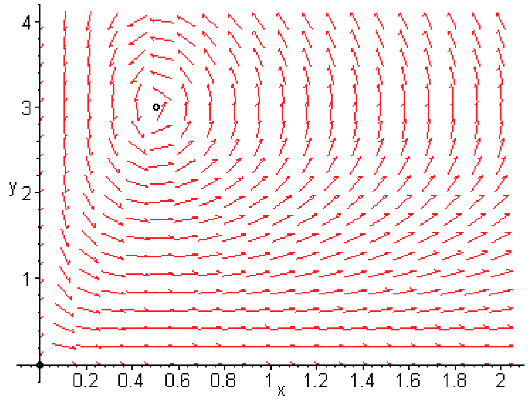
Section 9.5

$1(a)$.



(b). The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,y) = 0$$
$$y(\,-0.5 + x) = 0\,.$$

The two critical points are $(0\,,0)$ and $(0.5\,,3)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - y/2 & -\,x/2 \\ y & -\,1/2 + x \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & -\,1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ \ r_2 = \,-\,1/2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.

At the critical point $(0.5\,,3)$, the coefficient matrix of the linearized system is

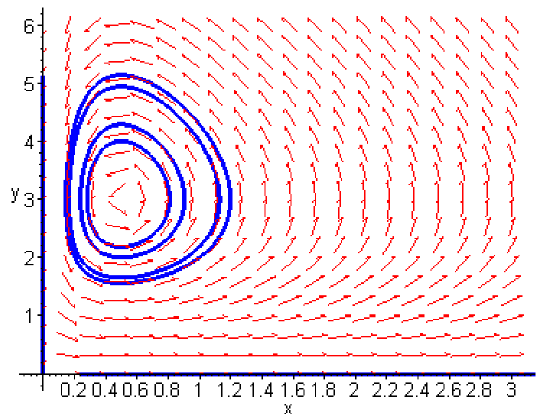
$$\mathbf{J}(0.5\,,3) = \begin{pmatrix} 0 & -\,1/4 \\ 3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i\,\frac{\sqrt{3}}{2}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\,2\,i\sqrt{3} \end{pmatrix}\ ;\ \ r_2 = \,-\,i\,\frac{\sqrt{3}}{2}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2\,i\sqrt{3} \end{pmatrix}.$$

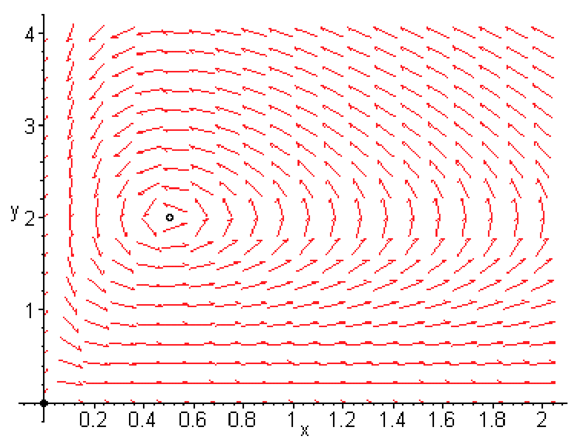
The eigenvalues are purely imaginary. Hence the critical point is a *center*, which is *stable*.

(d, e).



(f). Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5\,,3)$.

$2(a)$.



(b). The critical points are the solution set of the system of equations

$$x(1 - 0.5\,y) = 0$$
$$y(-0.25 + 0.5\,x) = 0\,.$$

The two critical points are $(0\,,0)$ and $(0.5\,,2)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 - y/2 & -x/2 \\ y/2 & -1/4 + x/2 \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & -\,1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,;\ \ r_2 = -\,1/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.

At the critical point $(0.5\,,2)$, the coefficient matrix of the linearized system is

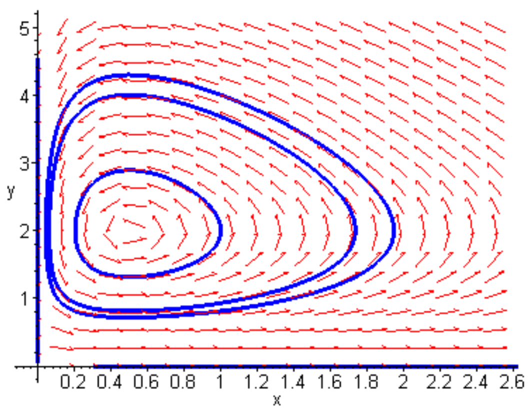
$$\mathbf{J}(0.5\,,2) = \begin{pmatrix} 0 & -\,1/4 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\,2\,i \end{pmatrix}\,;\ \ r_2 = -\,i/2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2\,i \end{pmatrix}.$$

