Section 5.8

3. Here $x\,p(x) = 1$ and $x^2q(x) = 2x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + 2\sum_{n=0}^{\infty}a_nx^{r+n+1} = 0\,.$$

After adjusting the indices in the *last* series, we obtain

$$a_0[r(r-1)+r]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n + 2\,a_{n-1}]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 = 0$, with *double* root $r = 0$. Setting the remaining coefficients equal to *zero*, we have for $n \geq 1$,

$$a_n(r) = -\frac{2}{(n+r)^2}\,a_{n-1}(r)\,.$$

It follows that

$$a_n(r) = \frac{(-1)^n\,2^n}{[(n+r)(n+r-1)\cdots(1+r)]^2}\,a_0\,,\quad n \geq 1\,.$$

Since $r = 0$, one solution is given by

$$y_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\,2^n}{(n!)^2}\,x^n\,.$$

For a second linearly independent solution, we follow the discussion in Section 5.7 .
First
note that

$$\frac{a_n'(r)}{a_n(r)} = -2\left[\frac{1}{n+r} + \frac{1}{n+r-1} + \cdots + \frac{1}{1+r}\right].$$

Setting $r = 0$,

$$a_n'(0) = -2\,H_n\,a_n(0) = -2\,H_n\frac{(-1)^n\,2^n}{(n!)^2}\,.$$

Therefore,

$$y_2(x) = y_1(x)\,ln\,x - 2\sum_{n=0}^{\infty}\frac{(-1)^n\,2^nH_n}{(n!)^2}\,x^n\,.$$

4. Here $x\,p(x) = 4$ and $x^2q(x) = 2 + x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n}+\;4\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}+$$
$$+\sum_{n=0}^{\infty}a_nx^{r+n+1}+2\sum_{n=0}^{\infty}a_nx^{r+n}=0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1)+4r+2]x^r+\sum_{n=1}^{\infty}\big[(r+n)(r+n-1)a_n+4(r+n)a_n+2\,a_n+a_{n-1}\big]x^{r+n}=0.$$

Assuming $a_0\neq 0$, the *indicial equation* is $r^2+3r+2=0$, with roots $r_1=-1$ and $r_2=-2$. Setting the remaining coefficients equal to *zero*, we have for $n\geq 1$,

$$a_n(r)=-\frac{1}{(n+r+1)(n+r+2)}\,a_{n-1}(r)\,.$$

It follows that

$$a_n(r)=\frac{(-1)^n}{[(n+r+1)(n+r)\cdots(2+r)][(n+r+2)(n+r)\cdots(3+r)]}\,a_0\,,\;n\geq 1\,.$$

Since $r_1=-1$, one solution is given by

$$y_1(x)=x^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n)!(n+1)!}\,x^n\,.$$

For a second linearly independent solution, we follow the discussion in Section 5.7 . Since $r_1-r_2=N=1$, we find that

$$a_1(r)=-\frac{1}{(r+2)(r+3)}\,,$$

with $a_0=1$. Hence the leading coefficient in the solution is

$$a=\lim_{r\to-2}(r+2)\,a_1(r)=-1\,.$$

Further,

$$(r+2)\,a_n(r)=\frac{(-1)^n}{(n+r+2)\,[(n+r+1)(n+r)\cdots(3+r)]^2}\,.$$

Let $A_n(r)=(r+2)\,a_n(r)$. It follows that

$$\frac{A_n'(r)}{A_n(r)}=-\frac{1}{n+r+2}-2\left[\frac{1}{n+r+1}+\frac{1}{n+r}+\cdots+\frac{1}{3+r}\right].$$

Setting $r=r_2=-2$,

$$\begin{aligned}\frac{A_n'(-2)}{A_n(-2)} &= -\frac{1}{n} - 2\left[\frac{1}{n-1} + \frac{1}{n-2} + \cdots + 1\right] \\ &= -H_n - H_{n-1}\,.\end{aligned}$$

Hence

$$\begin{aligned}c_n(-2) &= -(H_n + H_{n-1})\,A_n(-2) \\ &= -(H_n + H_{n-1})\frac{(-1)^n}{n!(n-1)!}\,.\end{aligned}$$

