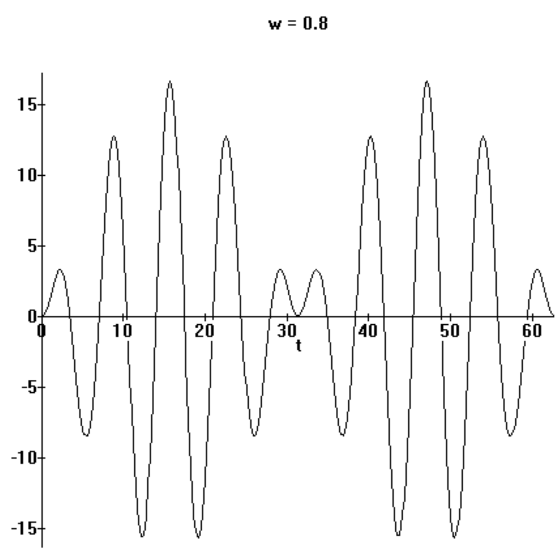
$$U(t) = \frac{3}{1-\omega^2}\,cos\,\omega t\,.$$

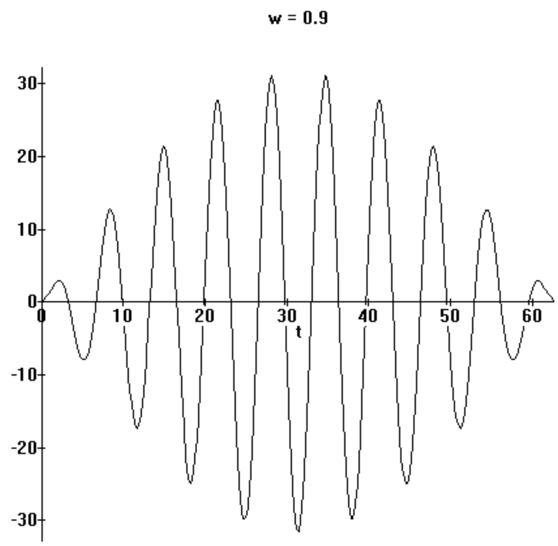
Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 3/(\omega^2 - 1)$ and $B = 0$. Hence the response is

$$u(t) = \frac{3}{1-\omega^2}[\,cos\,\omega t - cos\,t\,].$$

(b).







Note that

$$u(t) = \frac{6}{1-\omega^2}\, sin\left[\frac{(1-\omega)t}{2}\right] sin\left[\frac{(\omega+1)t}{2}\right].$$

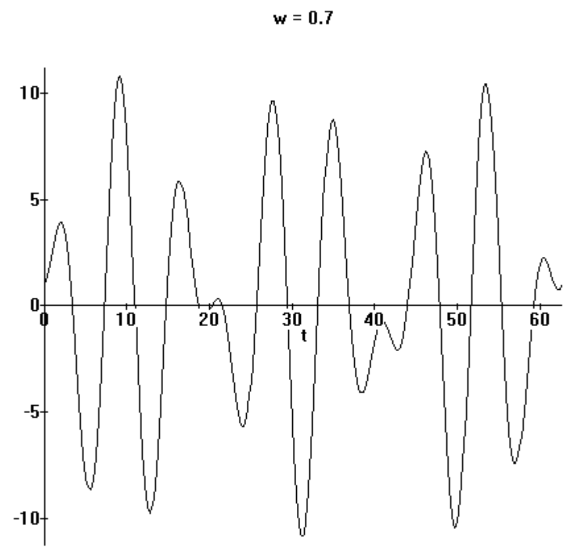
19(a). The homogeneous solution is $u_c(t) = Acos\, t + Bsin\, t$. Based on the method of *undetermined coefficients*, the particular solution is

$$U(t) = \frac{3}{1-\omega^2}\, cos\, \omega t\,.$$

Hence the general solution is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = (\omega^2+2)/(\omega^2-1)$ and $B = 1$. Hence the response is

$$u(t) = \frac{1}{1-\omega^2}\left[\,3\, cos\, \omega t - \left(\omega^2+2\right) cos\, t\,\right] + sin\, t\,.$$

