**Section 6.2**

1. Write the function as

$$\frac{3}{s^2+4}=\frac{3}{2}\,\frac{2}{s^2+4}\,.$$

Hence $\mathcal{L}^{-1}[Y(s)]=\frac{3}{2}\,sin\,2t\,.$

3. Using *partial fractions*,

$$\frac{2}{s^2+3s-4}=\frac{2}{5}\left[\frac{1}{s-1}-\frac{1}{s+4}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)]=\frac{2}{5}\left(e^t-e^{-4t}\right).$

5. Note that the denominator $s^2+2s+5$ is *irreducible* over the reals. Completing the square, $s^2+2s+5=(s+1)^2+4$. Now convert the function to a *rational function* of the variable $\xi=s+1$. That is,

$$\frac{2s+2}{s^2+2s+5}=\frac{2(s+1)}{(s+1)^2+4}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4}\right]=2\,cos\,2t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)]=\mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right]=2e^{-t}cos\,2t\,.$$

6. Using *partial fractions*,

$$\frac{2s-3}{s^2-4}=\frac{1}{4}\left[\frac{1}{s-2}+\frac{7}{s+2}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)]=\frac{1}{4}(e^{2t}+7e^{-2t})$. Note that we can also write

$$\frac{2s-3}{s^2-4}=2\,\frac{s}{s^2-4}-\frac{3}{2}\,\frac{2}{s^2-4}.$$

8. Using *partial fractions*,

$$\frac{8s^2-4s+12}{s(s^2+4)}=3\,\frac{1}{s}+5\,\frac{s}{s^2+4}-2\,\frac{2}{s^2+4}.$$

Hence $\mathcal{L}^{-1}[Y(s)] = 3 + 5\,cos\,2t - 2\,sin\,2t$ .

9. The denominator $s^2 + 4s + 5$ is *irreducible* over the reals. Completing the square, $s^2 + 4s + 5 = (s+2)^2 + 1$ . Now convert the function to a *rational function* of the variable $\xi = s + 2$ . That is,

$$\frac{1 - 2s}{s^2 + 4s + 5} = \frac{5 - 2(s+2)}{(s+2)^2 + 1} \, .$$

We find that

$$\mathcal{L}^{-1}\left[\frac{5}{\xi^2 + 1} - \frac{2\,\xi}{\xi^2 + 1}\right] = 5\,sin\,t - 2\,cos\,t \, .$$

Using the fact that $\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$ ,

$$\mathcal{L}^{-1}\left[\frac{1 - 2s}{s^2 + 4s + 5}\right] = e^{-2t}(5\,sin\,t - 2\,cos\,t) \, .$$

10. Note that the denominator $s^2 + 2s + 10$ is *irreducible* over the reals. Completing the square, $s^2 + 2s + 10 = (s+1)^2 + 9$ . Now convert the function to a *rational function* of the variable $\xi = s + 1$ . That is,

$$\frac{2s - 3}{s^2 + 2s + 10} = \frac{2(s+1) - 5}{(s+1)^2 + 9} \, .$$

We find that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2 + 9} - \frac{5}{\xi^2 + 9}\right] = 2\,cos\,3t - \frac{5}{3}\,sin\,3t \, .$$

Using the fact that $\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$ ,

$$\mathcal{L}^{-1}\left[\frac{2s - 3}{s^2 + 2s + 10}\right] = e^{-t}\left(2\,cos\,3t - \frac{5}{3}\,sin\,3t\right) .$$

12. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2\,Y(s) = 0 \, .$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - s - 3 = 0 \, .$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s + 3}{s^2 + 3s + 2} \, .$$

Using *partial fractions*,

$$\frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

Hence $y(t) = \mathcal{L}^{-1}[Y(s)] = 2\,e^{-t} - e^{-2t}$.

13. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 2\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - 1 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2-2s+2}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s-1$. That is,

$$\frac{1}{s^2-2s+2} = \frac{1}{(s-1)^2+1}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{1}{\xi^2+1}\right] = sin\,t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+2}\right] = e^t sin\,t\,.$$

Hence $y(t) = e^t sin\,t$.

15. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] - 2\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) - 2\,Y(s) - 2s + 4 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{2s-4}{s^2-2s-2}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s-1$. Completing the square,

$$\frac{2s-4}{s^2-2s-2} = \frac{2(s-1)-2}{(s-1)^2-3}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2-3} - \frac{2}{\xi^2-3}\right] = 2\,cosh\,\sqrt{3}\,t - \frac{2}{\sqrt{3}}\,sinh\,\sqrt{3}\,t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s-4}{s^2-2s-2}\right] = e^t\left(2\,cosh\,\sqrt{3}\,t - \frac{2}{\sqrt{3}}\,sinh\,\sqrt{3}\,t\right).$$

16. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + 5\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 2s\,Y(s) + 5\,Y(s) - 2s - 3 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{2s+3}{s^2+2s+5}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s+1$. That is,

$$\frac{2s+3}{s^2+2s+5} = \frac{2(s+1)+1}{(s+1)^2+4}.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4} + \frac{1}{\xi^2+4}\right] = 2\,cos\,2t + \frac{1}{2}\,sin\,2t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+2s+5}\right] = e^{-t}\left(2\,cos\,2t + \frac{1}{2}\,sin\,2t\right).$$

17. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - 4\left[s^3Y(s) - s^2\,y(0) - s\,y'(0) - y''(0)\right] +$$
$$+\,6\left[s^2\,Y(s) - s\,y(0) - y'(0)\right] - 4[s\,Y(s) - y(0)] + Y(s) = 0$$

Applying the *initial conditions*,

$$s^4Y(s) - 4s^3Y(s) + 6s^2\,Y(s) - 4s\,Y(s) + \,Y(s) - s^2 + 4s - 7 = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2\, - 4s\, + 1} = \frac{s^2 - 4s + 7}{(s-1)^4}\,.$$

Using *partial fractions*,

$$\frac{s^2 - 4s + 7}{(s-1)^4} = \frac{4}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2}.$$

Note that $\mathcal{L}[\,t^n\,] = (n!)/s^{n+1}$ and $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$. Hence the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2 - 4s + 7}{(s-1)^4}\right] = \frac{2}{3}\,t^3e^t - t^2e^t + t\,e^t.$$

18. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - \,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^4Y(s) - \,Y(s) - s^3 - s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 - 1}\,.$$

By inspection, it follows that $y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2-1}\right] = cosh\,t\,.$

19. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - 4\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^4Y(s) - \,4Y(s) - s^3 + 2s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 + 2}\,.$$

It follows that $y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+2}\right] = cos\,\sqrt{2}\,t\,.$

20. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \omega^2\,Y(s) = \frac{s}{s^2+4}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + \omega^2\,Y(s) - s = \frac{s}{s^2+4}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{1}{4-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+4}\right].$$

First note that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = cos\,\omega t \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = cos\,2t\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4-\omega^2}\,cos\,\omega t - \frac{1}{4-\omega^2}\,cos\,2t + cos\,\omega t \\ &= \frac{5-\omega^2}{4-\omega^2}\,cos\,\omega t - \frac{1}{4-\omega^2}\,cos\,2t\,. \end{aligned}$$

21. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 2\,Y(s) = \frac{s}{s^2+1}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - s + 2 = \frac{s}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2-2s+2)(s^2+1)} + \frac{s-2}{s^2-2s+2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} - \frac{s-4}{s^2-2s+2}\right].$$

Thus we can write

$$Y(s) = \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{2}{5}\,\frac{2s-3}{s^2-2s+2}\,.$$

For the *last term*, we note that $s^2 - 2s + 2 = (s-1)^2 + 1$. So that

$$\frac{2s-3}{s^2-2s+2} = \frac{2(s-1)-1}{(s-1)^2+1}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+1} - \frac{1}{\xi^2+1}\right] = 2\,cos\,t - sin\,t\,.$$

