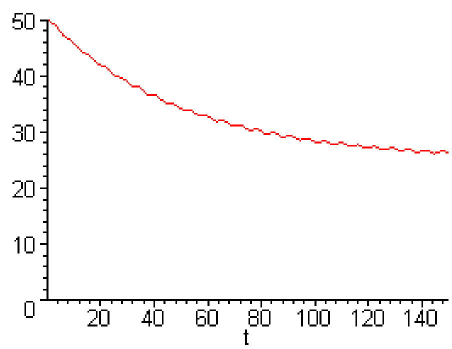
Section 2.3

$5(a)$. Let Q be the amount of salt in the tank. Salt enters the tank of water at a rate of $2\,\frac{1}{4}\left(1+\frac{1}{2}sin\,t\right)=\frac{1}{2}+\frac{1}{4}sin\,t\;\; oz/min$. It leaves the tank at a rate of $2\,Q/100\;\; oz/min$. Hence the differential equation governing the amount of salt at any time is

$$\frac{dQ}{dt}=\frac{1}{2}+\frac{1}{4}sin\,t-Q/50\,.$$

The initial amount of salt is $Q_0=50\;oz$. The governing ODE is *linear*, with integrating factor $\mu(t)=e^{t/50}$. Write the equation as $\left(e^{t/50}Q\right)'=e^{t/50}\left(\frac{1}{2}+\frac{1}{4}sin\,t\right)$. The specific solution is $Q(t)=25+\left[12.5sin\,t-625cos\,t+63150\,e^{-t/50}\right]/2501\;\;oz$.

(b).



(c). The amount of salt approaches a *steady state*, which is an oscillation of amplitude $1/4$ about a level of $25\;oz$.

$6(a)$. The equation governing the value of the investment is $dS/dt=r\,S$. The value of the investment, at any time, is given by $S(t)=S_0e^{rt}$. Setting $S(T)=2S_0$, the required time is $T=ln(2)/r$.

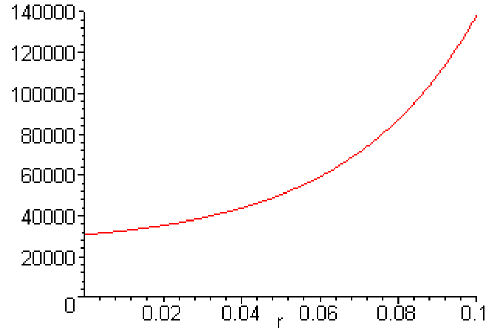
(b). For the case $r=7\%=.07$, $T\approx 9.9\;yrs$.

(c). Referring to Part(a), $r=ln(2)/T$. Setting $T=8$, the required interest rate is to be approximately $r=8.66\,\%$.

$8(a)$. Based on the solution in *Eq.*(16), with $S_0=0$, the value of the investments *with* contributions is given by $S(t)=25,000(e^{rt}-1)$. After *ten* years, person A has $S_A=\$25,000(1.226)=\$\,30,640$. Beginning at age 35, the investments can now be analyzed using the equations $S_A=30,640\,e^{.08t}$ and $S_B=25,000(e^{.08t}-1)$. After *thirty* years, the balances are $S_A=\$\,337,734$ and $S_B=\$\,250,579$.

(b). For an *unspecified* rate r, the balances after *thirty* years are $S_A=30,640\,e^{30r}$ and $S_B=25,000(e^{30r}-1)$.

(c).



(d). The two balances can *never* be equal.

$11(a)$. Let S be the value of the mortgage. The debt accumulates at a rate of rS, in which $r = .09$ is the *annual* interest rate. Monthly payments of $\$\,800$ are equivalent to $\$\,9,600$ *per year*. The differential equation governing the value of the mortgage is $dS/dt = .09\,S - 9,600$. Given that S_0 is the original amount borrowed, the debt is $S(t) = S_0 e^{.09t} - 106,667(e^{.09t} - 1)$. Setting $S(30) = 0$, it follows that $S_0 = \$\,99,500$.

(b). The *total* payment, over 30 years, becomes $\$\,288,000$. The interest paid on this purchase is $\$\,188,500$.

$13(a)$. The balance *increases* at a rate of $r\,S$ *\$/yr*, and *decreases* at a constant rate of k *\$ per year*. Hence the balance is modeled by the differential equation $dS/dt = rS - k$. The balance at any time is given by $S(t) = S_0 e^{rt} - \frac{k}{r}(e^{rt} - 1)$.

