Chapter Ten

Section 10.1

1. The general solution of the ODE is $y(x) = c_1\, cos\, x + c_2\, sin\, x$. Imposing the first boundary condition, it is necessary that $c_1 = 0$. Therefore $y(x) = c_2\, sin\, x$. Taking its derivative, $y'(x) = c_2\, cos\, x$. Imposing the second boundary condition, we require that $c_2\, cos\, \pi = 1$. The latter equation is satisfied only if $c_2 = -1$. Hence the solution of the boundary value problem is $y(x) = -sin\, x$.

4. The general solution of the differential equation is $y(x) = c_1\, cos\, x + c_2\, sin\, x$. It follows that $y'(x) = -c_1\, sin\, x + c_2\, cos\, x$. Imposing the first boundary condition, we find that $c_2 = 1$. Therefore $y(x) = c_1\, cos\, x + sin\, x$. Imposing the second boundary condition, we require that $c_1\, cos\, L + sin\, L = 0$. If $cos\, L \neq 0$, that is, as long as $L \neq (2k-1)\pi/2$, with k an integer, then $c_1 = -tanL$. The solution of the boundary value problem is

$$y(x) = -tanL\, cos\, x + sin\, x\, .$$

If $cos\, L = 0$, the boundary condition results in $sin\, L = 0$. The latter two equations are inconsistent, which implies that the BVP has no solution.

5. The general solution of the *homogeneous* differential equation is

$$y(x) = c_1\, cos\, x + c_2\, sin\, x\, .$$

Using any of a number of methods, including the *method of undetermined coefficients*, it is easy to show that a *particular solution* is $Y(x) = x$. Hence the general solution of the given differential equation is $y(x) = c_1\, cos\, x + c_2\, sin\, x + x$. The first boundary condition requires that $c_1 = 0$. Imposing the second boundary condition, it is necessary that $c_2\, sin\, \pi + \pi = 0$. The resulting equation has *no solution*. We conclude that the boundary value problem has no solution.

6. Using the *method of undetermined coefficients*, it is easy to show that the general solution of the ODE is $y(x) = c_1\, cos\sqrt{2}\, x + c_2\, sin\sqrt{2}\, x + x/2$. Imposing the first boundary condition, we find that $c_1 = 0$. The second boundary condition requires that $c_2\, sin\sqrt{2}\, \pi + \pi/2 = 0$. It follows that $c_2 = -\pi/2sin\sqrt{2}\, \pi$. Hence the solution of the boundary value problem is

$$y(x) = -\frac{\pi}{2sin\sqrt{2}\, \pi} sin\sqrt{2}\, x + \frac{x}{2}\, .$$

8. The general solution of the *homogeneous* differential equation is

$$y(x) = c_1\, cos\, 2x + c_2\, sin\, 2x\, .$$

Using the method of undetermined coefficients, a *particular solution* is $Y(x) = sin\, x/3$.

Hence the general solution of the given differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \sin x\,.$$

The first boundary condition requires that $c_1 = 0$. The second boundary requires that $c_2 \sin 2\pi + \frac{1}{3} \sin \pi = 0$. The latter equation is valid for *all* values of c_2. Therefore the solution of the boundary value problem is

$$y(x) = c_2 \sin 2x + \frac{1}{3} \sin x\,.$$

9. Using the *method of undetermined coefficients*, it is easy to show that the general solution of the ODE is $y(x) = c_1 \cos 2x + c_2 \sin 2x + \cos x/3$. It follows that $y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x - \sin x/3$. Imposing the first boundary condition, we find that $c_2 = 0$. The second boundary condition requires that

$$-2c_1 \sin 2\pi - \frac{1}{3} \sin \pi = 0\,.$$

The resulting equation is satisfied for all values of c_1. Hence the solution is the family of functions

$$y(x) = c_1 \cos 2x + \frac{1}{3} \cos x\,.$$

10. The general solution of the differential equation is

$$y(x) = c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x + \frac{1}{2} \cos x\,.$$

Its derivative is $y'(x) = -\sqrt{3}\,c_1 \sin \sqrt{3}\,x + \sqrt{3}\,c_2 \cos \sqrt{3}\,x - \sin x/2$. The first boundary condition requires that $c_2 = 0$. Imposing the second boundary condition, we obtain $-\sqrt{3}\,c_1 \sin \sqrt{3}\,\pi = 0$. It follows that $c_1 = 0$. Hence the solution of the BVP is $y(x) = \cos x/2$.