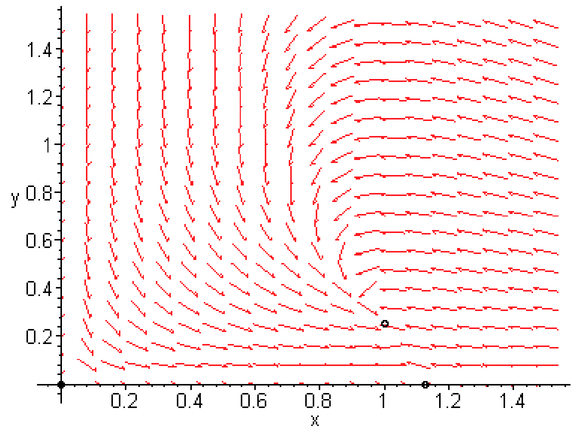
The eigenvalues are purely imaginary. Hence the critical point is a *center*, which is *stable*.

(d, e).



(f). Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5\,,2)$.

$4(a)$.



(b). The critical points are the solution set of the system of equations

$$x(9/8 - x - y/2) = 0$$
$$y(-1 + x) = 0\,.$$

The three critical points are $(0\,,0)$, $(9/8\,,0)$ and $(1\,,1/4)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 9/8 - 2x - y/2 & -x/2 \\ y & -1 + x \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 9/8 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 9/8\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,;\ \ r_2 = -1\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.

At the critical point $(9/8\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(9/8\,,0) = \begin{pmatrix} -9/8 & -9/16 \\ 0 & 1/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{9}{8}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,;\ \ r_2 = \frac{1}{8}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 9 \\ -20 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(9/8\,,0)$ is a *saddle*, which is *unstable*.

At the critical point $(1\,,1/4)$, the coefficient matrix of the linearized system is

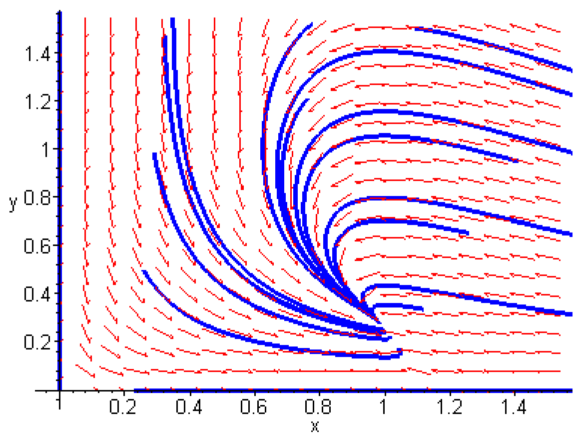
$$\mathbf{J}(1\,,1/4) = \begin{pmatrix} -1 & -1/2 \\ 1/4 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-2+\sqrt{2}}{4}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -2+\sqrt{2} \\ 1 \end{pmatrix}\,;\ r_2 = \frac{-2-\sqrt{2}}{4}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -2-\sqrt{2} \\ 1 \end{pmatrix}.$$

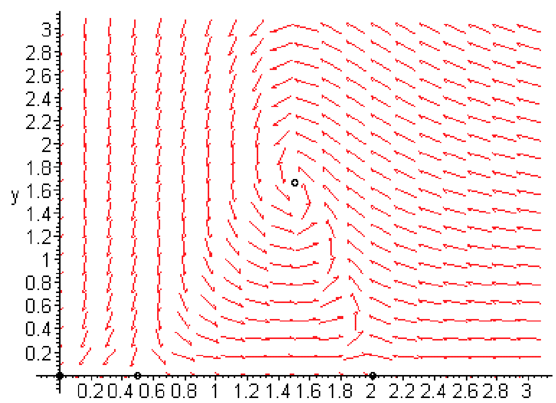
The eigenvalues are both negative. Hence the critical point is a *stable node*, which is *asymptotically stable*.

(d,e).



(f). Except for solutions along the coordinate axes, all solutions converge to the critical point $(1\,,1/4)$.