(b). The solution may also be expressed as $S(t) = (S_0 - \frac{k}{r})e^{rt} + \frac{k}{r}$. Note that if the withdrawal rate is $k_0 = r\,S_0$, the balance will remain at a constant level S_0.

(c). Assuming that $k > k_0$, $S(T_0) = 0$ for $T_0 = \frac{1}{r} ln\left[\frac{k}{k - k_0}\right]$.

(d). If $r = .08$ and $k = 2k_0$, then $T_0 = 8.66$ *years*.

(e). Setting $S(t) = 0$ and solving for e^{rt} in Part(b), $e^{rt} = \frac{k}{k - rS_0}$. Now setting $t = T$ results in $k = rS_0 e^{rT}/(e^{rT} - 1)$.

(f). In part(e), let $k = 12,000$, $r = .08$, and $T = 20$. The required investment becomes $S_0 = \$\,119,715$.

$14(a)$. Let $Q' = -r\,Q$. The general solution is $Q(t) = Q_0 e^{-rt}$. Based on the definition of *half-life*, consider the equation $Q_0/2 = Q_0 e^{-5730\,r}$. It follows that

$-5730\,r = ln(1/2)$, that is, $r = 1.2097 \times 10^{-4}$ *per year.*

(b). Hence the amount of carbon-14 is given by $Q(t) = Q_0\, e^{-1.2097\times 10^{-4}t}$.

(c). Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097\times 10^{-4}T}$. Solving for the *decay time*, the apparent age of the remains is approximately $T = 13,304.65$ *years*.

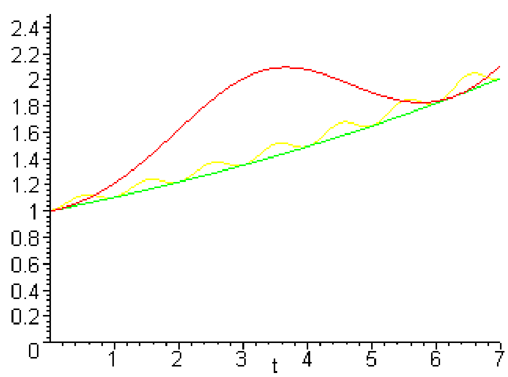
15. Let $P(t)$ be the population of mosquitoes at any time t. The rate of *increase* of the mosquito population is rP. The population *decreases* by $20,000$ *per day*. Hence the equation that models the population is given by $dP/dt = rP - 20,000$. Note that the variable t represents *days*. The solution is $P(t) = P_0 e^{rt} - \frac{20,000}{r}(e^{rt} - 1)$. In the absence of predators, the governing equation is $dP_1/dt = rP_1$, with solution $P_1(t) = P_0 e^{rt}$. Based on the data, set $P_1(7) = 2P_0$, that is, $2P_0 = P_0 e^{7r}$. The growth rate is determined as $r = ln(2)/7 = .09902$ *per day*. Therefore the population, including the *predation* by birds, is $P(t) = 2\times 10^5 e^{.099t} - 201,997(e^{.099t} - 1) =$ $= 201,997.3 - 1977.3\, e^{.099t}$.

16(a). $y(t) = exp[2/10 + t/10 - 2cos(t)/10]$. The *doubling-time* is $\tau \approx 2.9632$.

(b). The differential equation is $dy/dt = y/10$, with solution $y(t) = y(0)e^{t/10}$. The *doubling-time* is given by $\tau = 10ln(2) \approx 6.9315$.

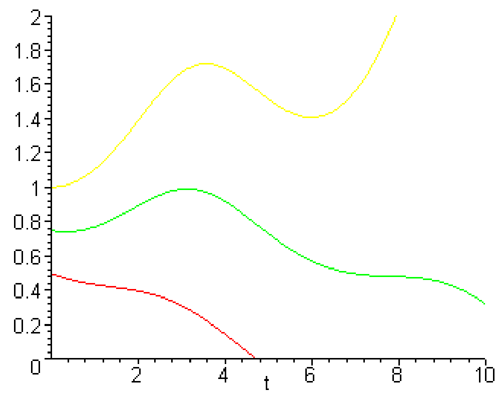
(c). Consider the differential equation $dy/dt = (0.5 + sin(2\pi t))\, y/5$. The equation is *separable*, with $\frac{1}{y}dy = \left(0.1 + \frac{1}{5}sin(2\pi t)\right)dt$. Integrating both sides, with respect to the appropriate variable, we obtain $ln\, y = (\pi t - cos(2\pi t))/10\pi + c$. Invoking the initial condition, the solution is $y(t) = exp[(1 + \pi t - cos(2\pi t))/10\pi]$. The *doubling-time* is $\tau \approx 6.3804$. The *doubling-time* approaches the value found in part(b).

