Section 4.2

1. The *magnitude* of $1+i$ is $R=\sqrt{2}$ and the *polar angle* is $\pi/4$. Hence the polar form is given by $1+i=\sqrt{2}\,e^{i\pi/4}$.

3. The *magnitude* of -3 is $R=3$ and the *polar angle* is π. Hence $-3=3\,e^{i\pi}$.

4. The *magnitude* of $-i$ is $R=1$ and the *polar angle* is $3\pi/2$. Hence $-i=e^{3\pi i/2}$.

5. The *magnitude* of $\sqrt{3}-i$ is $R=2$ and the *polar angle* is $-\pi/6=11\pi/6$. Hence the polar form is given by $\sqrt{3}-i=2\,e^{11\pi i/6}$.

6. The *magnitude* of $-1-i$ is $R=\sqrt{2}$ and the *polar angle* is $5\pi/4$. Hence the polar form is given by $-1-i=\sqrt{2}\,e^{5\pi i/4}$.

7. Writing the complex number in polar form, $1=e^{2m\pi i}$, where m may be any integer. Thus $1^{1/3}=e^{2m\pi i/3}$. Setting $m=0,1,2$ successively, we obtain the three roots as $1^{1/3}=1$, $1^{1/3}=e^{2\pi i/3}$, $1^{1/3}=e^{4\pi i/3}$. Equivalently, the roots can also be written as 1, $cos(2\pi/3)+i\,sin(2\pi/3)=\frac{1}{2}\left(-1+\sqrt{3}\right)$, $cos(4\pi/3)+isin(4\pi/3)=\frac{1}{2}\left(-1+\sqrt{3}\right)$.

9. Writing the complex number in polar form, $1=e^{2m\pi i}$, where m may be any integer. Thus $1^{1/4}=e^{2m\pi i/4}$. Setting $m=0,1,2,3$ successively, we obtain the three roots as $1^{1/4}=1$, $1^{1/4}=e^{\pi i/2}$, $1^{1/4}=e^{\pi i}$, $1^{1/4}=e^{3\pi i/2}$. Equivalently, the roots can also be written as 1, $cos(\pi/2)+i\,sin(\pi/2)=i$, $cos(\pi)+isin(\pi)=-1$, $cos(3\pi/2)+$ $+isin(3\pi/2)=-i$.

10. In polar form, $2(cos\,\pi/3+i\,sin\,\pi/3)=2\,e^{i\pi/3+2m\pi}$, in which m is any integer. Thus $[2(cos\,\pi/3+i\,sin\,\pi/3)]^{1/2}=2^{1/2}\,e^{i\pi/6+m\pi}$. With $m=0$, one square root is given by $2^{1/2}\,e^{i\pi/6}=\left(\sqrt{3}+i\right)/\sqrt{2}$. With $m=1$, the other root is given by $2^{1/2}\,e^{i7\pi/6}=\left(-\sqrt{3}-i\right)/\sqrt{2}$.

11. The characteristic equation is $r^3-r^2-r+1=0$. The roots are $r=-1,1,1$. One root is *repeated*, hence the general solution is $y=c_1e^{-t}+c_2e^t+c_3te^t$.

13. The characteristic equation is $r^3-2r^2-r+2=0$, with roots $r=-1,1,2$. The roots are real and *distinct*, hence the general solution is $y=c_1e^{-t}+c_2e^t+c_3e^{2t}$.

14. The characteristic equation can be written as $r^2(r^2-4r+4)=0$. The roots are $r=0,0,2,2$. There are two repeated roots, and hence the general solution is given by $y=c_1+c_2t+c_3e^{2t}+c_4te^{2t}$.

15. The characteristic equation is $r^6+1=0$. The roots are given by $r=(-1)^{1/6}$, that is, the six *sixth roots* of -1. They are $e^{-\pi i/6+m\pi i/3}$, $m=0,1,\cdots,5$. Explicitly,

$r = \left(\sqrt{3} - i\right)/2\,,\ \left(\sqrt{3} + i\right)/2\,,\ i\,,\ -i\,,\ \left(-\sqrt{3} + i\right)/2\,,\ \left(-\sqrt{3} - i\right)/2\,.$ Hence the general solution is given by $y = e^{\sqrt{3}\,t/2}[c_1 cos\,(t/2) + c_2 sin\,(t/2)] + c_3 cos\,t + c_4 sin\,t + e^{-\sqrt{3}\,t/2}[c_5 cos\,(t/2) + c_6 sin\,(t/2)]$.

