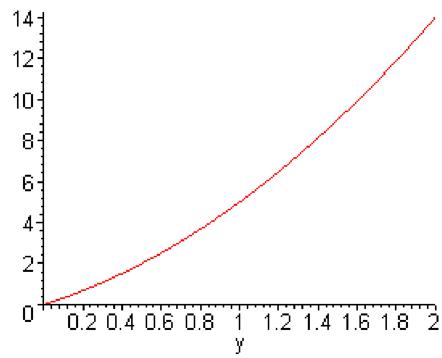
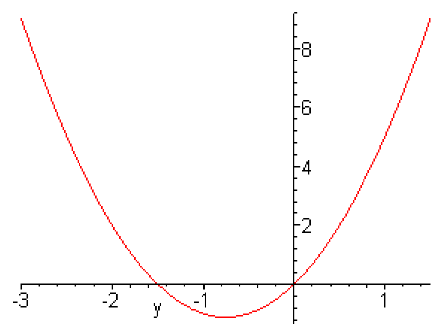
Section 2.5

1.



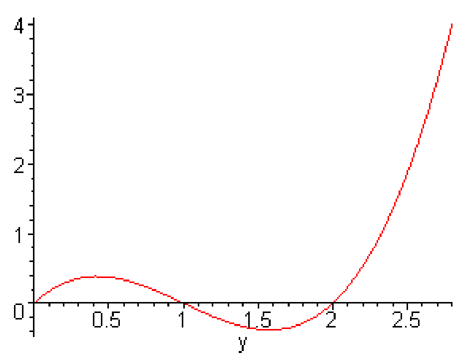
For $y_0 \geq 0$, the only equilibrium point is $y^* = 0$. $f'(0) = a > 0$, hence the equilibrium solution $\phi(t) = 0$ is *unstable*.

2.



The equilibrium points are $y^* = -a/b$ and $y^* = 0$. $f'(-a/b) < 0$, therefore the equilibrium solution $\phi(t) = -a/b$ is *asymptotically stable*.

3.



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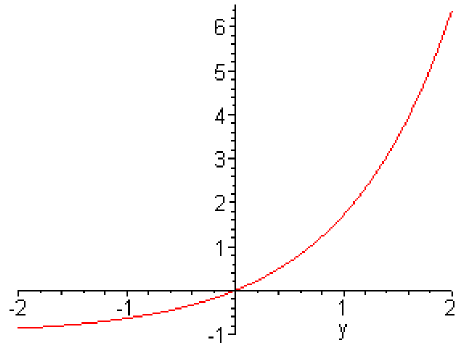
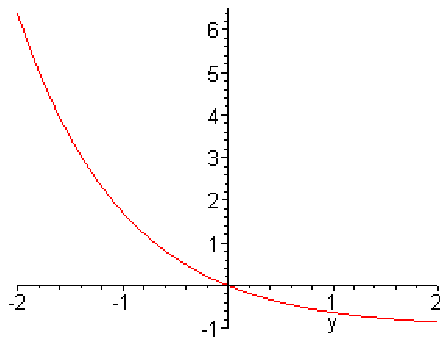


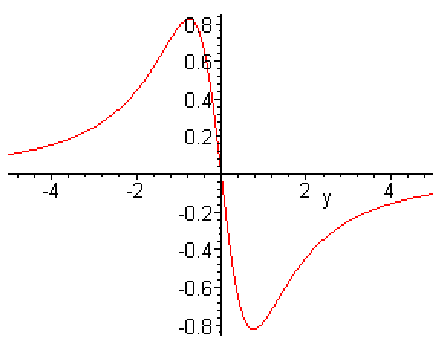
The only equilibrium point is $y^* = 0$. $f'(0) > 0$, hence the equilibrium solution $\phi(t) = 0$
is *unstable*.

5.

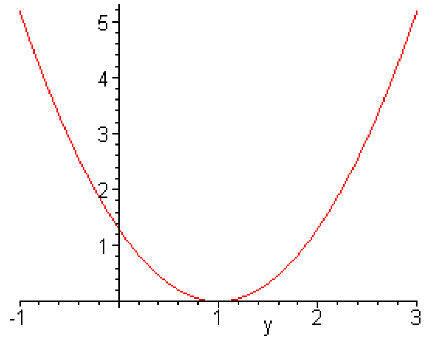


The only equilibrium point is $y^* = 0$. $f'(0) < 0$, hence the equilibrium solution $\phi(t) = 0$
is *asymptotically stable*.