(d).



17(a). The differential equation $dy/dt = r(t)\, y - k$ is *linear*, with integrating factor $\mu(t) = exp\left[-\int r(t)dt\right]$. Write the equation as $(\mu\, y)' = -k\, \mu(t)$. Integration of both

sides yields the general solution $y = \left[-k \int \mu(\tau) d\tau + y_0\, \mu(0) \right] / \mu(t)$. In this problem, the
integrating factor is $\mu(t) = exp[(cos\, t - t)/5]$.



(b). The population becomes *extinct*, if $y(t^*) = 0$, for some $t = t^*$. Referring to part(a),
we find that $y(t^*) = 0 \Rightarrow$

$$\int_0^{t^*} exp[(cos\, \tau - \tau)/5] d\tau = 5\, e^{1/5} y_c.$$

It can be shown that the integral on the left hand side increases *monotonically*, from *zero* to a limiting value of approximately 5.0893. Hence extinction can happen *only if* $5\, e^{1/5} y_c < 5.0893$, that is, $y_c < 0.8333$.

(c). Repeating the argument in part(b), it follows that $y(t^*) = 0 \Rightarrow$

$$\int_0^{t^*} exp[(cos\, \tau - \tau)/5] d\tau = \frac{1}{k}\, e^{1/5} y_c.$$

Hence extinction can happen *only if* $e^{1/5} y_c / k < 5.0893$, that is, $y_c < 4.1667\, k$.

(d). Evidently, y_c is a *linear* function of the parameter k.

19(a). Let $Q(t)$ be the *volume* of carbon monoxide in the room. The rate of *increase* of *CO* is $(.04)(0.1) = 0.004\ ft^3/min$. The amount of *CO leaves the room* at a rate of $(0.1)Q(t)/1200 = Q(t)/12000\ \ ft^3/min$. Hence the total rate of change is given by the differential equation $dQ/dt = 0.004 - Q(t)/12000$. This equation is *linear* and separable, with solution $Q(t) = 48 - 48\, exp(-t/12000)\ \ ft^3$. Note that $Q_0 = 0\ \ ft^3$. Hence the *concentration* at any time is given by $x(t) = Q(t)/1200 = Q(t)/12\ \%$.

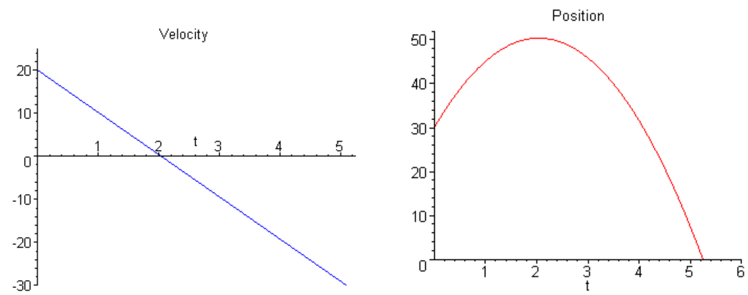
(b). The *concentration* of *CO* in the room is $x(t) = 4 - 4exp(-t/12000)\ \%$. A level of 0.00012 corresponds to 0.012 $\%$. Setting $x(\tau) = 0.012$, the solution of the equation $4 - 4exp(-t/12000) = 0.012$ is $\tau \approx 36\ \ minutes$.

20(a). The concentration is $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. It is easy to see that $c(t \to \infty) = k + P/r$.

(b). $c(t) = c_0\, e^{-rt/V}$. The *reduction times* are $T_{50} = ln(2)V/r$ and $T_{10} = ln(10)V/r$.

(c). The *reduction times*, in *years*, are $T_S = ln(10)(65.2)/12,200 = 430.85$
$T_M = ln(10)(158)/4,900 = 71.4$; $T_E = ln(10)(175)/460 = 6.05$
$T_O = ln(10)(209)/16,000 = 17.63$.