$5(a)$.



(b). The critical points are solutions of the system of equations

$$x\big(-1+2.5\,x-0.3\,y-x^2\big)=0$$
$$y(-1.5+\,x)=0\,.$$

The four critical points are $(0\,,0)$, $(1/2\,,0)$, $(2\,,0)$ and $(3/2\,,5/3)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix}-1+5x-3x^2-3y/10 & -3x/10\\ y & -3/2+x\end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix}-1 & 0\\ 0 & -3/2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=\,-1\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix};\ \ r_2=\,-3/2\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0\,,0)$ is a *stable node*, which is *asymptotically stable*.

At the critical point $(1/2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2\,,0)=\begin{pmatrix}3/4 & -3/20\\ 0 & -1\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=\frac{3}{4}\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix};\ \ r_2=\,-1\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}3\\35\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(1/2\,,0)$ is a *saddle*, which is *unstable*.

At the critical point $(2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2\,,0)=\begin{pmatrix}-3 & -3/5\\ 0 & 1/2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=\,-3\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix};\ r_2=1/2\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}6\\-35\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(2\,,0)$ is a *saddle*, which is *unstable*.

At the critical point $(3/2\,,5/3)$, the coefficient matrix of the linearized system is

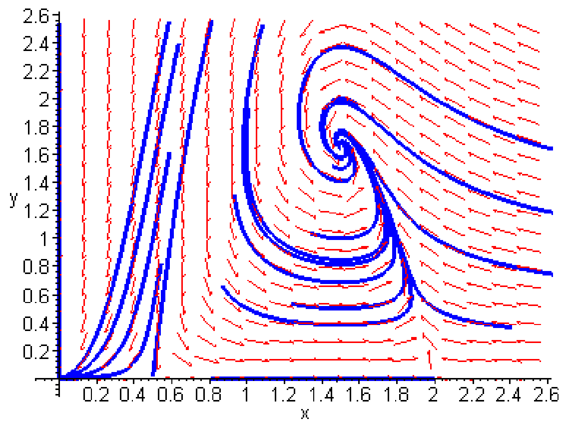
$$\mathbf{J}(3/2\,,5/3) = \begin{pmatrix} -\,3/4 & -\,9/20 \\ 5/3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-\,3 + i\sqrt{39}}{8}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \frac{-9+i\,3\sqrt{39}}{40} \\ 1 \end{pmatrix};\ r_2 = \frac{-\,3 - i\sqrt{39}}{8}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} \frac{-9-i\,3\sqrt{39}}{40} \\ 1 \end{pmatrix}.$$

The eigenvalues are complex conjugates. Hence the critical point $(3/2\,,5/3)$ is a *stable spiral*, which is *asymptotically stable*.

(d,e).



(f). The single solution curve that converges to the node at $(1/2\,,0)$ is a *separatrix.* Except for initial conditions on the coordinate axes, trajectories on either side of the separatrix converge to the node at $(0\,,0)$ or the stable spiral at $(3/2\,,5/3)$.

6. Given that t is measured from the time that x is a *maximum*, we have

$$x = \frac{c}{\gamma} + \frac{cK}{\gamma}\,cos\big(\sqrt{ac}\;t\big)$$
$$y = \frac{a}{\alpha} + K\frac{a}{\alpha}\sqrt{\frac{c}{\alpha}}\;sin\big(\sqrt{ac}\;t\big)\,.$$

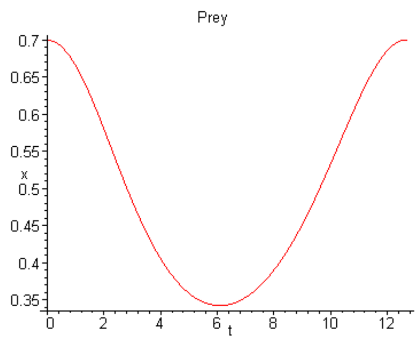
The *period* of oscillation is evidently $T = 2\pi/\sqrt{ac}$. Both populations oscillate about a mean value. The following is based on the properties of the cos and sin functions

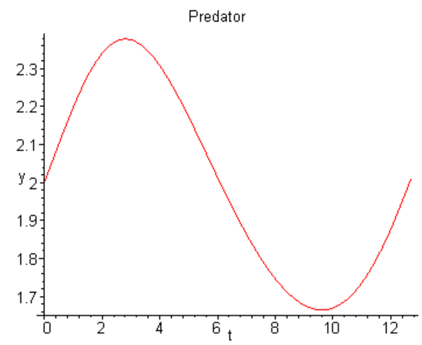
The prey population (x) is *maximum* at $t = 0$ and $t = T$. It is a *minimum* at $t = T/2$. Its rate of increase is greatest at $t = 3\,T/4$. The rate of *decrease* of the prey population is greatest at $t = T/4$.