16. The characteristic equation can be written as $(r^2 - 1)(r^2 - 4) = 0$. The roots are given by $r = \pm 1, \pm 2$. The roots are real and *distinct*, hence the general solution is $y = c_1 e^{-t} + c_2 e^{t} + c_3 e^{-2t} + c_4 e^{2t}$.

17. The characteristic equation can be written as $(r^2 - 1)^3 = 0$. The roots are given by $r = \pm 1$, each with *multiplicity three*. Hence the general solution is

$$y = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t} + c_4 e^{t} + c_5 t e^{t} + c_6 t^2 e^{t}.$$

18. The characteristic equation can be written as $r^2\left(r^4 - 1\right) = 0$. The roots are given by $r = 0, 0, \pm 1, \pm i$. The general solution is $y = c_1 + c_2 t + c_3 e^{-t} + c_4 e^{t} + c_5 cos\,t +$ $+\, c_6 sin\,t$.

19. The characteristic equation can be written as $r\left(r^4 - 3r^3 + 3r^2 - 3r + 2\right) = 0$. Examining the coefficients, it follows that $r^4 - 3r^3 + 3r^2 - 3r + 2 = (r - 1)(r - 2) \times (r^2 + 1)$. Hence the roots are $r = 0, 1, 2, \pm i$. The general solution of the ODE is given by $y = c_1 + c_2 e^{t} + c_3 e^{2t} + c_4 cos\,t + c_5 sin\,t$.

20. The characteristic equation can be written as $r(r^3 - 8) = 0$, with roots $r = 0$, $2\,e^{2m\pi i/3}, m = 0, 1, 2$. That is, $r = 0, 2, -1 \pm i\sqrt{3}$. Hence the general solution is $y = c_1 + c_2 e^{2t} + e^{-t}\left[c_3 cos\sqrt{3}\,t + c_4 sin\sqrt{3}\,t\right]$.

21. The characteristic equation can be written as $\left(r^4 + 4\right)^2 = 0$. The roots of the equation $r^4 + 4 = 0$ are $r = 1 \pm i\,,\ -1 \pm i$. Each of these roots has $multiplicity\,two$. The general solution is $y = e^{t}[c_1 cos\,t + c_2 sin\,t\,] + te^{t}[c_3 cos\,t + c_4 sin\,t\,] +$ $+\, e^{-t}[c_5 cos\,t + c_6 sin\,t\,] + te^{-t}[c_7 cos\,t + c_8 sin\,t\,]$.

22. The characteristic equation can be written as $(r^2 + 1)^2 = 0$. The roots are given by $r = \pm i$, each with $multiplicity\,two$. The general solution is $y = c_1 cos\,t + c_2 sin$ $t\,+$
$+\, t[c_3 cos\,t + c_4 sin\,t\,]$.

24. The characteristic equation is $r^3 + 5r^2 + 6r + 2 = 0$. Examining the coefficients, we find that $r^3 + 5r^2 + 6r + 2 = (r + 1)(r^2 + 4r + 2)$. Hence the roots are deduced as $r = -1\,,\ -2 \pm \sqrt{2}$. The general solution is $y = c_1 e^{-t} + c_2 e^{\left(-2+\sqrt{2}\right)t} + c_3 e^{\left(-2-\sqrt{2}\right)t}$.

25. The characteristic equation is $18r^3 + 21r^2 + 14r + 4 = 0$. By examining the first and last coefficients, we find that $18r^3 + 21r^2 + 14r + 4 = (2r + 1)(9r^2 + 6r + 4)$.

Hence the roots are $r = -1/2\,,\ \left(-1 \pm \sqrt{3}\right)/3$. The general solution of the ODE is given by $y = c_1 e^{-t/2} + e^{-t/3}\left[c_2 cos\left(t/\sqrt{3}\right) + c_3 sin\left(t/\sqrt{3}\right)\right]$.

26. The characteristic equation is $r^4 - 7r^3 + 6r^2 + 30r - 36 = 0$. By examining the first and last coefficients, we find that

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = (r-3)(r+2)\left(r^2 - 6r + 6\right).$$

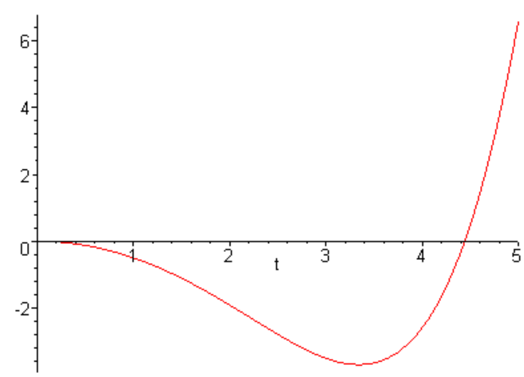
The roots are $r = -2, 3, 3 \pm \sqrt{3}$. The general solution is

$$y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{\left(3-\sqrt{3}\right)t} + c_4 e^{\left(3+\sqrt{3}\right)t}.$$