Based on the *translation property* of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-2s+2}\right] = e^t(2\,cos\,t - sin\,t).$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{5}\,cos\,t - \frac{2}{5}\,sin\,t + \frac{2}{5}\,e^t(2\,cos\,t - sin\,t)\,.$$

23. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + Y(s) = \frac{4}{s+1}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 2s\,Y(s) + Y(s) - 2s - 3 = \frac{4}{s+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}\,.$$

First write

$$\frac{2s+3}{(s+1)^2} = \frac{2(s+1)+1}{(s+1)^2} = \frac{2}{s+1} + \frac{1}{(s+1)^2}\,.$$

We note that

$$\mathcal{L}^{-1}\left[\frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2}\right] = 2\,t^2 + 2 + t\,.$$

So based on the *translation property* of the Laplace transform, the solution of the IVP is

$$y(t) = 2\,t^2 e^{-t} + t\,e^{-t} + 2\,e^{-t}.$$

25. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform
of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \,Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2\,Y(s) + \,Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}[f(t)] \;&= \int_0^\infty f(t)\,e^{-st}dt\\ &= \int_0^1 t\,e^{-st}dt\\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\,.\end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+1)} - e^{-s}\frac{s+1}{s^2(s^2+1)}\,.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

and

$$\frac{s}{s^2(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

We find, by inspection, that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = t - sin\,t\,.$$

Referring to *Line* 13 , in Table 6.2.1 ,

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Let

$$\mathcal{L}[\,g(t)\,] = \frac{s+1}{s^2(s^2+1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\,.$$

Then $g(t) = 1 + t - cos\, t - sin\, t$ . It follows, therefore, that

$$\mathcal{L}^{-1}\left[e^{-s} \cdot \frac{s+1}{s^2(s^2+1)}\right] = u_1(t)[\,1 + (t-1) - cos(t-1) - sin(t-1)\,]\,.$$

Combining the above, the solution of the IVP is

$$y(t) = t - sin\, t - u_1(t)[\,1 + (t-1) - cos(t-1) - sin(t-1)\,]\,.$$

26. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform
of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 4\, Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2\, Y(s) + \ 4\, Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned} \mathcal{L}[f(t)] \ &= \int_0^\infty f(t)\, e^{-st} dt \\ &= \int_0^1 t\, e^{-st} dt + \int_1^\infty e^{-st} dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}\,. \end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+4)} - e^{-s}\frac{1}{s^2(s^2+4)}\,.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right].$$

We find that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}\, t - \frac{1}{8}\, sin\, t\,.$$

Referring to *Line* 13 , in Table 6.2.1 ,

$$\mathcal{L}[\, u_c(t) f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

It follows that

$$\mathcal{L}^{-1}\left[e^{-s}\cdot\frac{1}{s^2(s^2+4)}\right]=u_1(t)\left[\frac{1}{4}(t-1)-\frac{1}{8}\,sin(t-1)\right].$$

Combining the above, the solution of the IVP is

$$y(t)=\frac{1}{4}\,t-\frac{1}{8}\,sin\,t-u_1(t)\left[\frac{1}{4}(t-1)-\frac{1}{8}\,sin(t-1)\right].$$

$28(a)$. Assuming that the conditions of Theorem 6.2.1 are satisfied,

$$\begin{aligned}F'(s)&=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt\\&=\int_0^\infty\frac{\partial}{\partial s}\big[e^{-st}f(t)\big]dt\\&=\int_0^\infty\big[-t\,e^{-st}f(t)\big]dt\\&=\int_0^\infty e^{-st}[-tf(t)]dt\,.\end{aligned}$$

$(b)$. Using *mathematical induction*, suppose that for some $k\geq 1$,

$$F^{(k)}(s)=\int_0^\infty e^{-st}\Big[(-t)^k f(t)\Big]dt\,.$$

Differentiating both sides,

$$\begin{aligned}F^{(k+1)}(s)&=\frac{d}{ds}\int_0^\infty e^{-st}\Big[(-t)^k f(t)\Big]dt\\&=\int_0^\infty\frac{\partial}{\partial s}\Big[e^{-st}(-t)^k f(t)\Big]dt\\&=\int_0^\infty\Big[-t\,e^{-st}(-t)^k f(t)\Big]dt\\&=\int_0^\infty e^{-st}\Big[(-t)^{k+1}f(t)\Big]dt\,.\end{aligned}$$