$(b.)$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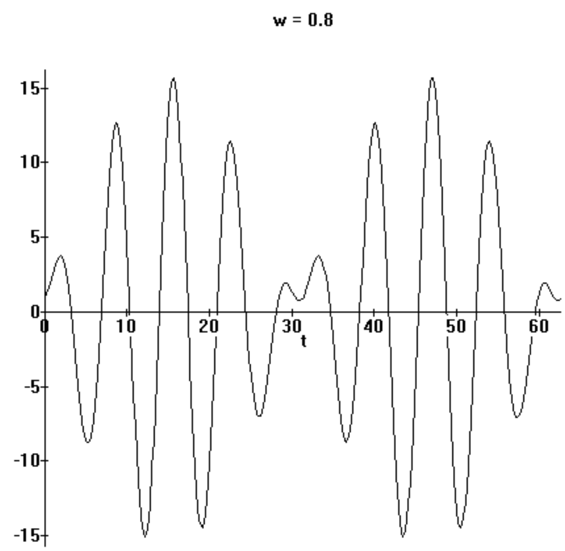
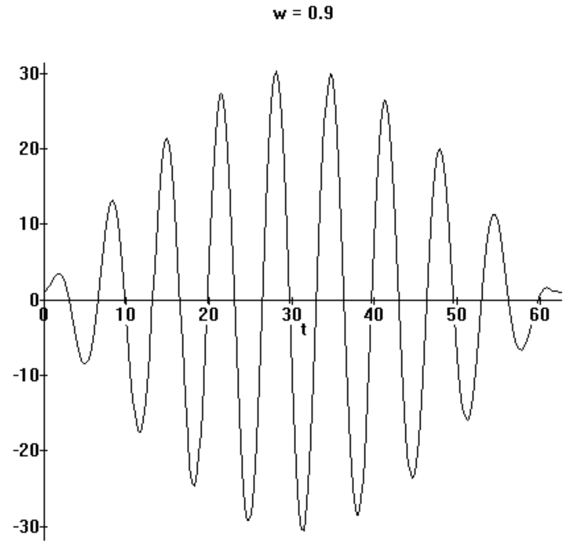
10
5
0
-5
-10
0
10
20
30
40
50
60
t