28. The characteristic equation is $r^4 + 6r^3 + 17r^2 + 22r + 14 = 0$. It can be shown that $r^4 + 6r^3 + 17r^2 + 22r + 14 = (r^2 + 2r + 2)(r^2 + 4r + 7)$. Hence the roots are $r = -1 \pm i\,,\ -2 \pm i\sqrt{3}$. The general solution is

$$y = e^{-t}[c_1 cos\, t + c_2 sin\, t] + e^{-2t}\left[c_3 cos\sqrt{3}\,t + c_4 sin\sqrt{3}\,t\right].$$

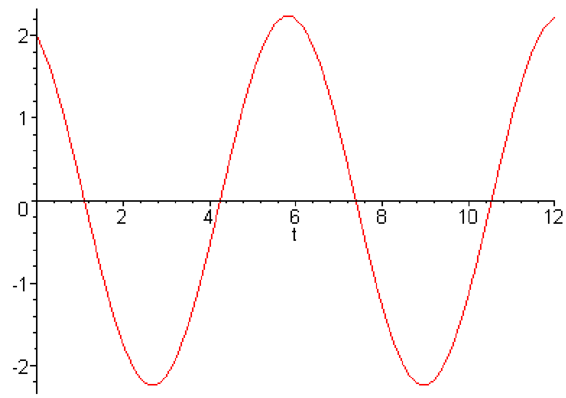
30. $y(t) = \frac{1}{2} e^{-t/\sqrt{2}} sin\left(t/\sqrt{2}\right) - \frac{1}{2} e^{t/\sqrt{2}} sin\left(t/\sqrt{2}\right)$.



32. The characteristic equation is $r^3 - r^2 + r - 1 = 0$, with roots $r = 1\,, \pm i$. Hence the general solution is $y(t) = c_1 e^t + c_2 cos\, t + c_3 sin\, t$. Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + c_3 &= -1 \\ c_1 - c_2 &= -2 \end{aligned}$$

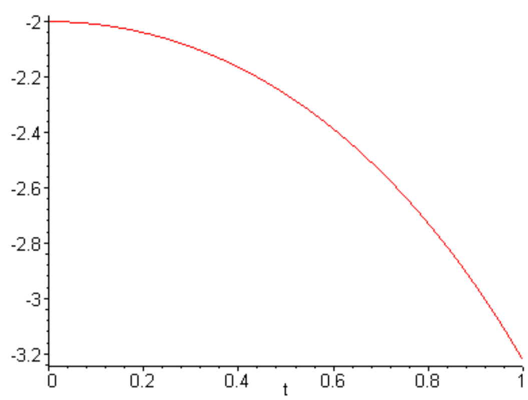
with solution $c_1 = 0\,, c_2 = 2\,, c_3 = -1$. Therefore the solution of the initial value problem is $y(t) = 2cos\, t - sin\, t$.



33. The characteristic equation is $2r^4 - r^3 - 9r^2 + 4r + 4 = 0$, with roots $r = -1/2$, $1, \pm 2$. Hence the general solution is $y(t) = c_1 e^{-t/2} + c_2 e^t + c_3 e^{-2t} + c_4 e^{2t}$. Applying the initial conditions, we obtain the system of equations

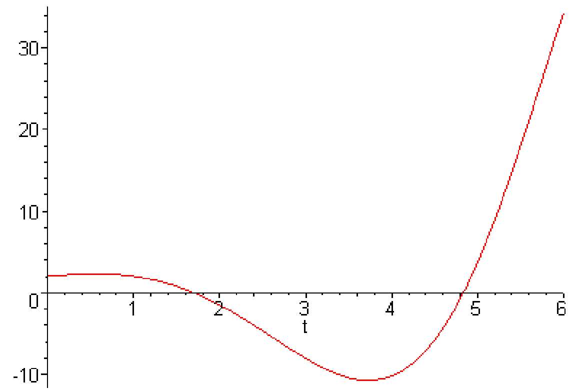
$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= -2 \\ -\frac{1}{2}c_1 + c_2 - 2c_3 + 2c_4 &= 0 \\ \frac{1}{4}c_1 + c_2 + 4c_3 + 4c_4 &= -2 \\ -\frac{1}{8}c_1 + c_2 - 8c_3 + 8c_4 &= 0 \end{aligned}$$

with solution $c_1 = -16/15$, $c_2 = -2/3$, $c_3 = -1/6$, $c_4 = -1/10$. Therefore the solution of the initial value problem is $y(t) = -\frac{16}{15}e^{-t/2} - \frac{2}{3}e^t - \frac{1}{6}e^{-2t} - \frac{1}{10}e^{2t}$.