29. We know that

$$\mathcal{L}\big[e^{at}\big]=\frac{1}{s-a}\,.$$

Based on Prob. $28$,

$$\mathcal{L}\big[-t\,e^{at}\big]=\frac{d}{ds}\left[\frac{1}{s-a}\right].$$

Therefore,

$$\mathcal{L}\big[t\,e^{at}\big] = \frac{1}{(s-a)^2}.$$

31. Based on Prob. 28,

$$\begin{aligned}\mathcal{L}[(-t)^n] &= \frac{d^n}{ds^n}\mathcal{L}[1] \\ &= \frac{d^n}{ds^n}\left[\frac{1}{s}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}[t^n] &= (-1)^n\frac{(-1)^n n!}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}}.\end{aligned}$$

33. Using the *translation property* of the Laplace transform,

$$\mathcal{L}\big[e^{at}\,sin\,bt\big] = \frac{b}{(s-a)^2+b^2}.$$

Therefore,

$$\begin{aligned}\mathcal{L}\big[t\,e^{at}\,sin\,bt\big] &= -\frac{d}{ds}\left[\frac{b}{(s-a)^2+b^2}\right] \\ &= \frac{2b(s-a)}{(s^2-2as+a^2+b^2)^2}.\end{aligned}$$

34. Using the *translation property* of the Laplace transform,

$$\mathcal{L}\big[e^{at}\,cos\,bt\big] = \frac{s-a}{(s-a)^2+b^2}.$$

Therefore,

$$\begin{aligned}\mathcal{L}\big[t\,e^{at}\,cos\,bt\big] &= -\frac{d}{ds}\left[\frac{s-a}{(s-a)^2+b^2}\right] \\ &= \frac{(s-a)^2-b^2}{(s^2-2as+a^2+b^2)^2}.\end{aligned}$$

35$(a)$. Taking the Laplace transform of the given *Bessel equation*,

$$\mathcal{L}[\,t\,y''] + \mathcal{L}[\,y'] + \mathcal{L}[\,t\,y\,] = 0\,.$$

Using the *differentiation property* of the transform,

$$-\frac{d}{ds}\mathcal{L}[\,y''] + \mathcal{L}[\,y'] - \frac{d}{ds}\mathcal{L}[\,y\,] = 0\,.$$

That is,

$$-\frac{d}{ds}\big[s^2Y(s) - s\,y(0) - y'(0)\big] + s\,Y(s) - y(0) - \frac{d}{ds}Y(s) = 0\,.$$

It follows that

$$\big(1+s^2\big)Y'(s) + s\,Y(s) = 0\,.$$

$(b)$. We obtain a *first-order linear* ODE in $Y(s)$:

$$Y'(s) + \frac{s}{s^2+1}Y(s) = 0\,,$$

with *integrating factor*

$$\mu(s) = exp\bigg(\int \frac{s}{s^2+1}ds\bigg) = \sqrt{s^2+1}\,.$$

The first-order ODE can be written as

$$\frac{d}{ds}\Big[\sqrt{s^2+1}\cdot Y(s)\Big] = 0\,,$$

with solution

$$Y(s) = \frac{c}{\sqrt{s^2+1}}\,.$$

$(c)$. In order to obtain *negative* powers of $s$, first write

$$\frac{1}{\sqrt{s^2+1}} = \frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-1/2}.$$

Expanding $\left(1+\frac{1}{s^2}\right)^{-1/2}$ in a *binomial series*,

$$\frac{1}{\sqrt{1+(1/s^2)}} = 1 - \frac{1}{2}\,s^{-2} + \frac{1\cdot 3}{2\cdot 4}\,s^{-4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,s^{-6} + \cdots,$$

valid for $s^{-2} < 1$. Hence, we can formally express $Y(s)$ as

$$Y(s) = c\left[\frac{1}{s} - \frac{1}{2}\,\frac{1}{s^3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{s^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{s^7} + \cdots\right].$$