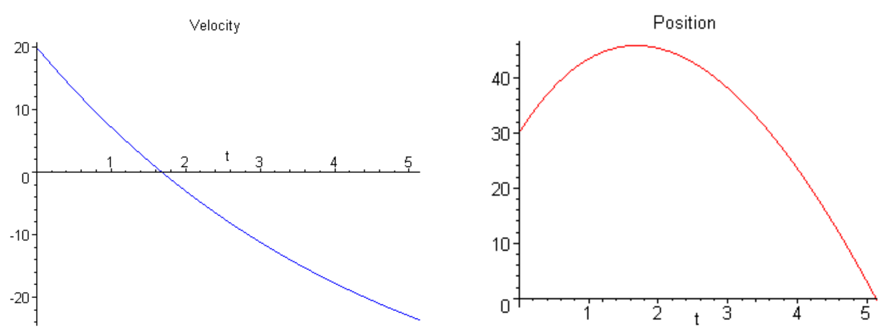
21(c).



22(a). The differential equation for the motion is $m\, dv/dt = -v/30 - mg$. Given the initial condition $v(0) = 20$ *m/s* , the solution is $v(t) = -44.1 + 64.1\, exp(-t/4.5)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.683$ *sec*. Integrating $v(t)$, the position is given by $x(t) = 318.45 - 44.1\, t - 288.45\, exp(-t/4.5)$. Hence the *maximum height* is $x(t_1) = 45.78$ *m*.

(b). Setting $x(t_2) = 0$, the ball hits the ground at $t_2 = 5.128$ *sec*.

(c).

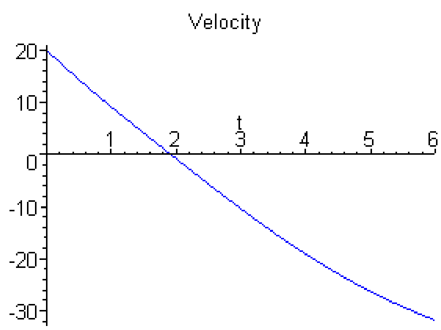


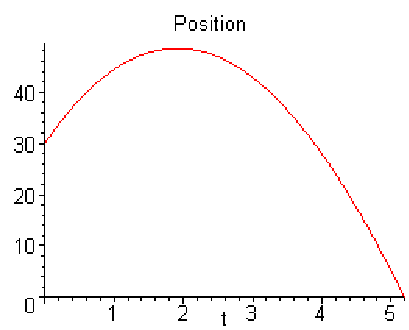
23(a). The differential equation for the *upward* motion is $m\, dv/dt = -\mu v^2 - mg$, in which $\mu = 1/1325$. This equation is *separable*, with $\frac{m}{\mu v^2 + mg} dv = -dt$. Integrating both sides and invoking the initial condition, $v(t) = 44.133\, tan(.425 - .222\, t)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.916\, sec$. Integrating $v(t)$, the position is given by $x(t) = 198.75\, ln[cos(0.222\, t - 0.425)] + 48.57$. Therefore the *maximum height* is $x(t_1) = 48.56\ m$.

(b). The differential equation for the *downward* motion is $m\, dv/dt = +\mu v^2 - mg$. This equation is also separable, with $\frac{m}{mg - \mu v^2} dv = -dt$. For convenience, set $t = 0$ at the *top* of the trajectory. The new initial condition becomes $v(0) = 0$. Integrating both sides and invoking the initial condition, we obtain $ln[(44.13 - v)/(44.13 + v)] = t/2.25$.

Solving for the velocity, $v(t) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, the position is given by $x(t) = 99.29\, ln\left[e^{t/2.25}/(1 + e^{t/2.25})^2\right] + 186.2$. To estimate the *duration* of the downward motion, set $x(t_2) = 0$, resulting in $t_2 = 3.276\, sec$. Hence the *total time* that the ball remains in the air is $t_1 + t_2 = 5.192\ sec$.

(c).





$24(a)$. Measure the positive direction of motion *downward*. Based on Newton's $2nd$ law,
the equation of motion is given by

$$m\,\frac{dv}{dt} = \begin{cases} -0.75\, v + mg & ,\ 0 < t < 10 \\ -12\, v + mg & ,\ t > 10 \end{cases} .$$

Note that gravity acts in the *positive* direction, and the drag force is *resistive*. During the first ten seconds of fall, the initial value problem is $dv/dt = -v/7.5 + 32$, with initial velocity $v(0) = 0$ *fps*. This differential equation is separable and linear, with solution $v(t) = 240(1 - e^{-t/7.5})$. Hence $v(10) = 176.7$ *fps*.