12. Assuming that $\lambda > 0$, we can set $\lambda = \mu^2$. The general solution of the differential equation is

$$y(x) = c_1 \cos \mu x + c_2 \sin \mu x\,,$$

so that $y'(x) = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$. Imposing the first boundary condition, it follows that $c_2 = 0$. Therefore $y(x) = c_1 \cos \mu x$. The second boundary condition requires that $c_1 \cos \mu\pi = 0$. For a nontrivial solution, it is necessary that $\cos \mu\pi = 0$, that is, $\mu\pi = (2n-1)\pi/2$, with $n = 1, 2, \cdots$. Therefore the *eigenvalues* are

$$\lambda_n = \frac{(2n-1)^2}{4}\,,\ n = 1, 2, \cdots.$$

The corresponding *eigenfunctions* are given by

$$y_n = cos\,\frac{(2n-1)x}{2}\,,\ n = 1, 2, \cdots.$$

Assuming that $\lambda < 0$, we can set $\lambda = -\mu^2$. The general solution of the differential equation is

$$y(x) = c_1\,cosh\,\mu x + c_2\,sinh\,\mu x\,,$$

so that $y'(x) = \mu c_1\,sinh\,\mu x + \mu c_2\,cosh\,\mu x$. Imposing the first boundary condition, it follows that $c_2 = 0$. Therefore $y(x) = c_1\,cosh\,\mu x$. The second boundary condition requires that $c_1\,cosh\,\mu\pi = 0$, which results in $c_1 = 0$. Hence the only solution is the trivial solution. Finally, with $\lambda = 0$, the general solution of the ODE is

$$y(x) = c_1 x + c_2\,.$$

It is easy to show that the boundary conditions require that $c_1 = c_2 = 0$. Therefore all of the eigenvalues are *positive*.

13. Assuming that $\lambda > 0$, we can set $\lambda = \mu^2$. The general solution of the differential equation is

$$y(x) = c_1\,cos\,\mu x + c_2\,sin\,\mu x\,,$$

so that $y'(x) = -\mu c_1\,sin\,\mu x + \mu c_2\,cos\,\mu x$. Imposing the first boundary condition, it follows that $c_2 = 0$. The second boundary condition requires that $c_1\,sin\,\mu\pi = 0$. For a nontrivial solution, we must have $\mu\pi = n\pi$, $n = 1, 2, \cdots$. It follows that the *eigenvalues* are

$$\lambda_n = n^2,\ n = 1, 2, \cdots,$$

and the corresponding *eigenfunctions* are

$$y_n = cos\,nx\,,\ n = 1, 2, \cdots.$$

Assuming that $\lambda < 0$, we can set $\lambda = -\mu^2$. The general solution of the differential equation is

$$y(x) = c_1\,cosh\,\mu x + c_2\,sinh\,\mu x\,,$$

so that $y'(x) = \mu c_1\,sinh\,\mu x + \mu c_2\,cosh\,\mu x$. Imposing the first boundary condition, it follows that $c_2 = 0$. The second boundary condition requires that $c_1\,sinh\,\mu\pi = 0$. The latter equation is satisfied only for $c_1 = 0$.

Finally, for $\lambda = 0$, the solution is $y(x) = c_1 x + c_2$. Imposing the boundary conditions, we find that $y(x) = c_2$. Therefore $\lambda = 0$ is *also* an eigenvalue, with corresponding eigenfunction $y_0(x) = 1$.

