Section 2.9

1. Writing the equation for each $n \geq 0$, $y_1 = -0.9\,y_0$, $y_2 = -0.9\,y_1$, $y_3 = -0.9\,y_2$ and so on, it is apparent that $y_n = (-0.9)^n\,y_0$. The terms constitute an *alternating series*, which converge to *zero*, regardless of y_0.

3. Write the equation for each $n \geq 0$, $y_1 = \sqrt{3}\,y_0$, $y_2 = \sqrt{4/2}\,y_1$, $y_3 = \sqrt{5/3}\,y_2, \cdots$. Upon substitution, we find that $y_2 = \sqrt{(4\cdot 3)/2}\,y_1$, $y_3 = \sqrt{(5\cdot 4\cdot 3)/(3\cdot 2)}\,y_0, \cdots$. It can be proved by mathematical induction, that

$$\begin{aligned} y_n &= \frac{1}{\sqrt{2}}\sqrt{\frac{(n+2)!}{n!}}\,y_0 \\ &= \frac{1}{\sqrt{2}}\sqrt{(n+1)(n+2)}\,y_0 . \end{aligned}$$

This sequence is *divergent*, except for $y_0 = 0$.

4. Writing the equation for each $n \geq 0$, $y_1 = -y_0$, $y_2 = y_1$, $y_3 = -y_2$, $y_4 = y_3$, and so on, it can be shown that

$$y_n = \begin{cases} y_0 & , \text{ for } n = 4k \text{ or } n = 4k-1 \\ -y_0 & , \text{ for } n = 4k-2 \text{ or } n = 4k-3 \end{cases}$$

The sequence is convergent *only* for $y_0 = 0$.

6. Writing the equation for each $n \geq 0$,

$$\begin{aligned} y_1 &= 0.5\,y_0 + 6 \\ y_2 &= 0.5\,y_1 + 6 = 0.5(0.5\,y_0 + 6) + 6 = (0.5)^2 y_0 + 6 + (0.5)6 \\ y_3 &= 0.5\,y_2 + 6 = 0.5(0.5\,y_1 + 6) + 6 = (0.5)^3 y_0 + 6[1 + (0.5) + (0.5)^2] \\ &\vdots \\ y_n &= (0.5)^n y_0 + 12[1 - (0.5)^n\,] \end{aligned}$$

which can be verified by mathematical induction. The sequence is convergent for all y_0, and in fact $y_n \to 12$.

7. Let y_n be the balance at the end of the *n-th* day. Then $y_{n+1} = (1 + r/356)\,y_n$. The solution of this difference equation is $y_n = (1 + r/365)^n\,y_0$, in which y_0 is the initial balance. At the end of *one year*, the balance is $y_{365} = (1 + r/365)^{365}\,y_0$. Given that $r = .07$, $y_{365} = (1 + r/365)^{365}\,y_0 = 1.0725\,y_0$. Hence the effective annual yield is $(1.0725\,y_0 - y_0)/y_0 = 7.25\,\%$.

8. Let y_n be the balance at the end of the *n-th* month. Then $y_{n+1} = (1 + r/12)\,y_n + 25$. As in the previous solutions, we have

$$y_n = \rho^n \left[y_0 - \frac{25}{1-\rho} \right] + \frac{25}{1-\rho} \,,$$

in which $\rho = (1 + r/12)$. Here r is the annual interest rate, given as $8\,\%$. Therefore $y_{36} = (1.0066)^{36} \left[1000 + \frac{(12)25}{r}\right] - \frac{(12)25}{r} = 2,283.63$ *dollars*.

9. Let y_n be the balance due at the end of the n-*th* month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$. Here r is the annual interest rate and P is the monthly payment. The solution, in terms of the amount borrowed, is given by

$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,,$$

in which $\rho = (1 + r/12)$ and $y_0 = 8,000$. To figure out the monthly payment, P, we require that $y_{36} = 0$. That is,

$$\rho^{36} \left[y_0 + \frac{P}{1-\rho} \right] = \frac{P}{1-\rho} \,.$$

After the specified amounts are substituted, we find the $P = \$\,258.14$.

11. Let y_n be the balance due at the end of the n-*th* month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = .09$ and P is the monthly payment. The initial value of the mortgage is $y_0 = 100,000$ *dollars*. Then the balance due at the end of the n-*th* month is

$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,.$$

where $\rho = (1 + r/12)$. In terms of the specified values,

$$y_n = (0.0075)^n \left[10^5 - \frac{12P}{r} \right] + \frac{12P}{r} \,.$$

Setting $n = 30(12) = 360$, and $y_{360} = 0$, we find that $P = 804.62$ *dollars*. For the monthly payment corresponding to a 20 year mortgage, set $n = 240$ and $y_{240} = 0$.

