Section 7.3

4. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 2 & 1 & 1 & | & 0 \\ 1 & -1 & 2 & | & 0 \end{pmatrix}.$$

Adding -2 times the *first row* to the *second row* and subtracting the *first row* from the *third row* results in

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & -3 & 3 & | & 0 \end{pmatrix}.$$

Adding the *negative* of the *second row* to the *third row* results in

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We evidently end up with an equivalent system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0\,. \end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = \alpha$, and $x_1 = -\alpha$. Hence all solutions have the form

$$\mathbf{x} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

5. The augmented matrix is

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 3 & 1 & 1 & | & 0 \\ -1 & 1 & 2 & | & 0 \end{pmatrix}.$$

Adding -3 times the *first row* to the *second row* and adding the *first row* to the *last row* yields

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix}.$$

Now add the negative of the *second row* to the *third row* to obtain

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{pmatrix}.$$

We end up with an equivalent linear system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3\,x_3 &= 0 \\ x_3 &= 0\,. \end{aligned}$$

Hence the unique solution of the given system of equations is $x_1 = x_2 = x_3 = 0\,.$

7. Write the given vectors as *columns* of the matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is evident that $det(\mathbf{X}) = 0$. Hence the vectors are *linearly dependent*. In order to find a linear relationship between them, write $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}\,.$ The latter equation is equivalent to

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 1 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 1 & 5/2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We obtain the system of equations

$$\begin{aligned} c_1 - c_3/2 &= 0 \\ c_2 + 5c_3/2 &= 0\,. \end{aligned}$$

Setting $c_3 = 2$, it follows that $c_1 = 1$ and $c_3 = -5$. Hence

$$\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + 2\mathbf{x}^{(3)} = \mathbf{0}\,.$$

9. The matrix containing the given vectors as *columns* is

$$\mathbf{X} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix}.$$

We find that $det(\mathbf{X}) = -70$. Hence the given vectors are *linearly independent*.

10. Write the given vectors as *columns* of the matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix}.$$

The *four* vectors are necessarily *linearly dependent*. Hence there are nonzero scalars such that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ -2 & 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

We end up with an equivalent linear system

$$\begin{aligned} c_1 + c_4 &= 0 \\ c_2 + c_4 &= 0 \\ c_3 &= 0\,. \end{aligned}$$

Let $c_4 = -1$. Then $c_1 = 1$ and $c_2 = 1$. Therefore we find that

$$\mathbf{x}^{(1)} + \mathbf{x}^{(2)} - \mathbf{x}^{(4)} = \mathbf{0}\,.$$

11. The matrix containing the given vectors as *columns*, $\mathbf{X}$, is of size $n \times m$. Since $n < m$, we can augment the matrix with $m - n$ rows of *zeros*. The resulting matrix, $\widetilde{\mathbf{X}}$, is of size $m \times m$. Since $\widetilde{\mathbf{X}}$ is square matrix, with *at least* one row of *zeros*, it follows that $det(\widetilde{\mathbf{X}}) = 0$. Hence the column vectors of $\widetilde{\mathbf{X}}$ are linearly dependent. That is, there is a *nonzero* vector, $\mathbf{c}$, such that $\widetilde{\mathbf{X}}\,\mathbf{c} = \mathbf{0}_{m\times 1}$. If we write only the first n rows of the latter equation, we have $\mathbf{X}\,\mathbf{c} = \mathbf{0}_{n\times 1}$. Therefore the column vectors of $\mathbf{X}$ are *linearly dependent*.

12. By inspection, we find that

$$\mathbf{x}^{(1)}(t) - 2\,\mathbf{x}^{(2)}(t) = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Hence $3\,\mathbf{x}^{(1)}(t) - 6\,\mathbf{x}^{(2)}(t) + \mathbf{x}^{(3)}(t) = \mathbf{0}$, and the vectors are *linearly dependent*.

13. Two vectors are *linearly dependent* if and only if one is a *nonzero* scalar multiple

of the other. However, there is no *nonzero* scalar, c, such that $2\,sin\,t = c\,sin\,t$ and $sin\,t = 2c\,sin\,t$ for all $t \in (-\infty, \infty)$. Therefore the vectors are *linearly independent*.