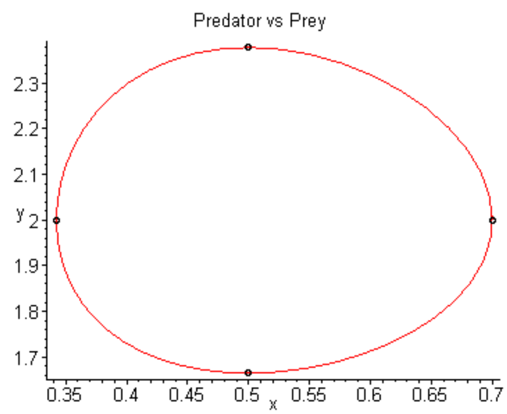
The predator population (y) is *maximum* at $t = T/4$. It is a *minimum* at $t = 3\,T/4$.

The rate of increase of the predator population is greatest at $t = 0$ and $t = T$. The rate of *decrease* of the predator population is greatest at $t = T/2$.

In the following example, the system in Problem 2 is solved numerically with the initial conditions $x(0) = 0.7$ and $y(0) = 2$. The critical point of interest is at $(0.5\,,2)$. Since $a = 1$ and $c = 1/4$, it follows that the period of oscillation is $T = 4\,\pi$.







$8(a)$. The *period* of oscillation for the linear system is $T = 2\pi/\sqrt{ac}$. In system (2), $a = 1$ and $c = 0.75$. Hence the period is estimated as $T = 2\pi/\sqrt{0.75} \approx 7.2552$.

(b). The estimated period appears to agree with the graphic in Figure 9.5.3 .

(c). The critical point of interest is at $(3\,,2)$. The system is solved numerically, with $y(0) = 2$ and $x(0) = 3.5\,,4.0\,,4.5\,,5.0$. The resulting periods are shown in the table:

	$x(0) = 3.5$	$x(0) = 4.0$	$x(0) = 4.5$	$x(0) = 5.0$
T	7.26	7.29	7.34	7.42

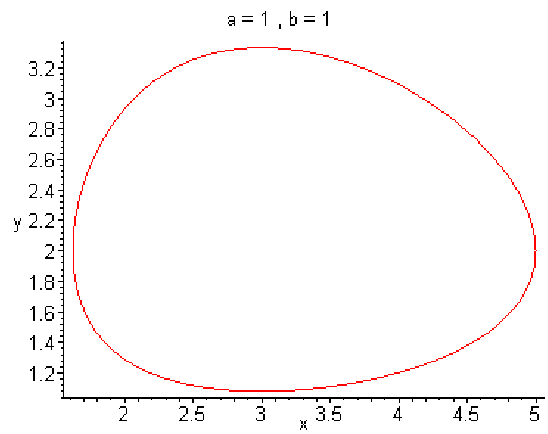
The actual amplitude steadily *increases* as the amplitude increases.

9. The system

$$\frac{dx}{dt} = a\,x\Big(1 - \frac{y}{2}\Big)$$
$$\frac{dy}{dt} = b\,y\Big(-1 + \frac{x}{3}\Big)$$

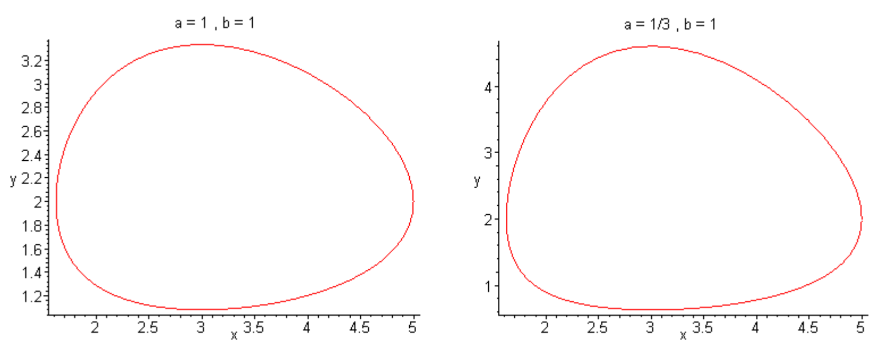
is solved numerically for various values of the parameters. The initial conditions are $x(0) = 5$, $y(0) = 2$.