Therefore,

$$y_2(x) = -y_1(x)\,ln\,x + x^{-2}\left[1 - \sum_{n=1}^{\infty}\frac{(-1)^n(H_n + H_{n-1})}{n!(n-1)!}\,x^n\right].$$

6. Let $y(x) = v(x)/\sqrt{x}$. Then $y' = x^{-1/2}\,v' - x^{-3/2}\,v/2$ and $y'' = x^{-1/2}\,v'' -$ $-\,x^{-3/2}\,v' + 3\,x^{-5/2}\,v/4$. Substitution into the ODE results in

$$\left[x^{3/2}\,v'' - x^{1/2}\,v' + 3\,x^{-1/2}\,v/4\right] + \left[x^{1/2}\,v' - x^{-1/2}\,v/2\right] + \left(x^2 - \frac{1}{4}\right)x^{-1/2}\,v = 0\,.$$

Simplifying, we find that

$$v'' + v = 0\,,$$

with *general solution* $v(x) = c_1 cos\,x + c_2\,sin\,x$. Hence

$$y(x) = c_1 x^{-1/2} cos\,x + c_2\,x^{-1/2} sin\,x\,.$$

8. The absolute value of the ratio of consecutive terms is

$$\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \frac{|x|^{2m+2}\,2^{2m}(m+1)!\,m!}{|x|^{2m}\,2^{2m+2}(m+2)!(m+1)!} = \frac{|x|^2}{4(m+2)(m+1)}\,.$$

Applying the *ratio test*,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \lim_{m\to\infty}\,\frac{|x|^2}{4(m+2)(m+1)} = 0\,.$$

Hence the series for $J_1(x)$ converges absolutely *for all* values of x . Furthermore, since the series for $J_0(x)$ also converges absolutely for all x, term-by-term differentiation results in

$$\begin{aligned} J_0'(x) &= \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1}\, m!(m-1)!} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, x^{2m+1}}{2^{2m+1}(m+1)!\, m!} \\ &= -\frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{2^{2m}(m+1)!\, m!}\,. \end{aligned}$$

Therefore, $J_0'(x) = -J_1(x)$.

$9(a)$. Note that $x\,p(x) = 1$ and $x^2 q(x) = x^2 - \nu^2$, which are *both* analytic at $x = 0$. Thus $x = 0$ is a *regular* singular point. Furthermore, $p_0 = 1$ and $q_0 = -\nu^2$. Hence the *indicial equation* is $r^2 - \nu^2 = 0$, with roots $r_1 = \nu$ and $r_2 = -\nu$.

(b). Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty} (r+n)a_n\, x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0\left[r(r-1) + r - \nu^2\right]x^r + a_1\left[(r+1)r + (r+1) - \nu^2\right] + \\ + \sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + (r+n)a_n - \nu^2 a_n + a_{n-2}\right]x^{r+n} = 0.$$

Setting the coefficients equal to *zero*, we find that $a_1 = 0$, and

$$a_n = \frac{-1}{(r+n)^2 - \nu^2}\, a_{n-2}\,,$$

for $n \geq 2$. It follows that $a_3 = a_5 = \cdots = a_{2m+1} = \cdots = 0$. Furthermore, with $r = \nu$,

$$a_n = \frac{-1}{n(n+2\nu)}\, a_{n-2}\,.$$

So for $m = 1, 2, \cdots$,

$$\begin{aligned} a_{2m} &= \frac{-1}{2m(2m+2\nu)}\, a_{2m-2} \\ &= \frac{(-1)^m}{2^{2m}\, m!(1+\nu)(2+\nu)\cdots(m-1+\nu)(m+\nu)}\, a_0\,. \end{aligned}$$

Hence one solution is

$$y_1(x) = x^{\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\cdots(m-1+\nu)(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

(c). Assuming that $r_1 - r_2 = 2\nu$ is *not* an integer, simply setting $r = -\nu$ in the above results in a second *linearly independent* solution

$$y_2(x) = x^{-\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\cdots(m-1-\nu)(m-\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

(d). The absolute value of the ratio of consecutive terms in $y_1(x)$ is

$$\begin{aligned}\left|\frac{a_{2m+2}\, x^{2m+2}}{a_{2m}\, x^{2m}}\right| &= \frac{|x|^{2m+2}\, 2^{2m}\, m!(1+v)\cdots(m+\nu)}{|x|^{2m}\, 2^{2m+2}(m+1)!(1+v)\cdots(m+1+\nu)} \\ &= \frac{|x|^2}{4(m+1)(m+1+\nu)}\,.\end{aligned}$$