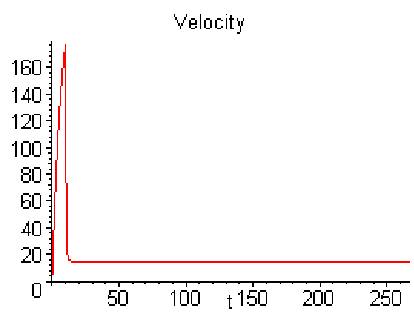
(b). Integrating the velocity, with $x(t) = 0$, the distance fallen is given by

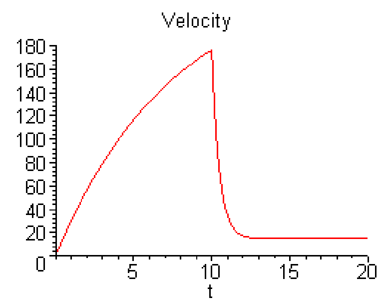
$$x(t) = 240\, t + 1800\, e^{-t/7.5} - 1800 .$$

Hence $x(10) = 1074.5\, ft$.

(c). For computational purposes, reset time to $t = 0$. For the remainder of the motion, the initial value problem is $dv/dt = -32v/15 + 32$, with specified initial velocity $v(0) = 176.7\,fps$. The solution is given by $v(t) = 15 + 161.7\,e^{-32t/15}$. As $t \to \infty$, $v(t) \to v_L = 15\,fps$. Integrating the velocity, with $x(0) = 1074.5$, the distance fallen after the parachute is open is given by $x(t) = 15t - 75.8\,e^{-32t/15} + 1150.3$. To find the duration of the second part of the motion, estimate the root of the transcendental equation $15T - 75.8\,e^{-32T/15} + 1150.3 = 5000$. The result is $T = 256.6$ sec.

(d).





$25(a)$. Measure the positive direction of motion *upward*. The equation of motion is given by $m dv/dt = -kv - mg$. The initial value problem is $dv/dt = -kv/m - g$, with $v(0) = v_0$. The solution is $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Setting $v(t_m) = 0$, the maximum height is reached at time $t_m = (m/k)ln[(mg + k\,v_0)/mg]$. Integrating the velocity, the position of the body is

$$x(t) = -mg\,t/k + \left[\left(\frac{m}{k}\right)^2 g + \frac{m\,v_0}{k}\right](1 - e^{-kt/m}).$$

Hence the maximum height reached is

$$x_m = x(t_m) = \frac{m\,v_0}{k} - g\left(\frac{m}{k}\right)^2 ln\left[\frac{mg + k\,v_0}{mg}\right].$$

(b). Recall that for $\delta \ll 1$, $ln(1+\delta) = \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4 + \ldots$

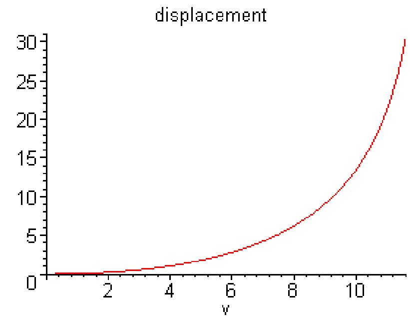
$26(b)$. $\lim\limits_{k\to 0} \frac{-mg + (k\,v_0 + mg)e^{-kt/m}}{k} = \lim\limits_{k\to 0} -\frac{t}{m}(k\,v_0 + mg)e^{-kt/m} = -gt$.

(c). $\lim\limits_{m\to 0}\left[-\frac{mg}{k} + \left(\frac{mg}{k} + v_0\right)e^{-kt/m}\right] = 0$, since $\lim\limits_{m\to 0} e^{-kt/m} = 0$.

$28(a)$. In terms of displacement, the differential equation is $mv\,dv/dx = -kv + mg$. This follows from the *chain rule*: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dt}$. The differential equation is separable, with

$$x(v) = -\frac{mv}{k} - \frac{m^2 g}{k^2} ln\left|\frac{mg - k\,v}{mg}\right|.$$

The inverse *exists*, since both x and v are monotone increasing. In terms of the given parameters, $x(v) = -1.25\,v - 15.31\,ln|0.0816\,v - 1|$.



(b). $x(10) = 13.45$ *meters*. The required value is $k = 0.24$.

(c). In part(a), set $v = 10$ *m/s* and $x = 10$ *meters*.