Assuming that *term-by-term* inversion is valid,

$$\begin{aligned} y(t) &= c\left[1 - \frac{1}{2}\frac{t^2}{2!} + \frac{1\cdot 3}{2\cdot 4}\frac{t^4}{4!} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^6}{6!} + \cdots\right] \\ &= c\left[1 - \frac{2!}{2^2}\frac{t^2}{2!} + \frac{4!}{2^2\cdot 4^2}\frac{t^4}{4!} - \frac{6!}{2^2\cdot 4^2\cdot 6^2}\frac{t^6}{6!} + \cdots\right]. \end{aligned}$$

It follows that

$$\begin{aligned} y(t) &= c\left[1 - \frac{1}{2^2}t^2 + \frac{1}{2^2\cdot 4^2}t^4 - \frac{1}{2^2\cdot 4^2\cdot 6^2}t^6 + \cdots\right] \\ &= c\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}t^{2n}. \end{aligned}$$

The series is evidently the expansion, about $x = 0$, of $J_0(t)$.

36$(b)$. Taking the Laplace transform of the given *Legendre equation*,

$$\mathcal{L}[\,y''] - \mathcal{L}\left[t^2 y''\right] - 2\,\mathcal{L}[t\,y'] + \alpha(\alpha+1)\mathcal{L}[\,y\,] = 0\,.$$

Using the *differentiation property* of the transform,

$$\mathcal{L}[\,y''] - \frac{d^2}{ds^2}\mathcal{L}[\,y''] + 2\frac{d}{ds}\mathcal{L}[\,y'] + \alpha(\alpha+1)\mathcal{L}[\,y\,] = 0\,.$$

That is,

$$\begin{aligned} \left[s^2Y(s) - s\,y(0) - y'(0)\right] - \frac{d^2}{ds^2}\left[s^2Y(s) - s\,y(0) - y'(0)\right] + \\ + 2\frac{d}{ds}[s\,Y(s) - y(0)] + \alpha(\alpha+1)Y(s) = 0\,. \end{aligned}$$

Invoking the *initial conditions*, we have

$$s^2Y(s) - 1 - \frac{d^2}{ds^2}\left[s^2Y(s) - 1\right] + 2\frac{d}{ds}[s\,Y(s)] + \alpha(\alpha+1)Y(s) = 0\,.$$

After carrying out the differentiation, the equation simplifies to

$$\frac{d^2}{ds^2}\left[s^2Y(s)\right] - 2\frac{d}{ds}[s\,Y(s)] - \left[s^2 + \alpha(\alpha+1)\right]Y(s) = -1\,.$$

That is,

$$s^2\frac{d^2}{ds^2}Y(s) + 2s\,\frac{d}{ds}Y(s) - \left[s^2 + \alpha(\alpha+1)\right]Y(s) = -1\,.$$

37. By definition of the Laplace transform, given the appropriate conditions,

$$\begin{aligned}\mathcal{L}[\,g(t)] &= \int_0^\infty e^{-st}\left[\int_0^t f(\tau)d\tau\right]dt \\ &= \int_0^\infty \int_0^t e^{-st} f(\tau)d\tau dt\,.\end{aligned}$$

Assuming that the order of integration can be exchanged,

$$\begin{aligned}\mathcal{L}[\,g(t)] &= \int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st}dt\right]d\tau \\ &= \int_0^\infty f(\tau)\left[\frac{e^{-s\tau}}{s}\right]d\tau.\end{aligned}$$

[Note the *region* of integration is the area between the lines $\tau(t) = t$ and $\tau(t) = 0$ .]
Hence

$$\begin{aligned}\mathcal{L}[\,g(t)] &= \frac{1}{s}\int_0^\infty f(\tau)\,e^{-s\tau}d\tau \\ &= \frac{1}{s}\,\mathcal{L}[\,f(t)].\end{aligned}$$