Section 2.6

1. $M(x,y) = 2x + 3$ and $N(x,y) = 2y - 2$. Since $M_y = N_x = 0$, the equation is *exact*. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) =$ $= x^2 + 3x + h(y)$. Now $\psi_y = h'(y)$, and equating with N results in the possible function $h(y) = y^2 - 2y$. Hence $\psi(x,y) = x^2 + 3x + y^2 - 2y$, and the solution is defined *implicitly* as $x^2 + 3x + y^2 - 2y = c$.

2. $M(x,y) = 2x + 4y$ and $N(x,y) = 2x - 2y$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

4. First divide both sides by $(2xy + 2)$. We now have $M(x,y) = y$ and $N(x,y) = x$. Since $M_y = N_x = 0$, the resulting equation is *exact*. Integrating M with respect to x, while holding y constant, results in $\psi(x,y) = xy + h(y)$. Differentiating with respect to y, $\psi_y = x + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 0$, and hence $h(y) = 0$ is acceptable. Therefore the solution is defined *implicitly* as $xy = c$. Note that if $xy + 1 = 0$, the equation is trivially satisfied.

6. Write the given equation as $(ax - by)dx + (bx - cy)dy$. Now $M(x,y) = ax - by$ and $N(x,y) = bx - cy$. Since $M_y \neq N_x$, the differential equation is *not* exact.

8. $M(x,y) = e^x sin\, y + 3y$ and $N(x,y) = -3x + e^x sin\, y$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

10. $M(x,y) = y/x + 6x$ and $N(x,y) = ln\, x - 2$. Since $M_y = N_x = 1/x$, the given equation is *exact*. Integrating N with respect to y, while holding x constant, results in $\psi(x,y) = y\, ln\, x - 2y + h(x)$. Differentiating with respect to x, $\psi_x = y/x + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = 6x$, and hence $h(x) = 3x^2$. Therefore the solution
is defined *implicitly* as $3x^2 + y\, ln\, x - 2y = c$.

11. $M(x,y) = x\, ln\, y + xy$ and $N(x,y) = y\, ln\, x + xy$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

13. $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. Since $M_y = N_x = -1$, the equation is *exact*. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) =$ $= x^2 - xy + h(y)$. Now $\psi_y = -x + h'(y)$. Equating ψ_y with N results in $h'(y) = 2y$, and hence $h(y) = y^2$. Thus $\psi(x,y) = x^2 - xy + y^2$, and the solution is given *implicitly* as $x^2 - xy + y^2 = c$. Invoking the initial condition $y(1) = 3$, the specific solution is $x^2 - xy + y^2 = 7$. The *explicit* form of the solution is $y(x) = \frac{1}{2}\left[x + \sqrt{28 - 3x^2}\,\right]$. Hence the solution is valid as long as $3x^2 \leq 28$.

16. $M(x,y) = y\, e^{2xy} + x$ and $N(x,y) = bx\, e^{2xy}$. Note that $M_y = e^{2xy} + 2xy\, e^{2xy}$, and $N_x = b\, e^{2xy} + 2bxy\, e^{2xy}$. The given equation is *exact*, as long as $b = 1$. Integrating

N with respect to y, while holding x constant, results in $\psi(x,y) = e^{2xy}/2 + h(x)$. Now differentiating with respect to x, $\psi_x = y\,e^{2xy} + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = x$, and hence $h(x) = x^2/2$. Conclude that $\psi(x,y) = e^{2xy}/2 + x^2/2$. Hence the solution is given *implicitly* as $e^{2xy} + x^2 = c$.

17. Integrating $\psi_y = N$, while holding x constant, yields
$\psi(x,y) = \int N(x,y)dy + h(x)$.
Taking the partial derivative with respect to x, $\psi_x = \int \frac{\partial}{\partial x} N(x,y)dy + h'(x)$. Now set $\psi_x = M(x,y)$ and therefore $h'(x) = M(x,y) - \int \frac{\partial}{\partial x} N(x,y)dy$. Based on the fact that $M_y = N_x$, it follows that $\frac{\partial}{\partial y}[h'(x)] = 0$. Hence the expression for $h'(x)$ can be integrated to obtain

$$h(x) = \int M(x,y)dx - \int \left[\int \frac{\partial}{\partial x} N(x,y)dy \right] dx\,.$$

18. Observe that $\frac{\partial}{\partial y}[M(x)] = \frac{\partial}{\partial x}[N(y)] = 0$.

20. $M_y = y^{-1}cos\,y - y^{-2}sin\,y$ and $N_x = -2\,e^{-x}(cos\,x + sin\,x)/y$. Multiplying both sides by the integrating factor $\mu(x,y) = y\,e^x$, the given equation can be written as $(e^x sin\,y - 2y\,sin\,x)dx + (e^x cos\,y + 2cos\,x)dy = 0$. Let $\overline{M} = \mu M$ and $\overline{N} = \mu N$. Observe that $\overline{M}_y = \overline{N}_x$, and hence the latter ODE is *exact*. Integrating $\overline{N}$ with respect to y, while holding x constant, results in $\psi(x,y) = e^x sin\,y + 2y\,cos\,x + h(x)$. Now differentiating with respect to x, $\psi_x = e^x sin\,y - 2y\,sin\,x + h'(x)$. Setting $\psi_x = \overline{M}$, we find that $h'(x) = 0$, and hence $h(x) = 0$ is feasible. Hence the solution of the given equation is defined *implicitly* by $e^x sin\,y + 2y\,cos\,x = \beta$.