16. The eigenvalues λ and eigenvectors **x** satisfy the equation

$$\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(3-\lambda)(-1-\lambda)+8=0$, that is,

$$\lambda^2 - 2\lambda + 5 = 0\,.$$

The eigenvalues are $\lambda_1 = 1-2\,i$ and $\lambda_2 = 1+2\,i$. The components of the eigenvector $\mathbf{x}^{(1)}$ are solutions of the system

$$\begin{pmatrix} 2+2\,i & -2 \\ 4 & -2+2\,i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two equations reduce to $(1+i)x_1 = x_2$. Hence $\mathbf{x}^{(1)} = (1\,,1+i)^T$. Now setting $\lambda = \lambda_2 = 1+2\,i$, we have

$$\begin{pmatrix} 2-2\,i & -2 \\ 4 & -2-2\,i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with solution given by $\mathbf{x}^{(2)} = (1\,,1-i)^T$.

17. The eigenvalues λ and eigenvectors **x** satisfy the equation

$$\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-2-\lambda)(-2-\lambda)-1=0$, that is,

$$\lambda^2 + 4\lambda + 3 = 0\,.$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. For $\lambda_1 = -3$, the system of equations becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,,-1)^T$. Substituting $\lambda = \lambda_2 = -1$, we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equations reduce to $x_1 = x_2$. Hence a solution vector is given by $\mathbf{x}^{(2)} = (1\,,1)^T$.

19. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, the determinant of the coefficient matrix must be zero. That is,

$$\lambda^2 - 4 = 0\,.$$

Hence the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$. Substituting the first eigenvalue, $\lambda = -2$, yields

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The system is equivalent to the equation $\sqrt{3}\,x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = \left(1\,,\,-\sqrt{3}\right)^T$. Substitution of $\lambda = 2$ results in

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 = \sqrt{3}\,x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = \left(\sqrt{3}\,,1\right)^T$.

20. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix} -3-\lambda & 3/4 \\ -5 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-3-\lambda)(1-\lambda) + 15/4 = 0$, that is,

$$\lambda^2 + 2\lambda + 3/4 = 0\,.$$

Hence the eigenvalues are $\lambda_1 = -3/2$ and $\lambda_2 = -1/2$. In order to determine the eigenvector corresponding to λ_1, set $\lambda = -3/2$. The system of equations becomes

$$\begin{pmatrix} -3/2 & 3/4 \\ -5 & 5/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $-2\,x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,,2)^T$. Substitution of $\lambda = \lambda_2 = -1/2$ results in

$$\begin{pmatrix} -5/2 & 3/4 \\ -5 & 3/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $10\,x_1 = 3\,x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = (3\,,10)^T$.

22. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, with roots $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Setting $\lambda = \lambda_1 = 1$, we have

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1\,,0\,,\,-1)^T$. Setting $\lambda = \lambda_2 = 2$, the *reduced* system of equations is

$$\begin{aligned} x_1 + 2\,x_2 &= 0 \\ x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (\,-2\,,1\,,0)^T$. Finally, setting $\lambda = \lambda_3 = 3$, the *reduced* system of equations is

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (0\,,1\,,\,-1)^T$.

23. For computational purposes, note that if λ is an eigenvalue of $\mathbf{B}$, then $c\,\lambda$ is an eigenvalue of the matrix $\mathbf{A} = c\,\mathbf{B}$. Eigenvectors are unaffected, since they are only determined up to a scalar multiple. So with

$$\mathbf{B} = \begin{pmatrix} 11 & -2 & 8 \\ -2 & 2 & 10 \\ 8 & 10 & 5 \end{pmatrix},$$

the associated characteristic equation is $\mu^3 - 18\mu^2 - 81\mu + 1458 = 0$, with roots $\mu_1 = -9$, $\mu_2 = 9$ and $\mu_3 = 18$. Hence the eigenvalues of the given matrix, $\mathbf{A}$, are $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$. Setting $\lambda = \lambda_1 = -1$, (which corresponds to using $\mu_1 = -9$ in the *modified* problem) the *reduced* system of equations is