29(a). Let x represent the height above the earth's surface. The equation of motion is given by $m\frac{dv}{dt} = -G\frac{Mm}{(R+x)^2}$, in which G is the universal gravitational constant. The symbols M and R are the *mass* and *radius* of the earth, respectively. By the chain rule,

$$mv\frac{dv}{dx} = -G\frac{Mm}{(R+x)^2}.$$

This equation is separable, with $v\,dv = -GM(R+x)^{-2}dx$. Integrating both sides, and
invoking the initial condition $v(0) = \sqrt{2gR}$, the solution is $v^2 = 2GM(R+x)^{-1} +$ $+ 2gR - 2GM/R$. From elementary physics, it follows that $g = GM/R^2$. Therefore $v(x) = \sqrt{2g}\left[R/\sqrt{R+x}\right]$. (*Note that* $g = 78,545$ *mi/hr*^{2}.)

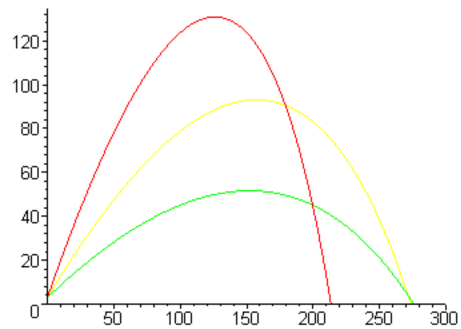
(b). We now consider $dx/dt = \sqrt{2g}\left[R/\sqrt{R+x}\right]$. This equation is also separable, with $\sqrt{R+x}\,dx = \sqrt{2g}\,R\,dt$. By definition of the variable x, the initial condition is $x(0) = 0$. Integrating both sides, we obtain $x(t) = \left[\frac{3}{2}\left(\sqrt{2g}\,R\,t + \frac{2}{3}R^{3/2}\right)\right]^{2/3} - R$. Setting the distance $x(T) + R = 240,000$, and solving for T, the duration of such a flight would be $T \approx 49$ *hours*.

32(a). Both equations are linear and separable. The initial conditions are $v(0) = u\,cos$ A
and $w(0) = u\,sin\,A$. The two solutions are $v(t) = u\,cos\,A\,e^{-rt}$ and $w(t) = -g/r +$ $+ (u\,sin\,A + g/r)e^{-rt}$.

(b). Integrating the solutions in part(a), and invoking the initial conditions, the coordinates are $x(t) = \frac{u}{r} cos\, A(1 - e^{-rt})$ and

$$y(t) = -gt/r + \left(g + ur\, sin\, A + hr^2\right)/r^2 - \left(\frac{u}{r} sin\, A + g/r^2\right)e^{-rt}.$$

(c).



(d). Let T be the time that it takes the ball to go $350\, ft$ horizontally. Then from above, $e^{-T/5} = (u\, cos\, A - 70)/u\, cos\, A$. At the same time, the height of the ball is given by $y(T) = -160\, T + 267 + 125 u sin\, A - (800 + 5u\, sin\, A)[(u\, cos\, A - 70)/u\, cos\, A]$. Hence A and u must satisfy the inequality

$$800 ln\left[\frac{u\, cos\, A - 70}{u\, cos\, A}\right] + 267 + 125 u sin\, A - (800 + 5u\, sin\, A)[(u\, cos\, A - 70)/u\, cos\, A] \geq 10\, .$$

$33(a)$. Solving equation (i), $y'(x) = [(k^2 - y)/y]^{1/2}$. The *positive* answer is chosen, since y is an *increasing* function of x.

(b). Let $y = k^2 sin^2 t$. Then $dy = 2k^2 sin\, t\, cos\, t\, dt$. Substituting into the equation in part(a), we find that

$$\frac{2k^2 sin\, t\, cos\, t\, dt}{dx} = \frac{cos\, t}{sin\, t}\, .$$

Hence $2k^2 sin^2 t\, dt = dx$.

(c). Letting $\theta = 2t$, we further obtain $k^2 sin^2 \frac{\theta}{2}\, d\theta = dx$. Integrating both sides of the equation and noting that $t = \theta = 0$ corresponds to the *origin*, we obtain the solutions $x(\theta) = k^2(\theta - sin\, \theta)/2$ and [from part(b)] $y(\theta) = k^2(1 - cos\, \theta)/2$.

(d). Note that $y/x = (1 - cos\, \theta)/(\theta - sin\, \theta)$. Setting $x = 1$, $y = 2$, the solution of the equation $(1 - cos\, \theta)/(\theta - sin\, \theta) = 2$ is $\theta \approx 1.401$. Substitution into either of the expressions yields $k \approx 2.193$.