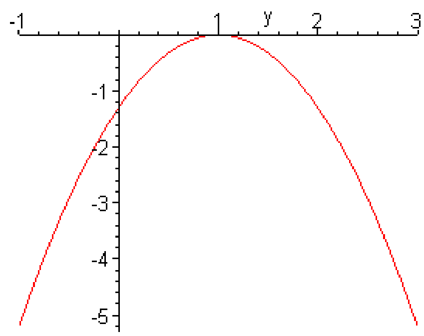
6.



7(b).

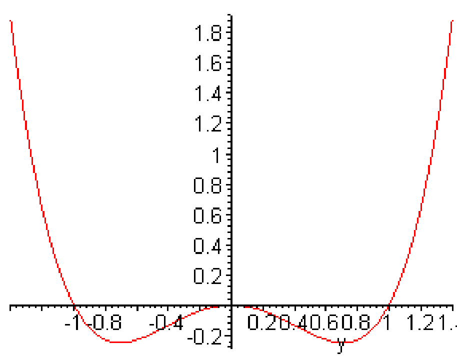


8.

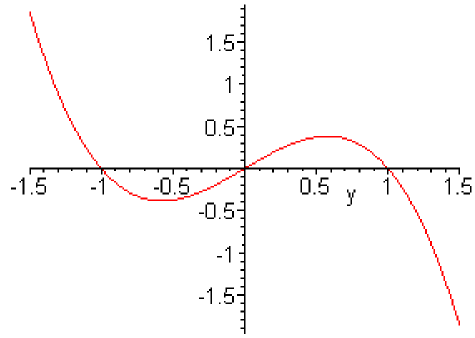


The only equilibrium point is $y^* = 1$. Note that $f'(1) = 0$, and that $y' < 0$ for $y \neq 1$. As long as $y_0 \neq 1$, the corresponding solution is *monotone decreasing*. Hence the equilibrium solution $\phi(t) = 1$ is *semistable*.

9.

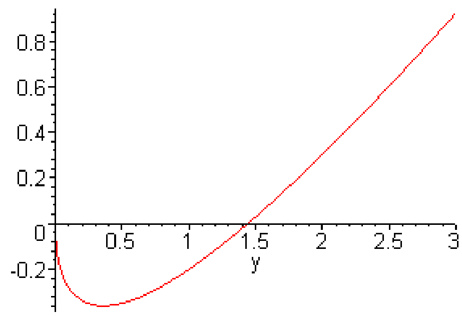


10.

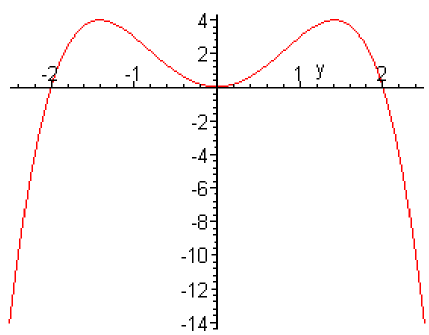


The equilibrium points are $y^* = 0, \pm 1$. $f'(y) = 1 - 3y^2$. The equilibrium solution $\phi(t) = 0$ is *unstable*, and the remaining two are *asymptotically stable*.

11.

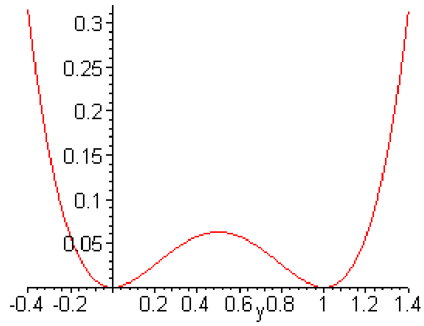


12.



The equilibrium points are $y^* = 0, \pm 2$. $f'(y) = 8y - 4y^3$. The equilibrium solutions $\phi(t) = -2$ and $\phi(t) = +2$ are *unstable* and *asymptotically stable*, respectively. The equilibrium solution $\phi(t) = 0$ is *semistable*.

13.



The equilibrium points are $y^* = 0$ and 1. $f'(y) = 2y - 6y^2 + 4y^3$. Both equilibrium solutions are *semistable*.

$15(a)$. Inverting the Solution (11), Eq. (13) shows t as a function of the population y and
the carrying capacity K. With $y_0 = K/3$,

$$t = -\frac{1}{r} ln \left| \frac{(1/3)[1-(y/K)]}{(y/K)[1-(1/3)]} \right| .$$

Setting $y = 2y_0$,

$$\tau = -\frac{1}{r} ln \left| \frac{(1/3)[1-(2/3)]}{(2/3)[1-(1/3)]} \right| .$$

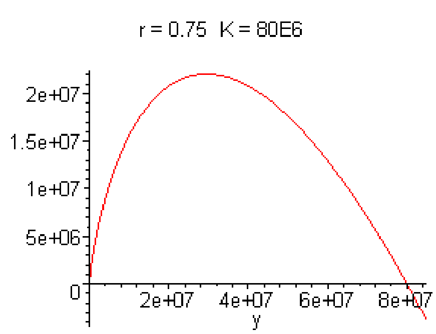
That is, $\tau = \frac{1}{r} ln\, 4$. If $r = 0.025$ *per year*, $\tau = 55.45$ *years*.