12. Let y_n be the balance due at the end of the n-*th* month, with y_0 the initial value of the mortgage. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = 0.1$ and $P = 900$ *dollars* is the *maximum* monthly payment. Given that the life of the mortgage is 20 *years*, we require that $y_{240} = 0$. The balance due at the end of the n-*th* month is

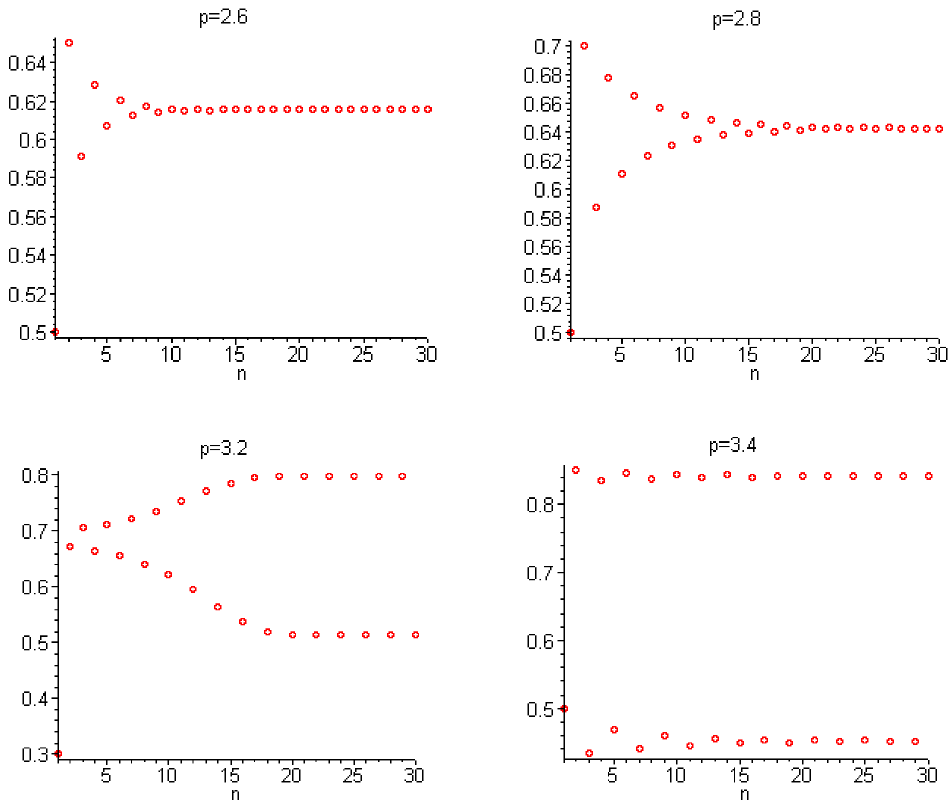
$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,.$$

In terms of the specified values for the parameters, the solution of

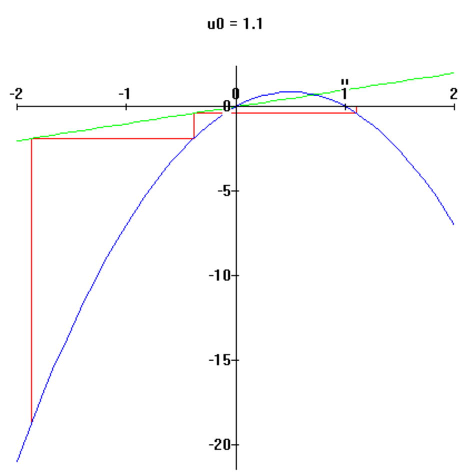
$$(.00833)^{240}\left[y_0 - \frac{12(1000)}{0.1}\right] = -\frac{12(1000)}{0.1}$$

is $y_0 = 103,624.62$ *dollars*.

15.



16. For example, take $\rho = 3.5$ and $u_0 = 1.1$:

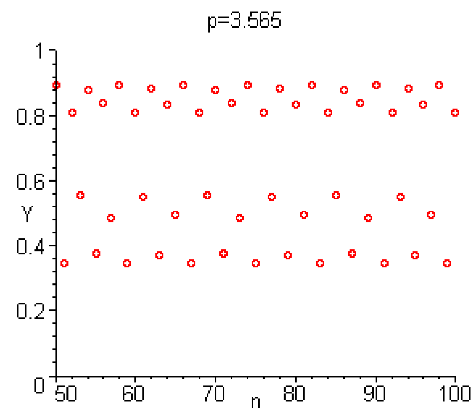


$19(a)$. $\delta_2 = (\rho_2 - \rho_1)/(\rho_3 - \rho_2) = (3.449 - 3)/(3.544 - 3.449) = 4.7263$.