$$\begin{aligned} 2\,x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1\,,2\,,\,-2)^T$. Setting $\lambda = \lambda_2 = 1$, the *reduced* system of equations is

$$x_1 + 2\,x_3 = 0$$
$$x_2 - 2\,x_3 = 0\,.$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (2\,,\,-2\,,\,-1)^T$. Finally, setting $\lambda = \lambda_2 = 1$, the *reduced* system of equations is

$$x_1 - x_3 = 0$$
$$2\,x_2 - x_3 = 0\,.$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (2\,,1\,,2)^T$.

25. Suppose that $\mathbf{Ax} = \mathbf{0}$, but that $\mathbf{x} \neq \mathbf{0}$. Let $\mathbf{A} = (a_{ij})$. Using elementary row operations, it is possible to transform the matrix into one that is *not* upper triangular. If it were upper triangular, backsubstitution would imply that $\mathbf{x} = \mathbf{0}$. Hence a linear combination of all the rows results in a row containing only *zeros*. That is, there are n scalars, β_i, one for each row and not all zero, such that for each for column j,

$$\sum_{i=1}^{n} \beta_i\, a_{ij} = 0\,.$$

Now consider $\mathbf{A}^* = (b_{ij})$. By definition, $b_{ij} = \overline{a_{ji}}$, or $a_{ij} = \overline{b_{ji}}$. It follows that for each j,

$$\sum_{i=1}^{n} \beta_i\, \overline{b_{ji}} = \sum_{k=1}^{n} \overline{b_{jk}}\, \beta_k = \sum_{k=1}^{n} b_{jk}\, \overline{\beta_k} = 0\,.$$

Let $\mathbf{y} = \left(\overline{\beta_1}, \overline{\beta_2}, \cdots, \overline{\beta_n}\right)^T$. We therefore have *nonzero* vector, $\mathbf{y}$, such that $\mathbf{A}^*\mathbf{y} = \mathbf{0}$.

26. By definition,

$$\begin{aligned}(\mathbf{Ax}\,,\mathbf{y}) &= \sum_{i=0}^{n} (\mathbf{Ax})_i\, \overline{y_i} \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij}\, x_j\, \overline{y_i}\,.\end{aligned}$$

Let $b_{ij} = \overline{a_{ji}}$, so that $a_{ij} = \overline{b_{ji}}$. Now interchanging the order or summation,

$$\begin{aligned}(\mathbf{Ax}\,,\mathbf{y}) &= \sum_{j=0}^{n} x_j \sum_{i=0}^{n} a_{ij}\, \overline{y_i} \\ &= \sum_{j=0}^{n} x_j \sum_{i=0}^{n} \overline{b_{ji}}\, \overline{y_i}\,.\end{aligned}$$

Now note that

$$\sum_{i=0}^{n} \overline{b_{ji}}\,\overline{y_i} = \overline{\sum_{i=0}^{n} b_{ji}\,y_i} = \overline{(\mathbf{A}^*\mathbf{y})}_j\,.$$

Therefore

$$(\mathbf{Ax}\,,\mathbf{y}) = \sum_{j=0}^{n} x_j\,\overline{(\mathbf{A}^*\mathbf{y})}_j = (\mathbf{x}\,,\mathbf{A}^*\mathbf{y})\,.$$

28. By linearity,

$$\begin{aligned}\mathbf{A}\big(\mathbf{x}^{(0)} + \alpha\,\boldsymbol{\xi}\big) &= \mathbf{Ax}^{(0)} + \alpha\,\mathbf{A}\boldsymbol{\xi} \\ &= \mathbf{b} + \mathbf{0} \\ &= \mathbf{b}\,.\end{aligned}$$

29. Let $c_{ij} = \overline{a_{ji}}$. By the hypothesis, there is a nonzero vector, $\mathbf{y}$, such that

$$\sum_{j=1}^{n} c_{ij}\,y_j = \sum_{j=1}^{n} \overline{a_{ji}}\;y_j = 0\,,\ i = 1, 2, \cdots, n\,.$$