(b). In Eq. (13), set $y_0/K = \alpha$ and $y/K = \beta$. As a result, we obtain

$$T = -\frac{1}{r} ln \left| \frac{\alpha[1-\beta]}{\beta[1-\alpha]} \right| .$$

Given $\alpha = 0.1$, $\beta = 0.9$ and $r = 0.025$ *per year*, $\tau = 175.78$ *years*.

$16(a)$.



17. Consider the change of variable $u = ln(y/K)$. Differentiating both sides with respect
to t, $u' = y'/y$. Substitution into the Gompertz equation yields $u' = -ru$, with solution $u = u_0 e^{-rt}$. It follows that $ln(y/K) = ln(y_0/K)e^{-rt}$. That is,

$$\frac{y}{K} = exp\left[ln(y_0/K)e^{-rt}\right].$$

(a). Given $K = 80.5 \times 10^6$, $y_0/K = 0.25$ and $r = 0.71$ *per year*, $y(2) = 57.58 \times 10^6$.

(b). Solving for t,

$$t = -\frac{1}{r}ln\left[\frac{ln(y/K)}{ln(y_0/K)}\right].$$

Setting $y(\tau) = 0.75K$, the corresponding time is $\tau = 2.21$ *years*.

$19(a)$. The rate of *increase* of the volume is given by rate of *flow in* $-$ rate of *flow out*. That is, $dV/dt = k - \alpha a\sqrt{2gh}$. Since the cross section is *constant*, $dV/dt = Adh/dt$. Hence the governing equation is $dh/dt = \left(k - \alpha a\sqrt{2gh}\right)/A$.

(b). Setting $dh/dt = 0$, the equilibrium height is $h_e = \frac{1}{2g}\left(\frac{k}{\alpha a}\right)^2$. Furthermore, since $f'(h_e) < 0$, it follows that the equilibrium height is *asymptotically stable*.

(c). Based on the answer in part(b), the water level will intrinsically tend to approach h_e. Therefore the height of the tank must be *greater* than h_e; that is, $h_e < V/A$.

$22(a)$. The equilibrium points are at $y^* = 0$ and $y^* = 1$. Since $f'(y) = \alpha - 2\alpha y$, the equilibrium solution $\phi = 0$ is *unstable* and the equilibrium solution $\phi = 1$ is *asymptotically stable*.

(b). The ODE is separable, with $[y(1-y)]^{-1}dy = \alpha\, dt$. Integrating both sides and invoking the initial condition, the solution is

$$y(t) = \frac{y_0\, e^{\alpha t}}{1 - y_0 + y_0\, e^{\alpha t}}.$$

It is evident that (independent of y_0) $\lim_{t\to-\infty} y(t) = 0$ and $\lim_{t\to\infty} y(t) = 1$.

$23(a)$. $y(t) = y_0\, e^{-\beta t}$.

(b). From part(a), $dx/dt = \alpha\, x\, y_0\, e^{-\beta t}$. Separating variables, $dx/x = \alpha\, y_0\, e^{-\beta t}dt$. Integrating both sides, the solution is $x(t) = x_0\, exp\left[\alpha\, y_0/\beta\left(1 - e^{-\beta t}\right)\right]$.

(c). As $t\to\infty$, $y(t)\to 0$ and $x(t)\to x_0\, exp(\alpha\, y_0/\beta)$. Over a *long* period of time, the proportion of carriers *vanishes*. Therefore the proportion of the population that escapes the epidemic is the proportion of *susceptibles* left at that time, $x_0\, exp(\alpha\, y_0/\beta)$.

$25(a)$. Note that $f(x) = x[(R - R_c) - a\,x^2]$, and $f'(x) = (R - R_c) - 3a\,x^2$. So if $(R - R_c) < 0$, the only equilibrium point is $x^* = 0$. $f'(0) < 0$, and hence the solution $\phi(t) = 0$ is *asymptotically stable.*

(b). If $(R - R_c) > 0$, there are *three* equilibrium points $x^* = 0, \pm\sqrt{(R - R_c)/a}$. Now $f'(0) > 0$, and $f'\left(\pm\sqrt{(R - R_c)/a}\right) < 0$. Hence the solution $\phi = 0$ is *unstable*, and the solutions $\phi = \pm\sqrt{(R - R_c)/a}$ are *asymptotically stable.*

(c).