21. $M_y = 1$ and $N_x = 2$. Multiply both sides by the integrating factor $\mu(x,y) = y$ to obtain $y^2dx + (2xy - y^2e^y)dy = 0$. Let $\overline{M} = yM$ and $\overline{N} = yN$. It is easy to see that $\overline{M}_y = \overline{N}_x$, and hence the latter ODE is *exact*. Integrating $\overline{M}$ with respect to x yields $\psi(x,y) = xy^2 + h(y)$. Equating ψ_y with $\overline{N}$ results in $h'(y) = -y^2e^y$, and hence $h(y) = -e^y(y^2 - 2y + 2)$. Thus $\psi(x,y) = xy^2 - e^y(y^2 - 2y + 2)$, and the solution is defined *implicitly* by $xy^2 - e^y(y^2 - 2y + 2) = c$.

24. The equation $\mu M + \mu N y' = 0$ has an integrating factor if $(\mu M)_y = (\mu N)_x$, that is, $\mu_y M - \mu_x N = \mu N_x - \mu M_y$. Suppose that $N_x - M_y = R\,(xM - yN)$, in which R is some function depending *only* on the quantity $z = xy$. It follows that the modified form of the equation is *exact*, if $\mu_y M - \mu_x N = \mu R\,(xM - yN) = R\,(\mu\,xM - \mu\,yN)$. This relation is satisfied if $\mu_y = (\mu\,x)R$ and $\mu_x = (\mu\,y)R$. Now consider $\mu = \mu(xy)$. Then the partial derivatives are $\mu_x = \mu' y$ and $\mu_y = \mu' x$. Note that $\mu' = d\mu/dz$. Thus μ must satisfy $\mu'(z) = R(z)$. The latter equation is *separable*, with $d\mu = R(z)dz$, and $\mu(z) = \int R(z)dz$. Therefore, given $R = R(xy)$, it is possible to determine $\mu = \mu(xy)$ which becomes an integrating factor of the differential equation.

28. The equation is not exact, since $N_x - M_y = 2y - 1$. However, $(N_x - M_y)/M =$ $= (2y-1)/y$ is a function of y alone. Hence there exists $\mu = \mu(y)$, which is a solution of the differential equation $\mu' = (2 - 1/y)\mu$. The latter equation is *separable*, with $d\mu/\mu = 2 - 1/y$. One solution is $\mu(y) = exp(2y - ln\, y) = e^{2y}/y$. Now rewrite the given ODE as $e^{2y}dx + (2x\, e^{2y} - 1/y)dy = 0$. This equation is *exact*, and it is easy to see that $\psi(x,y) = x\, e^{2y} - ln\, y$. Therefore the solution of the given equation is defined implicitly by $x\, e^{2y} - ln\, y = c$.

30. The given equation is not exact, since $N_x - M_y = 8x^3/y^3 + 6/y^2$. But note that $(N_x - M_y)/M = 2/y$ is a function of y alone, and hence there is an integrating factor $\mu = \mu(y)$. Solving the equation $\mu' = (2/y)\mu$, an integrating factor is $\mu(y) = y^2$. Now rewrite the differential equation as $(4x^3 + 3y)dx + (3x + 4y^3)dy = 0$. By inspection, $\psi(x,y) = x^4 + 3xy + y^4$, and the solution of the given equation is defined implicitly by $x^4 + 3xy + y^4 = c$.

32. Multiplying both sides of the ODE by $\mu = [xy(2x+y)]^{-1}$, the given equation is equivalent to $[(3x+y)/(2x^2+xy)]dx + [(x+y)/(2xy+y^2)]dy = 0$. Rewrite the differential equation as

$$\left[\frac{2}{x} + \frac{2}{2x+y}\right]dx + \left[\frac{1}{y} + \frac{1}{2x+y}\right]dy = 0\,.$$

It is easy to see that $M_y = N_x$. Integrating M with respect to x, while keeping y constant, results in $\psi(x,y) = 2ln|x| + ln|2x+y| + h(y)$. Now taking the partial derivative with respect to y , $\psi_y = (2x+y)^{-1} + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 1/y$, and hence $h(y) = ln\,|y|$. Therefore

$$\psi(x,y) = 2ln|x| + ln|2x+y| + ln\,|y|\,,$$

and the solution of the given equation is defined implicitly by $2x^3y + x^2y^2 = c$.