Applying the *ratio test*,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\, x^{2m+2}}{a_{2m}\, x^{2m}}\right| = \lim_{m\to\infty} \frac{|x|^2}{4(m+1)(m+1+\nu)} = 0\,.$$

Hence the series for $y_1(x)$ converges absolutely *for all* values of x. The same can be shown for $y_2(x)$. Note also, that if ν is a *positive* integer, then the coefficients in the series for $y_2(x)$ are *undefined*.

$10(a)$. It suffices to calculate $L[J_0(x)\, ln\, x]$. Indeed,

$$[J_0(x)\, ln\, x]' = J_0'(x)\, ln\, x + \frac{J_0(x)}{x}$$

and

$$[J_0(x)\, ln\, x]'' = J_0''(x)\, ln\, x + 2\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2}\,.$$

Hence

$$\begin{aligned}L[J_0(x)\, ln\, x] &= x^2 J_0''(x)\, ln\, x + 2x\, J_0'(x) - J_0(x) + \\ &\quad + x\, J_0'(x)\, ln\, x + J_0(x) + x^2 J_0(x)\, ln\, x\,.\end{aligned}$$

Since $x^2 J_0''(x) + x\, J_0'(x) + x^2 J_0(x) = 0$,

$$L[J_0(x)\, ln\, x] = 2x\, J_0'(x)\,.$$

(b). Given that $L[y_2(x)] = 0$, after adjusting the indices in Part (a), we have

$$b_1x + 2^2b_2\,x^2 + \sum_{n=3}^{\infty}\big(n^2b_n + b_{n-2}\big)x^n = \,-\,2x\,J_0'(x)\,.$$

Using the series representation of $J_0'(x)$ in Problem 8,

$$b_1x + 2^2b_2\,x^2 + \sum_{n=3}^{\infty}\big(n^2b_n + b_{n-2}\big)x^n = \,-\,2\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n}}{2^{2n}(n!)^2}\,.$$

(c). Equating the coefficients on both sides of the equation, we find that

$$b_1 = b_3 = \cdots = b_{2m+1} = \cdots = 0\,.$$

Also, with $n = 1$, $2^2b_2 = 1/(1!)^2$, that is, $b_2 = 1/\big[2^2(1!)^2\big]$. Furthermore, for $m \geq 2$,

$$(2m)^2b_{2m} + b_{2m-2} = \,-\,2\frac{(-1)^m(2m)}{2^{2m}(m!)^2}\,.$$

More explicitly,

$$\begin{aligned} b_4 &= -\,\frac{1}{2^2\,4^2}\left(1+\frac{1}{2}\right) \\ b_6 &= \frac{1}{2^2\,4^2\,6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right) \\ &\;\vdots \end{aligned}$$

It can be shown, in general, that

$$b_{2m} = (-1)^{m+1}\frac{H_m}{2^{2m}(m!)^2}\,.$$

11. Bessel's equation of *order one* is

$$x^2\,y'' + x\,y' + \big(x^2-1\big)y = 0\,.$$

Based on Problem 9, the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \\ + \sum_{n=0}^{\infty}a_nx^{r+n+2} - \sum_{n=0}^{\infty}a_nx^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1)+r-1]x^r + a_1[(r+1)r+(r+1)-1] +$$
$$+\sum_{n=2}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n - a_n + a_{n-2}]x^{r+n} = 0.$$

Setting the coefficients equal to *zero*, we find that $a_1 = 0$, and

$$\begin{aligned} a_n(r) &= \frac{-1}{(r+n)^2-1}\,a_{n-2}(r) \\ &= \frac{-1}{(n+r+1)(n+r-1)}\,a_{n-2}(r)\,, \end{aligned}$$

for $n \geq 2$. It follows that $a_3 = a_5 = \cdots = a_{2m+1} = \cdots = 0$. Solving the recurrence relation,

$$a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2\cdots(r+3)^2(r+1)}\,a_0.$$