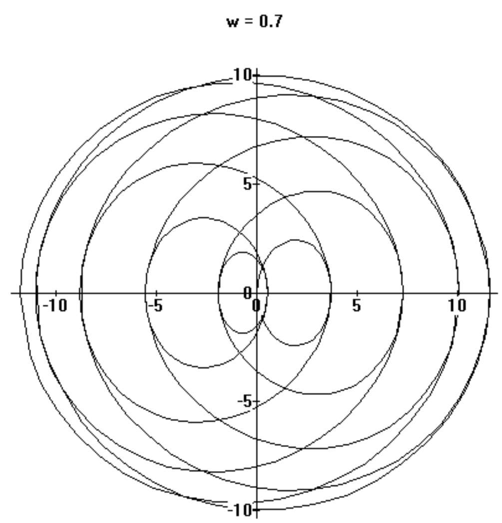

w = 0.8
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60
t

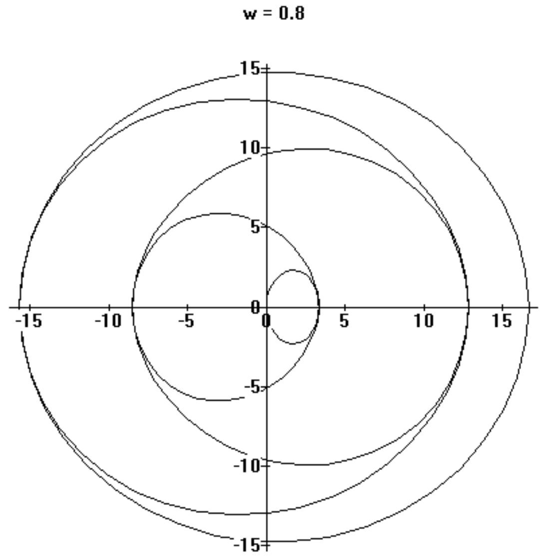


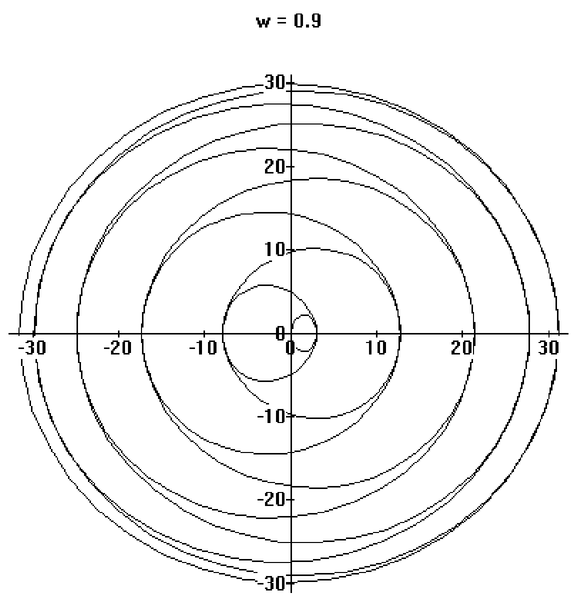
Note that

$$u(t) = \frac{6}{1-\omega^2}\,sin\left[\frac{(1-\omega)t}{2}\right]sin\left[\frac{(\omega+1)t}{2}\right] + cos\,t + sin\,t\,.$$

20.







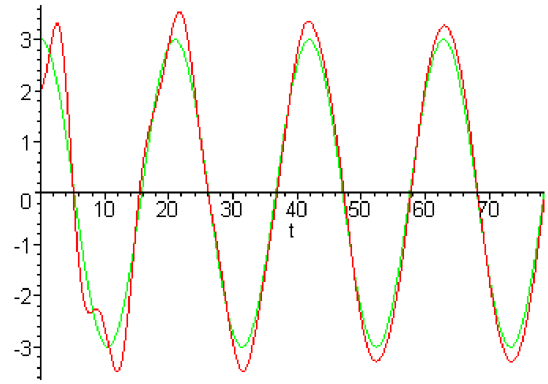
21. The general solution is $u(t) = u_c(t) + U(t)$, in which

$$u_c(t) = e^{-t/16}\left[-\frac{171358}{132721}\,cos\,\frac{\sqrt{255}}{16}t - \frac{257758}{132721\sqrt{255}}\,sin\,\frac{\sqrt{255}}{16}t\right]$$

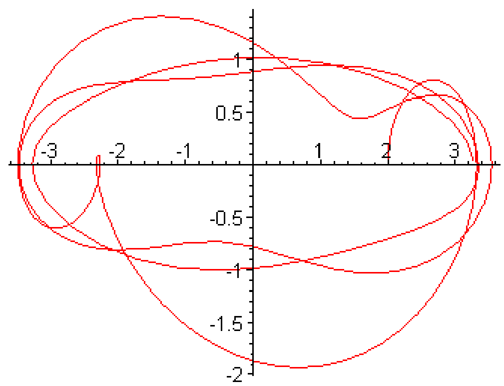
and

$$U(t) = \frac{1}{132721}[436800\,cos(.3t) + 18000\,sin(.3t)]\,.$$

(a).



(b).