(a). $a = 1$ and $b = 1$:



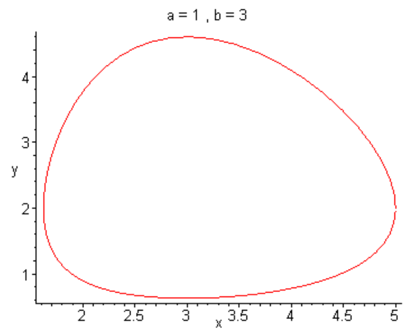
The period is estimated by observing when the trajectory becomes a closed curve. In this case, $T \approx 6.45$.

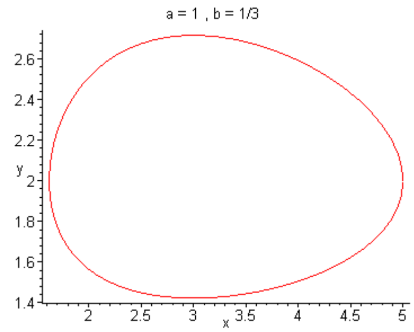
(b). $a = 3$ and $a = 1/3$, with $b = 1$:



For $a = 3$, $T \approx 3.69$. For $a = 1/3$, $T \approx 11.44$.

(c). $b = 3$ and $b = 1/3$, with $a = 1$:





For $b = 3$, $T \approx 3.82$. For $b = 1/3$, $T \approx 11.06$.

(d). It appears that if one of the parameters is fixed, the period varies *inversely* with the other parameter. Hence one might postulate the relation

$$T = \frac{k}{f(a\,,b)}\,.$$

$10(a)$. Since $T = 2\pi/\sqrt{ac}$, we first note that

$$\int_A^{A+T} cos\big(\sqrt{ac}\,t + \phi\big)dt = \int_A^{A+T} sin\big(\sqrt{ac}\,t + \phi\big)dt = 0\,.$$

Hence

$$\overline{x} = \frac{1}{T}\int_A^{A+T}\frac{c}{\gamma}\,dt = \frac{c}{\gamma} \text{ and } \overline{y} = \frac{1}{T}\int_A^{A+T}\frac{a}{\alpha}\,dt = \frac{a}{\alpha}\,.$$

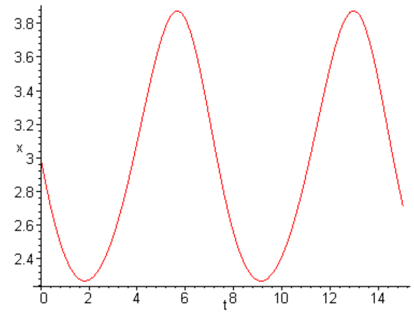
(b). One way to estimate the mean values is to find a horizontal line such that the area above the line is approximately equal to the area under the line. From Figure $9.5.3$, it appears that $\overline{x} \approx 3.25$ and $\overline{y} \approx 2.0$. In Example 1, $a = 1$, $c = 0.75$, $\alpha = 0.5$ and $\gamma = 0.25$. Using the result in Part (a), $\overline{x} = 3$ and $\overline{y} = 2$.

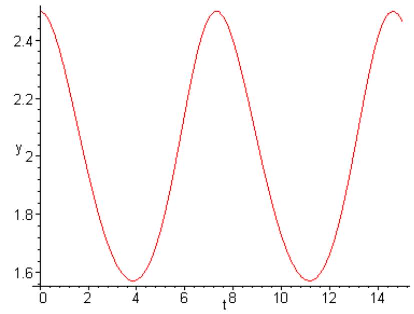
(c). The system

$$\frac{dx}{dt} = x\Big(1 - \frac{y}{2}\Big)$$
$$\frac{dy}{dt} = y\Big(-\frac{3}{4} + \frac{x}{4}\Big)$$

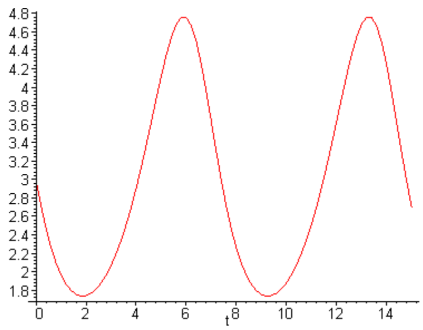
is solved numerically for various initial conditions.