(b). $\%$ *diff* $= \frac{|\delta - \delta_2|}{\delta} \times 100 = \frac{|4.6692 - 4.7363|}{4.6692} \times 100 \approx 1.22\ \%$.

(c). Assuming $(\rho_3 - \rho_2)/(\rho_4 - \rho_3) = \delta$, $\rho_4 \approx 3.5643$

(d). A period 16 solutions appears near $\rho \approx 3.565$.



(e). Note that $(\rho_{n+1} - \rho_n) = \delta_n^{-1}(\rho_n - \rho_{n-1})$. With the assumption that $\delta_n = \delta$, we have $(\rho_{n+1} - \rho_n) = \delta^{-1}(\rho_n - \rho_{n-1})$, which is of the form $y_{n+1} = \alpha\, y_n$, $n \geq 3$. It follows that $(\rho_k - \rho_{k-1}) = \delta^{3-k}(\rho_3 - \rho_2)$ for $k \geq 4$. Then

$$\begin{aligned}
\rho_n &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + (\rho_4 - \rho_3) + \cdots + (\rho_n - \rho_{n-1}) \\
&= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)[1 + \delta^{-1} + \delta^{-2} + \cdots + \delta^{3-n}] \\
&= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[\frac{1 - \delta^{4-n}}{1 - \delta^{-1}}\right].
\end{aligned}$$

Hence $\lim_{n\to\infty} \rho_n = \rho_2 + (\rho_3 - \rho_2)\left[\frac{\delta}{\delta - 1}\right]$. Substitution of the appropriate values yields

$$\lim_{n\to\infty} \rho_n = 3.5699$$

Miscellaneous Problems

1. Linear $[\,y = c/x^2 + x^3/5\,]$.
2. Homogeneous $[arctan(y/x) - ln\sqrt{x^2+y^2} = c\,]$.
3. Exact $[\,x^2 + xy - 3y - y^3 = 0\,]$.
4. Linear in $x(y)$ $[\,x = c\,e^y + y\,e^y\,]$.
5. Exact $[\,x^2y + xy^2 + x = c\,]$.
6. Linear $[\,y = x^{-1}(1 - e^{1-x})\,]$.
7. Let $u = x^2$ $[\,x^2 + y^2 + 1 = c\,e^{y^2}\,]$.
8. Linear $[\,y = (4 + cos\,2 - cos\,x)/x^2\,]$.
9. Exact $[\,x^2y + x + y^2 = c\,]$.
10. $\mu = \mu(x)$ $[\,y^2/x^3 + y/x^2 = c\,]$.
11. Exact $[\,x^3/3 + xy + e^y = c\,]$.
12. Linear $[\,y = c\,e^{-x} + e^{-x}ln(1 + e^x)\,]$.
13. Homogeneous $[\,2\sqrt{y/x} - ln\,|x| = c\,]$.
14. Exact/Homogeneous $[\,x^2 + 2xy + 2y^2 = 34\,]$.
15. Separable $[\,y = c/cosh^2(x/2)\,]$.
16. Homogeneous $[\,\left(2/\sqrt{3}\right)arctan\left[(2y-x)/\sqrt{3}\,x\right] - ln\,|x| = c\,]$.
17. Linear $[\,y = c\,e^{3x} - e^{2x}\,]$.
18. Linear/Homogeneous $[\,y = c\,x^{-2} - x\,]$.
19. $\mu = \mu(x)$ $[\,3y - 2xy^3 - 10x = 0\,]$.
20. Separable $[\,e^x + e^{-y} = c\,]$.
21. Homogeneous $[\,e^{-y/x} + ln\,|x| = c\,]$.
22. Separable $[\,y^3 + 3y - x^3 + 3x = 2\,]$.
23. Bernoulli $[\,1/y = -x\int x^{-2}e^{2x}\,dx + cx\,]$.
24. Separable $[\,sin^2x\,sin\,y = c\,]$.
25. Exact $[\,x^2/y + arctan(y/x) = c\,]$.
26. $\mu = \mu(x)$ $[\,x^2 + 2x^2y - y^2 = c\,]$.
27. $\mu = \mu(x)$ $[\,sin\,x\,cos\,2y - \frac{1}{2}\,sin^2x = c\,]$.
28. Exact $[\,2xy + xy^3 - x^3 = c\,]$.
29. Homogeneous $[\,arcsin(y/x) - ln\,|x| = c\,]$.
30. Linear in $x(y)$ $[\,xy^2 - ln\,|y| = 0\,]$.
31. Separable $[\,x + ln\,|x| + x^{-1} + y - 2\,ln\,|y| = c\,]$.
32. $\mu = \mu(y)$ $[\,x^3y^2 + xy^3 = -4\,]$.