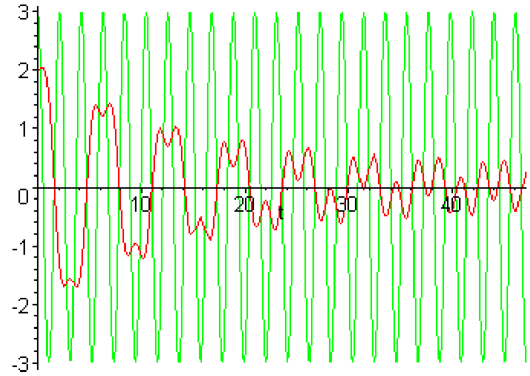
23. The general solution is $u(t) = u_c(t) + U(t)$, in which

$$u_c(t) = e^{-t/16}\left[\frac{9746}{4105}\,cos\,\frac{\sqrt{255}}{16}t + \frac{1258}{821\sqrt{255}}\,sin\,\frac{\sqrt{255}}{16}t\right]$$

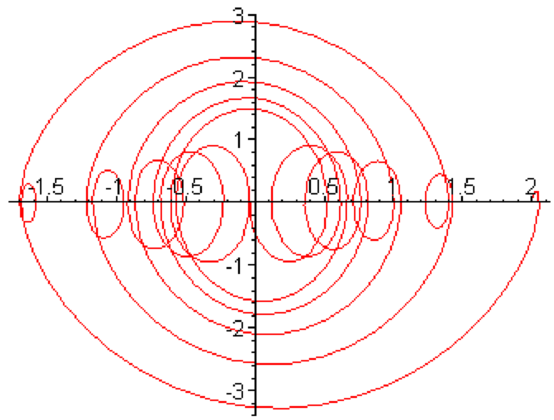
and

$$U(t) = \frac{1}{4105}[-1536\,cos(3t) + 72\,sin(3t)]\,.$$

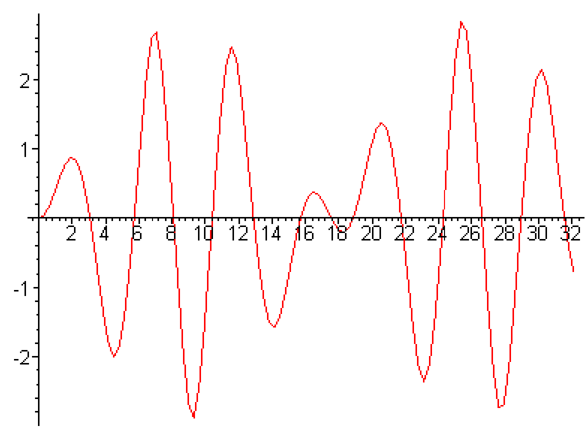
(a).


3
2
1
0
-1
-2
-3
10
20
t
30
40

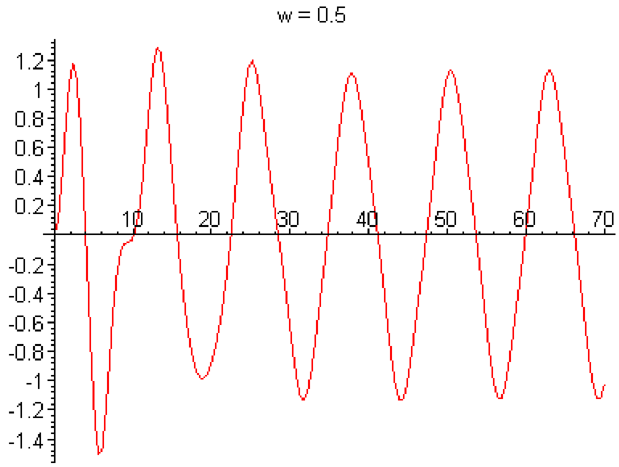
(b).

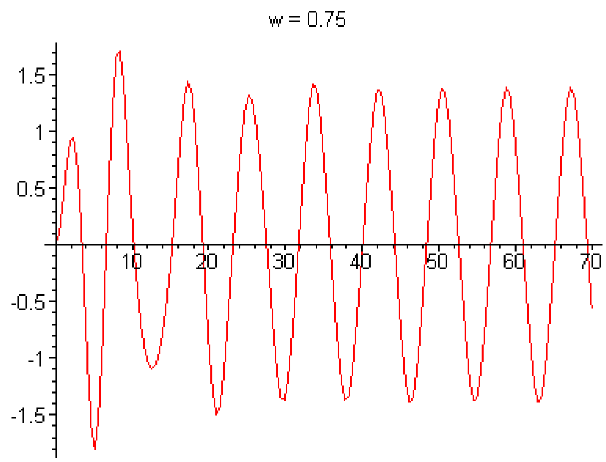

3
2
1
-1
-2
-3
-1.5
-1
-0.5
0.5
1
1.5
2

24.