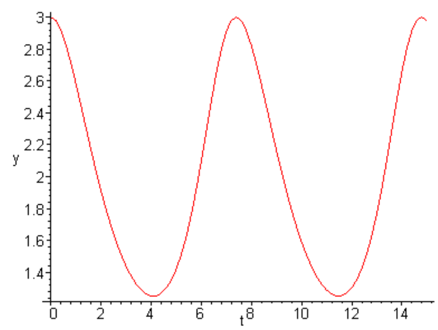
$x(0) = 3$ and $y(0) = 2.5$:



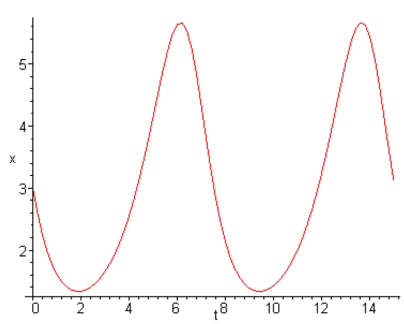


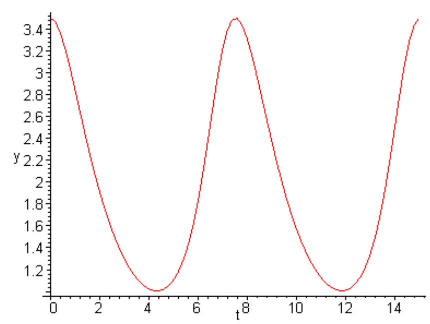
$x(0) = 3$ and $y(0) = 3.0$:



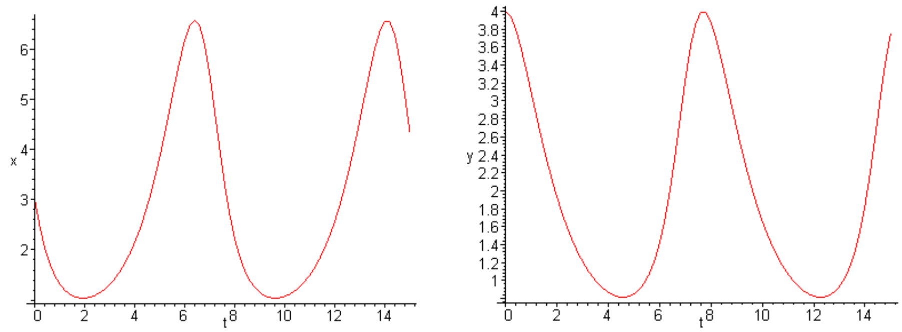


$x(0) = 3$ and $y(0) = 3.5$:





$x(0) = 3$ and $y(0) = 4.0$:



It is evident that the mean values *increase* as the amplitude increases. That is, the mean values increase as the initial conditions move farther from the critical point.

12. The system of equations in model (1) is given by

$$\frac{dx}{dt} = x(a - \alpha\, y)$$
$$\frac{dy}{dt} = y(\,-c + \gamma\, x)\,.$$

Based on the hypothesis, let the *death* rate of the insect population and the predators be $p\,x$ and $q\,y$, respectively. The modified system of equations becomes

$$\frac{dx}{dt} = x(a - \alpha\, y) - p\,x$$
$$\frac{dy}{dt} = y(\,-c + \gamma\, x) - q\,y\,,$$

in which $p > 0\,, q > 0\,.$ The critical points are solutions of the system of equations

$$x(a - p - \alpha\, y) = 0$$
$$y(\,-c - q + \gamma\, x) = 0\,.$$

It is easy to see that the critical points are now at $(0\,,0)$ and $\left(\frac{c+q}{\gamma}\,,\,\frac{a-p}{\alpha}\right)$. Furthermore, since $(c+q)/\gamma > c/\gamma$, the equilibrium level of the insect population has *increased*. On the other hand, since $(a-p)/\alpha < a/\alpha$, equilibrium level of the predators has *decreased*. Indeed, the introduction of insecticide creates a potential to significantly affect the predator population $(a \approx p)$.