14. It can be shown, as in Prob. 12 , that $\lambda > 0$. Setting $\lambda = \mu^2$, the general solution of the resulting ODE is

$$y(x) = c_1 \cos \mu x + c_2 \sin \mu x \,,$$

with $y'(x) = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$. Imposing the first boundary condition, we find that $c_2 = 0$. Therefore $y(x) = c_1 \cos \mu x$. The second boundary condition requires that $c_1 \cos \mu L = 0$. For a nontrivial solution, it is necessary that $\cos \mu L = 0$, that is, $\mu = (2n-1)\pi/(2L)$, with $n = 1, 2, \cdots$. Therefore the *eigenvalues* are

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{4L^2} \,,\ n = 1, 2, \cdots .$$

The corresponding *eigenfunctions* are given by

$$y_n = \cos \frac{(2n-1)\pi x}{2L} \,,\ n = 1, 2, \cdots .$$

16. Assuming that $\lambda > 0$, we can set $\lambda = \mu^2$. The general solution of the differential equation is

$$y(x) = c_1 \cosh \mu x + c_2 \sinh \mu x \,.$$

The first boundary condition requires that $c_1 = 0$. Therefore $y(x) = c_2 \sinh \mu x$ and $y'(x) = c_2 \cosh \mu x$. Imposing the second boundary condition, it is necessary that $c_2 \cosh \mu L = 0$. The latter equation is valid only for $c_2 = 0$. The only solution is the trivial solution.

Assuming that $\lambda > 0$, we set $\lambda = -\mu^2$. The general solution of the resulting ODE is

$$y(x) = c_1 \cos \mu x + c_2 \sin \mu x \,.$$

Imposing the first boundary condition, we find that $c_1 = 0$. Hence $y(x) = c_2 \sin \mu x$ and $y'(x) = c_2 \cos \mu x$. In order to satisfy the second boundary condition, it is necessary that $c_2 \cos \mu L = 0$. For a nontrivial solution, $\mu = (2n-1)\pi/(2L)$, with $n = 1, 2, \cdots$. Therefore the *eigenvalues* are

$$\lambda_n = -\frac{(2n-1)^2 \pi^2}{4L^2} \,,\ n = 1, 2, \cdots .$$

The corresponding *eigenfunctions* are given by

$$y_n = \sin \frac{(2n-1)\pi x}{2L} \,,\ n = 1, 2, \cdots .$$

Finally, for $\lambda = 0$, the general solution is *linear*. Based on the boundary conditions, it follows that $y(x) = 0$. Therefore all of the eigenvalues are negative.

$17(a)$. Setting $\lambda = \mu^2$, write the general solution of the ODE $y'' + \mu^2 y = 0$ as

$$y(x) = k_1 e^{i\mu x} + k_2 e^{-i\mu x}.$$

Imposing the boundary conditions $y(0) = y(\pi) = 0$, we obtain the system of equations

$$\begin{gathered} k_1 + k_2 = 0 \\ k_1 e^{i\mu\pi} + k_2 e^{-i\mu\pi} = 0\,. \end{gathered}$$

The system has a *nontrivial* solution if and only if the coefficient matrix is *singular*. Set the determinant equal to zero to obtain

$$e^{-i\mu\pi} - e^{i\mu\pi} = 0\,.$$

(b). Let $\mu = \nu + i\sigma$. Then $i\mu\pi = i\nu\pi - \sigma\pi$, and the previous equation can be written as

$$e^{\sigma\pi} e^{-i\nu\pi} - e^{-\sigma\pi} e^{i\nu\pi} = 0\,.$$

Using Euler's relation, $e^{i\nu\pi} = cos\,\nu\pi + i\,sin\,\nu\pi$, we obtain

$$e^{\sigma\pi}(cos\,\nu - i\,sin\,\nu) - e^{-\sigma\pi}(cos\,\nu + i\,sin\,\nu) = 0\,.$$

Equating the real and imaginary parts of the equation,

$$\begin{gathered} (e^{\sigma\pi} - e^{-\sigma\pi})cos\,\nu\pi = 0 \\ (e^{\sigma\pi} + e^{-\sigma\pi})sin\,\nu\pi = 0\,. \end{gathered}$$

(c). Based on the second equation, $\nu = n\,,\ n \in \mathbb{I}$. Since $cos\,n\pi \neq 0$, it follows that $e^{\sigma\pi} = e^{-\sigma\pi}$, or $e^{2\sigma\pi} = 1$. Hence $\sigma = 0$, and $\mu = n\,, n \in \mathbb{I}$.