2
1
-1
-2
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32

$25(a)$.


w = 0.5
1.2
1
0.8
0.6
0.4
0.2
10
20
30
40
50
60
70
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4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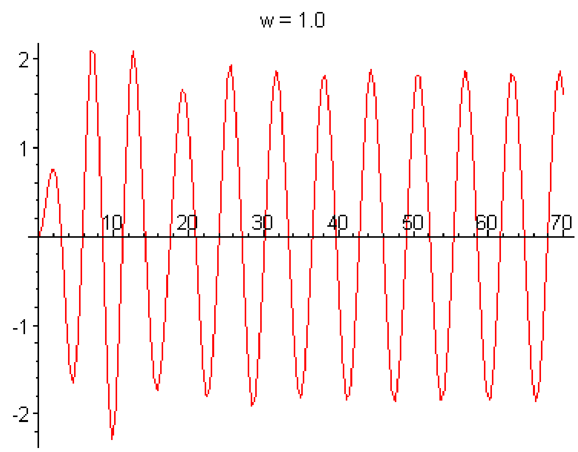
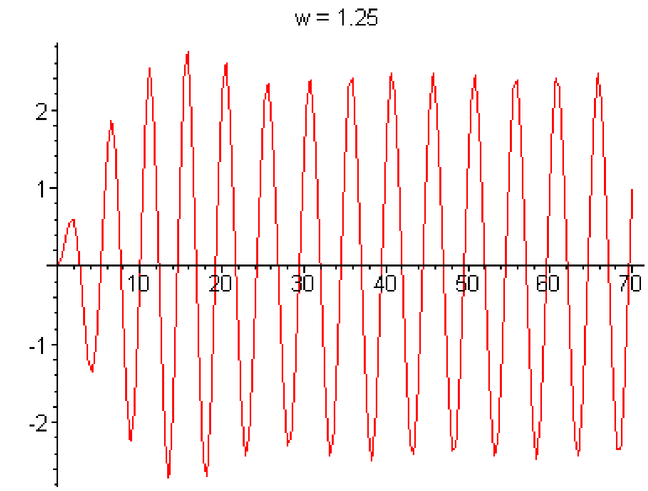
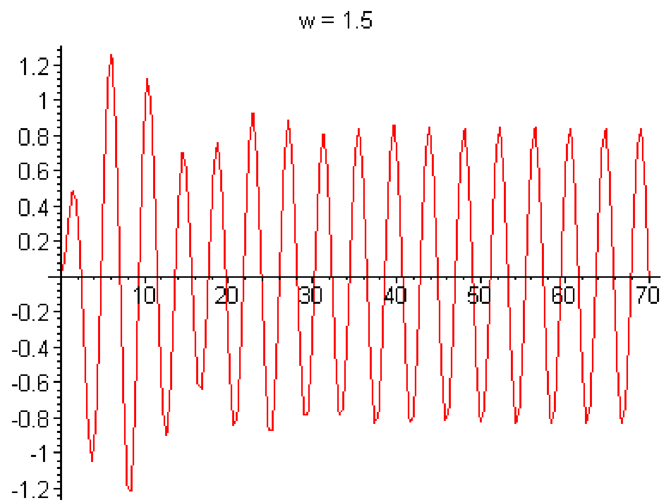
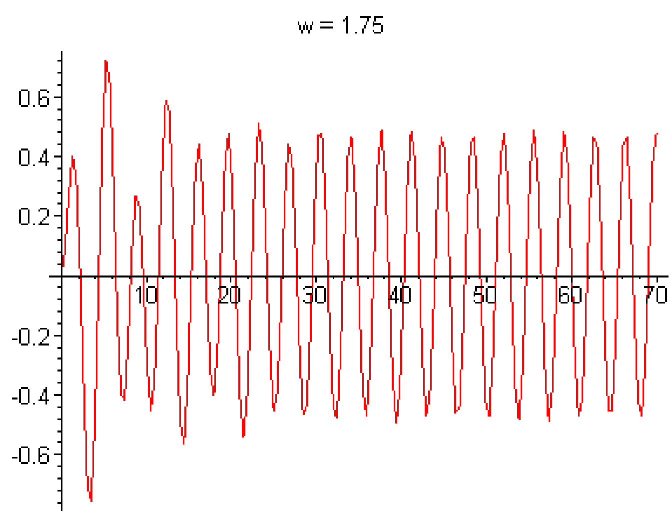
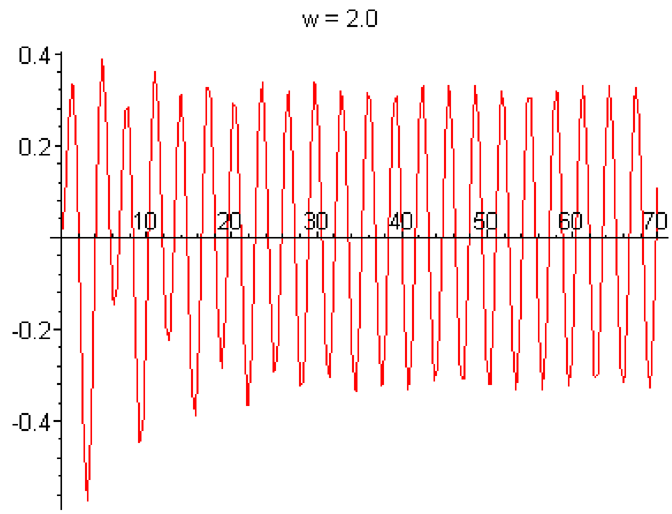
0.5
10
20
30
40
50
60
70
-0.5
-1
-1.5