Taking the *conjugate* of both sides, and interchanging the indices, we have

$$\sum_{i=1}^{n} a_{ij}\;\overline{y_i} = 0\,.$$

This implies that a linear combination of *each row* of $\mathbf{A}$ is equal to *zero*. Now consider the augmented matrix $[\mathbf{A}\,|\mathbf{b}]$. Replace the *last* row by

$$\sum_{i=1}^{n}\ \overline{y_i}\,[a_{i1}\,, a_{i2}\,, \cdots, a_{in}\,, b_i] = \left[0\,, 0\,, \cdots, 0\,, \sum_{i=1}^{n}\ \overline{y_i}\,b_i\right].$$

We find that if $(\mathbf{b}\,,\mathbf{y}) = 0$, then the last row of the augmented matrix contains only zeros. Hence there are $n-1$ remaining equations. We can now set $x_n = \alpha$, some parameter, and solve for the other variables in terms of α . Therefore the system of equations $\mathbf{Ax} = \mathbf{b}$ has a solution.

30. If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then there is a nonzero vector, $\mathbf{x}$, such that

$$\mathbf{Ax} = \lambda\,\mathbf{x} = \mathbf{0}\,.$$

That is, $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution. This implies that the mapping defined by $\mathbf{A}$ is *not 1-to-1*, and hence not invertible. On the other hand, if $\mathbf{A}$ is singular, then $det(\mathbf{A}) = 0$.
Thus, $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution. The latter equation can be written as $\mathbf{Ax} = 0\,\mathbf{x}$.

31. As shown in Prob. 26, $(\mathbf{Ax}\,,\mathbf{y}) = (\mathbf{x}\,,\mathbf{A}^*\mathbf{y})$. By definition of a *Hermitian* matrix,

$\mathbf{A} = \mathbf{A}^*$.

32(a). Based on Prob. 31 , $(\mathbf{Ax}\,,\mathbf{x}) = (\mathbf{x}\,,\mathbf{Ax})$.

(b). Let **x** be an eigenvector corresponding to an eigenvalue λ . It then follows that $(\mathbf{Ax}\,,\mathbf{x}) = (\lambda\mathbf{x}\,,\mathbf{x})$ and $(\mathbf{x}\,,\mathbf{Ax}) = (\mathbf{x}\,,\lambda\mathbf{x})$. Based on the properties of the inner product, $(\lambda\mathbf{x}\,,\mathbf{x}) = \lambda(\mathbf{x}\,,\mathbf{x})$ and $(\mathbf{x}\,,\lambda\mathbf{x}) = \overline{\lambda}(\mathbf{x}\,,\mathbf{x})$. Then from Part (a),

$$\lambda(\mathbf{x}\,,\mathbf{x}) = \overline{\lambda}(\mathbf{x}\,,\mathbf{x})\,.$$

(c). From Part (b),

$$\big(\lambda - \overline{\lambda}\big)(\mathbf{x}\,,\mathbf{x}) = 0\,.$$

Based on the definition of an eigenvector, $(\mathbf{x}\,,\mathbf{x}) = \|\mathbf{x}\|^2 > 0$. Hence we must have $\lambda - \overline{\lambda} = 0$, which implies that λ is *real*.

33. From Prob. 31 ,

$$\big(\mathbf{Ax}^{(1)}\,,\mathbf{x}^{(2)}\big) = \big(\mathbf{x}^{(1)}\,,\mathbf{Ax}^{(2)}\big)\,.$$

Hence

$$\lambda_1\big(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}\big) = \overline{\lambda_2}\big(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}\big) = \lambda_2\big(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}\big)\,,$$

since the eigenvalues are real. Therefore

$$(\lambda_1 - \lambda_2)\big(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}\big) = 0\,.$$

Given that $\lambda_1 \neq \lambda_2$, we must have $\big(\mathbf{x}^{(1)}\,,\mathbf{x}^{(2)}\big) = 0$.