The solution decreases without bound.

34. $y(t) = \frac{2}{13}e^{-t} + e^{t/2}\left[\frac{24}{13}cos\, t + \frac{3}{13}sin\, t\right]$.

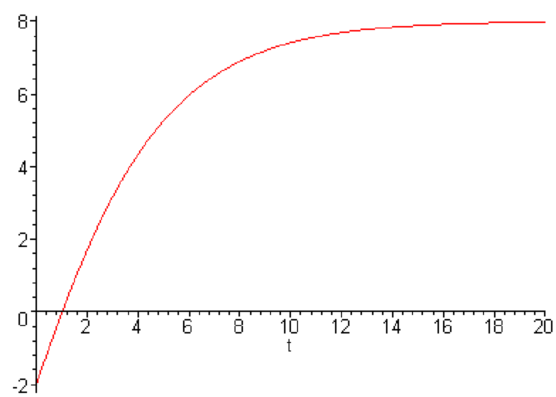


The solution is an oscillation with *increasing* amplitude.

35. The characteristic equation is $6\,r^3 + 5r^2 + r = 0$, with roots $r = 0\,, -1/3\,, -1/2$. The general solution is $y(t) = c_1 + c_2 e^{-t/3} + c_3 e^{-t/2}$. Invoking the initial conditions, we require that

$$\begin{aligned} c_1 + c_2 + c_3 &= -2 \\ -\frac{1}{3}c_2 - \frac{1}{2}c_3 &= 2 \\ \frac{1}{9}c_2 + \frac{1}{4}c_3 &= 0 \end{aligned}$$

with solution $c_1 = 8\,, c_2 = -18\,, c_3 = 8$. Therefore the solution of the initial value problem is $y(t) = 8 - 18e^{-t/3} + 8e^{-t/2}$.



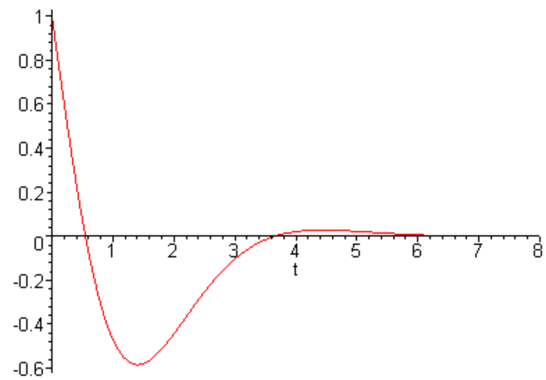
36. The general solution is derived in Prob.(28) as

$$y(t) = e^{-t}[c_1 cos\,t + c_2 sin\,t] + e^{-2t}\left[c_3 cos\sqrt{3}\,t + c_4 sin\sqrt{3}\,t\right].$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_3 = 1$$
$$-c_1 + c_2 - 2c_3 + \sqrt{3}\, c_4 = -2$$
$$-2c_2 + c_3 - 4\sqrt{3}\, c_4 = 0$$
$$2c_1 + 2c_2 + 10c_3 + 9\sqrt{3}\, c_4 = 3$$

with solution $c_1 = 21/13$, $c_2 = -38/13$, $c_3 = -8/13$, $c_4 = 17\sqrt{3}/39$.



The solution is a rapidly-decaying oscillation.

38.

$$W(e^t, e^{-t}, cos\, t, sin\, t) = -8$$
$$W(cosh\, t, sinh\, t, cos\, t, sin\, t) = 4$$

40. Suppose that $c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t} = 0$, and each of the r_k are real and different. Multiplying this equation by $e^{-r_1 t}$, $c_1 + c_2 e^{(r_2 - r_1)t} + \cdots + c_n e^{(r_n - r_1)t} = 0$. Differentiation results in

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0.$$

Now multiplying the latter equation by $e^{-(r_2 - r_1)t}$, and differentiating, we obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3 - r_2)t} + \cdots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n - r_2)t} = 0.$$

Following the above steps in a similar manner, it follows that

$$c_n(r_n - r_{n-1})\cdots(r_n - r_1)e^{(r_n - r_{n-1})t} = 0.$$

Since these equations hold for all t, and all the r_k are different, we have $c_n = 0$. Hence

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_{n-1} e^{r_{n-1} t} = 0, \quad -\infty < t < \infty.$$

The same procedure can now be repeated, successively, to show that

$$c_1 = c_2 = \cdots = c_n = 0.$$