Section 1.3

1. The differential equation is *second order*, since the highest derivative in the equation is of order *two*. The equation is *linear*, since the left hand side is a linear function of y and
its derivatives.

3. The differential equation is *fourth order*, since the highest derivative of the function y is of order *four*. The equation is also *linear*, since the terms containing the dependent variable is linear in y and its derivatives.

4. The differential equation is *first order*, since the only derivative is of order *one*. The dependent variable is *squared*, hence the equation is *nonlinear*.

5. The differential equation is *second order*. Furthermore, the equation is *nonlinear*, since the dependent variable y is an argument of the *sine function*, which is *not* a linear function.

7. $y_1(t) = e^t \Rightarrow y_1'(t) = y_1''(t) = e^t$. Hence $y_1'' - y_1 = 0$.
Also, $y_2(t) = cosh\, t \Rightarrow y_1'(t) = sinh\, t$ and $y_2''(t) = cosh\, t$. Thus $y_2'' - y_2 = 0$.

9. $y(t) = 3t + t^2 \Rightarrow y'(t) = 3 + 2t$. Substituting into the differential equation, we have $t(3 + 2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2$. Hence the given function is a solution.

10. $y_1(t) = t/3 \Rightarrow y_1'(t) = 1/3$ and $y_1''(t) = y_1'''(t) = y_1''''(t) = 0$. Clearly, $y_1(t)$ is a solution. Likewise, $y_2(t) = e^{-t} + t/3 \Rightarrow y_2'(t) = -e^{-t} + 1/3$, $y_2''(t) = e^{-t}$, $y_2'''(t) = -e^{-t}$, $y_2''''(t) = e^{-t}$. Substituting into the left hand side of the equation, we find that $e^{-t} + 4(-e^{-t}) + 3(e^{-t} + t/3) = e^{-t} - 4e^{-t} + 3e^{-t} + t = t$. Hence both functions are solutions of the differential equation.

11. $y_1(t) = t^{1/2} \Rightarrow y_1'(t) = t^{-1/2}/2$ and $y_1''(t) = -t^{-3/2}/4$. Substituting into the left hand side of the equation, we have

$$\begin{aligned} 2t^2\big(-t^{-3/2}/4\big) + 3t\big(t^{-1/2}/2\big) - t^{1/2} &= -t^{1/2}/2 + 3\,t^{1/2}/2 - t^{1/2} \\ &= 0 \end{aligned}$$

Likewise, $y_2(t) = t^{-1} \Rightarrow y_2'(t) = -t^{-2}$ and $y_2''(t) = 2\,t^{-3}$. Substituting into the left hand side of the differential equation, we have $2t^2(2\,t^{-3}) + 3t(-t^{-2}) - t^{-1} = 4\,t^{-1} -$ $- 3\,t^{-1} - t^{-1} = 0$. Hence both functions are solutions of the differential equation.

12. $y_1(t) = t^{-2} \Rightarrow y_1'(t) = -2t^{-3}$ and $y_1''(t) = 6\,t^{-4}$. Substituting into the left hand side of the differential equation, we have $t^2(6\,t^{-4}) + 5t(-2t^{-3}) + 4\,t^{-2} = 6\,t^{-2} -$ $- 10\,t^{-2} + 4\,t^{-2} = 0$. Likewise, $y_2(t) = t^{-2}ln\, t \Rightarrow y_2'(t) = t^{-3} - 2t^{-3}ln\, t$ and $y_2''(t) = -5\,t^{-4} + 6\,t^{-4}ln\, t$. Substituting into the left hand side of the equation, we have $t^2(-5\,t^{-4} + 6\,t^{-4}ln\, t) + 5t(t^{-3} - 2t^{-3}ln\, t) + 4(t^{-2}ln\, t) = -5\,t^{-2} + 6\,t^{-2}ln\, t +$

$+\,5\,t^{-2} - 10\,t^{-2}ln\,t + 4\,t^{-2}ln\,t = 0$. Hence both functions are solutions of the differential equation.