w = 1.0
2
1
10
20
30
40
50
60
70
-1
-2

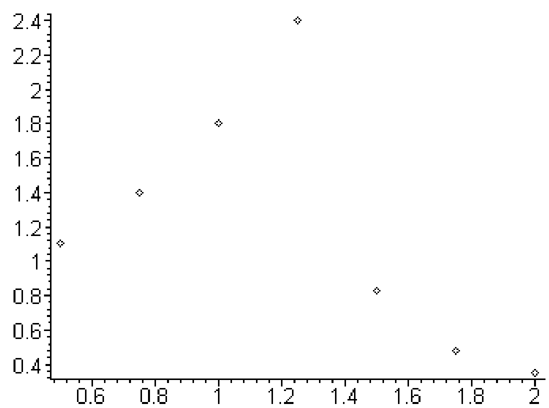

w = 1.25
2
1
-1
-2
10
20
30
40
50
60
70


w = 1.5
1.2
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1
-1.2
10
20
30
40
50
60
70


w = 1.75
0.6
0.4
0.2
-0.2
-0.4
-0.6
10
20
30
40
50
60
70

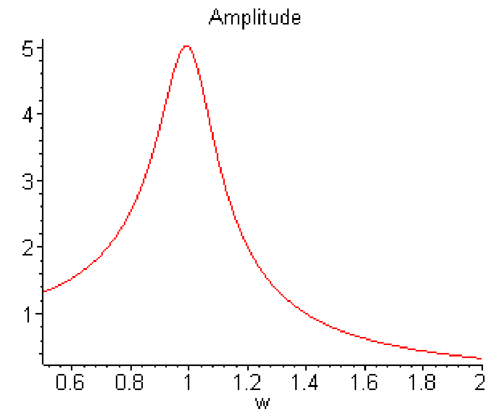


(b).



(c). The amplitude for a similar system with a *linear* spring is given by

$$R = \frac{5}{\sqrt{25 - 49\omega^2 + 25\omega^4}}\,.$$


Amplitude
5
4
3
2
1
0.6
0.8
1
1.2
1.4
1.6
1.8
2
w