With $r = r_1 = 1$,

$$a_{2m}(1) = \frac{(-1)^m}{2^{2m}(m+1)!\,m!}\,a_0.$$

For a *second* linearly independent solution, we follow the discussion in Section 5.7 . Since $r_1 - r_2 = N = 2$, we find that

$$a_2(r) = -\frac{1}{(r+3)(r+1)}\,,$$

with $a_0 = 1$. Hence the leading coefficient in the solution is

$$a = \lim_{r\to -1}(r+1)\,a_2(r) = -\frac{1}{2}\,.$$

Further,

$$(r+1)\,a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)\,[(2m+r-1)\cdots(3+r)]^2}\,.$$

Let $A_n(r) = (r+1)\,a_n(r)$. It follows that

$$\frac{A'_{2m}(r)}{A_{2m}(r)} = -\frac{1}{2m+r+1} - 2\left[\frac{1}{2m+r-1} + \cdots + \frac{1}{3+r}\right].$$

Setting $r = r_2 = -1$, we calculate

$$\begin{aligned} c_{2m}(-1) &= -\frac{1}{2}(H_m + H_{m-1})A_{2m}(-1) \\ &= -\frac{1}{2}(H_m + H_{m-1})\frac{(-1)^m}{2m\,[(2m-2)\cdots 2]^2} \\ &= -\frac{1}{2}(H_m + H_{m-1})\frac{(-1)^m}{2^{2m-1}\,m!(m-1)!}\,. \end{aligned}$$

Note that $a_{2m+1}(r) = 0$ implies that $A_{2m+1}(r) = 0$, so

$$c_{2m+1}(-1) = \left[\frac{d}{dr}A_{2m+1}(r)\right]_{r=r_2} = 0\,.$$

Therefore,

$$y_2(x) = -\frac{1}{2}\left[x\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)!\,m!}\left(\frac{x}{2}\right)^{2m}\right] ln\, x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty}\frac{(-1)^m(H_m + H_{m-1})}{m!(m-1)!}\left(\frac{x}{2}\right)^{2m}\right].$$

Based on the definition of $J_1(x)$,

$$y_2(x) = -J_1(x)\, ln\, x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty}\frac{(-1)^m(H_m + H_{m-1})}{m!(m-1)!}\left(\frac{x}{2}\right)^{2m}\right].$$

12. Consider a solution of the form

$$y(x) = \sqrt{x}\; f\big(\alpha x^\beta\big).$$

Then

$$y' = \frac{df}{d\xi}\cdot\frac{\alpha\beta\, x^\beta}{\sqrt{x}} + \frac{f(\xi)}{2\sqrt{x}}$$

in which $\xi = \alpha x^\beta$. Hence

$$y'' = \frac{d^2 f}{d\xi^2}\cdot\frac{\alpha^2\beta^2\, x^{2\beta}}{x\sqrt{x}} + \frac{df}{d\xi}\cdot\frac{\alpha\beta^2\, x^\beta}{x\sqrt{x}} - \frac{f(\xi)}{4x\sqrt{x}}\,,$$

and

$$x^2\, y'' = \alpha^2\beta^2\, x^{2\beta}\sqrt{x}\,\frac{d^2 f}{d\xi^2} + \alpha\beta^2\, x^\beta\sqrt{x}\,\frac{df}{d\xi} - \frac{1}{4}\sqrt{x}\, f(\xi)\,.$$

Substitution into the ODE results in

$$\alpha^2\beta^2\, x^{2\beta}\,\frac{d^2 f}{d\xi^2} + \alpha\beta^2\, x^\beta\,\frac{df}{d\xi} - \frac{1}{4}\, f(\xi) + \left(\alpha^2\beta^2\, x^{2\beta} + \frac{1}{4} - \nu^2\beta^2\right) f(\xi) = 0\,.$$

Simplifying, and setting $\xi = \alpha x^\beta$, we find that

$$\xi^2 \frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} + \left(\xi^2 - \nu^2\right) f(\xi) = 0\,, \quad (*)$$

which is a *Bessel* equation of *order* ν. Therefore, the general solution of the given ODE is

$$y(x) = \sqrt{x}\,\left[c_1\, f_1\left(\alpha x^\beta\right) + c_2\, f_2\left(\alpha x^\beta\right)\right],$$

in which $f_1(\xi)$ and $f_2(\xi)$ are the linearly independent solutions of $(*)$.