13. $y(t) = (cos\,t)ln\,cos\,t + t\,sin\,t \Rightarrow y'(t) = -(sin\,t)ln\,cos\,t + t\,cos\,t$ and $y''(t) = -(cos\,t)ln\,cos\,t - t\,sin\,t + sec\,t$. Substituting into the left hand side of the differential equation, we have $(-(cos\,t)ln\,cos\,t - t\,sin\,t + sec\,t) + (cos\,t)ln\,cos\,t +$ $+\,t\,sin\,t = -(cos\,t)ln\,cos\,t - t\,sin\,t + sec\,t + (cos\,t)ln\,cos\,t + t\,sin\,t = sec\,t$. Hence the function $y(t)$ is a solution of the differential equation.

15. Let $y(t) = e^{rt}$. Then $y''(t) = r^2e^{rt}$, and substitution into the differential equation results in $r^2e^{rt} + 2\,e^{rt} = 0$. Since $e^{rt} \neq 0$, we obtain the algebraic equation $r^2 + 2 = 0$. The roots of this equation are $r_{1,2} = \pm\, i\sqrt{2}$.

17. $y(t) = e^{rt} \Rightarrow y'(t) = r\,e^{rt}$ and $y''(t) = r^2e^{rt}$. Substituting into the differential equation, we have $r^2e^{rt} + re^{rt} - 6\,e^{rt} = 0$. Since $e^{rt} \neq 0$, we obtain the algebraic equation $r^2 + r - 6 = 0$, that is, $(r-2)(r+3) = 0$. The roots are $r_{1,2} = -3\,, 2$.

18. Let $y(t) = e^{rt}$. Then $y'(t) = re^{rt}$, $y''(t) = r^2e^{rt}$ and $y'''(t) = r^3e^{rt}$. Substituting the derivatives into the differential equation, we have $r^3e^{rt} - 3r^2e^{rt} + 2re^{rt} = 0$. Since $e^{rt} \neq 0$, we obtain the algebraic equation $r^3 - 3r^2 + 2r = 0$. By inspection, it follows that $r(r-1)(r-2) = 0$. Clearly, the roots are $r_1 = 0$, $r_2 = 1$ and $r_3 = 2$.

20. $y(t) = t^r \Rightarrow y'(t) = r\,t^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Substituting the derivatives into the differential equation, we have $t^2[r(r-1)t^{r-2}] - 4t(r\,t^{r-1}) + 4\,t^r = 0$. After some algebra, it follows that $r(r-1)t^r - 4r\,t^r + 4\,t^r = 0$. For $t \neq 0$, we obtain the algebraic equation $r^2 - 5r + 4 = 0$. The roots of this equation are $r_1 = 1$ and $r_2 = 4$.

21. The order of the partial differential equation is *two*, since the highest derivative, in fact each one of the derivatives, is of *second order*. The equation is *linear*, since the left hand side is a linear function of the partial derivatives.

23. The partial differential equation is *fourth order*, since the highest derivative, and in fact each of the derivatives, is of order *four*. The equation is *linear*, since the left hand side is a linear function of the partial derivatives.

24. The partial differential equation is *second order*, since the highest derivative of the function $u(x,y)$ is of order *two*. The equation is *nonlinear*, due to the product $u \cdot u_x$ on the left hand side of the equation.

25. $u_1(x,y) = cos\,x\,cosh\,y \Rightarrow \frac{\partial^2 u_1}{\partial x^2} = -\,cos\,x\,cosh\,y$ and $\frac{\partial^2 u_1}{\partial y^2} = cos\,x\,cosh\,y$. It is evident that $\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0$. Likewise, given $u_2(x,y) = ln(x^2+y^2)$, the second derivatives are

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$$
$$\frac{\partial^2 u_2}{\partial y^2} = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}$$

Adding the partial derivatives,

$$\begin{aligned}\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} &= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \\ &= \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} \\ &= 0\,.\end{aligned}$$

Hence $u_2(x,y)$ is also a solution of the differential equation.

27. Let $u_1(x,t) = sin\,\lambda x\, sin\,\lambda at$. Then the second derivatives are

$$\frac{\partial^2 u_1}{\partial x^2} = -\lambda^2 sin\,\lambda x\, sin\,\lambda at$$
$$\frac{\partial^2 u_1}{\partial t^2} = -\lambda^2 a^2 sin\,\lambda x\, sin\,\lambda at$$

It is easy to see that $a^2 \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2}$. Likewise, given $u_2(x,t) = sin(x-at)$, we have

$$\frac{\partial^2 u_2}{\partial x^2} = -\,sin(x-at)$$
$$\frac{\partial^2 u_2}{\partial t^2} = -\,a^2 sin(x-at)$$

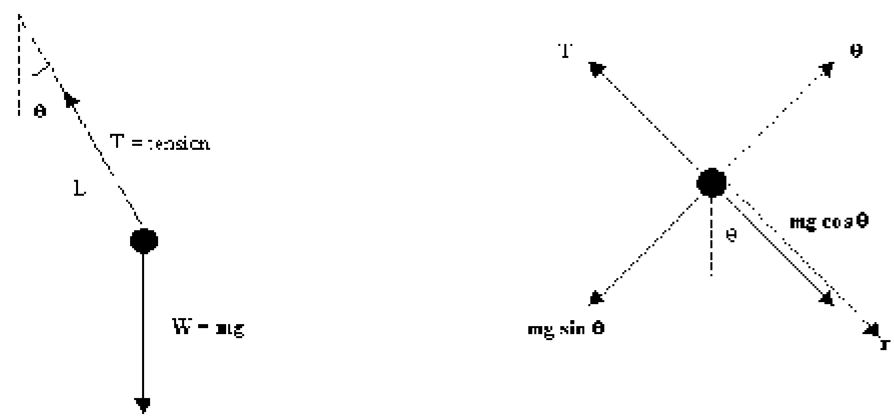
Clearly, $u_2(x,t)$ is also a solution of the partial differential equation.

28. Given the function $u(x,t) = \sqrt{\pi/t}\; e^{-x^2/4\alpha^2 t}$, the partial derivatives are

$$u_{xx} = -\frac{\sqrt{\pi/t}\; e^{-x^2/4\alpha^2 t}}{2\alpha^2 t} + \frac{\sqrt{\pi/t}\; x^2 e^{-x^2/4\alpha^2 t}}{4\alpha^4 t^2}$$
$$u_t = -\frac{\sqrt{\pi t}\; e^{-x^2/4\alpha^2 t}}{2t^2} + \frac{\sqrt{\pi}\; x^2 e^{-x^2/4\alpha^2 t}}{4\alpha^2 t^2 \sqrt{t}}$$

It follows that $\alpha^2\, u_{xx} = u_t = -\frac{\sqrt{\pi}\,(2\alpha^2 t - x^2) e^{-x^2/4\alpha^2 t}}{4\alpha^2 t^2 \sqrt{t}}$.
Hence $u(x,t)$ is a solution of the partial differential equation.

$29(a)$.



(b). The path of the particle is a circle, therefore *polar coordinates* are intrinsic to the problem. The variable r is radial distance and the angle θ is measured from the vertical. Newton's Second Law states that $\sum \mathbf{F} = m\mathbf{a}$. In the *tangential* direction, the equation of motion may be expressed as $\sum F_\theta = m\, a_\theta$, in which the *tangential acceleration*, that is, the linear acceleration *along* the path is $a_\theta = L\, d^2\theta/dt^2$. (a_θ is *positive* in the direction of increasing θ). Since the only force acting in the tangential direction is the component of weight, the equation of motion is

$$-\, mg\, sin\, \theta = mL\frac{d^2\theta}{dt^2}\, .$$

(Note that the equation of motion in the radial direction will include the tension in the rod).

(c). Rearranging the terms results in the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\, \theta = 0\, .$$