Section 3.3

1. Suppose that $\alpha\, f(t)+\beta\, g(t)=0$, that is, $\alpha(t^2+5t)+\beta(t^2-5t)=0$ on some interval I. Then $(\alpha+\beta)t^2+5(\alpha-\beta)t=0\,,\forall\, t\in I$. Since a quadratic .has at most *two*
roots, we must have $\alpha+\beta=0$ and $\alpha-\beta=0$. The *only* solution is $\alpha=\beta=0$. Hence the two functions are linearly *independent*.

3. Suppose that $e^{\lambda t}cos\,\mu t=A\,e^{\lambda t}sin\,\mu t$, for some $A\neq 0$, on an interval I. Since the function $sin\,\mu t\neq 0$ on some *subinterval* $I_0\subset I$, we conclude that $tan\,\mu t=A$ on I_0. This is clearly a contradiction, hence the functions are linearly *independent*.

4. Obviously, $f(x)=e\,g(x)$ for all real numbers x. Hence the functions are linearly *dependent*.

5. Here $f(x)=3g(x)$ for all real numbers. Hence the functions are linearly *dependent*.

8. Note that $f(x)=g(x)$ for $x\in[0\,,\infty)$, and $f(x)=-g(x)$ for $x\in(-\infty\,,0]$. It follows that the functions are linearly *dependent* on $\mathbb{R}^+$ and $\mathbb{R}^-$. Nevertheless, they are linearly *independent* on *any* open interval containing *zero*.

9. Since $W(t)=t\,sin^2 t$ has only *isolated* zeros, $W(t)$ cannot identically vanish on any open interval. Hence the functions are linearly *independent*.

10. Same argument as in Prob. 9.

11. By linearity of the differential operator, c_1y_1 and c_2y_2 are also solutions. Calculating
the Wronskian, $W(c_1y_1\,,c_2y_2)=(c_1y_1)(c_2y_2)'-(c_1y_1)'(c_2y_2)=c_1c_2\,W(y_1\,,y_2)$.
Since $W(y_1\,,y_2)$ is not *identically zero*, neither is $W(c_1y_1\,,c_2y_2)$.

13. Direct calculation results in

$$\begin{aligned}W(a_1y_1+a_2y_2,b_1y_1+b_2y_2)&=a_1b_2W(y_1\,,y_2)-b_1a_2W(y_1\,,y_2)\\&=(a_1b_2-a_2b_1)W(y_1\,,y_2)\,.\end{aligned}$$

Hence the combinations are also linearly independent as long as $a_1b_2-a_2b_1\neq 0$.

14. Let $\alpha(\mathbf{i}+\mathbf{j})+\beta(\mathbf{i}-\mathbf{j})=0\,\mathbf{i}+0\,\mathbf{j}$. Then $\alpha+\beta=0$ and $\alpha-\beta=0$. The only solution is $\alpha=\beta=0$. Hence the given vectors are linearly independent. Furthermore, any vector $a_1\mathbf{i}+a_2\mathbf{j}=\left(\frac{a_1}{2}+\frac{a_2}{2}\right)(\mathbf{i}+\mathbf{j})+\left(\frac{a_1}{2}-\frac{a_2}{2}\right)(\mathbf{i}-\mathbf{j})$.

16. Writing the equation in standard form, we find that $P(t)=sin\,t/cos\,t$. Hence the Wronskian is $W(t)=b\,exp\big(-\int\frac{sin\,t}{cos\,t}dt\big)=b\,exp(ln|cos\,t|)=b\,cos\,t$, in which b is some constant.

17. After writing the equation in standard form, we have $P(x) = 1/x$. The Wronskian is $W(t) = c\,exp\big(-\int \frac{1}{x}dx\big) = c\,exp(-ln|x|) = c/|x|$, in which c is some constant.

18. Writing the equation in standard form, we find that $P(x) = -2x/(1-x^2)$. The Wronskian is $W(t) = c\,exp\big(-\int \frac{-2x}{1-x^2}dx\big) = c\,exp(-ln|1-x^2|) = c|1-x^2|^{-1}$, in which c is some constant.

19. Rewrite the equation as $p(t)y'' + p'(t)y' + q(t)y = 0$. After writing the equation in standard form, we have $P(t) = p'(t)/p(t)$. Hence the Wronskian is

$$W(t) = c\,exp\left(-\int \frac{p'(t)}{p(t)}dt\right) = c\,exp(-ln\,p(t)) = c/p(t)\,.$$

21. The Wronskian associated with the solutions of the differential equation is given by $W(t) = c\,exp\big(-\int \frac{-2}{t^2}dt\big) = c\,exp(-2/t)$. Since $W(2) = 3$, it follows that for the hypothesized set of solutions, $c = 3\,e$. Hence $W(4) = 3\sqrt{e}$.

22. For the given differential equation, the Wronskian satisfies the first order differential equation $W' + p(t)W = 0$. Given that W is *constant*, it is necessary that $p(t) \equiv 0$.

23. Direct calculation shows that

$$\begin{aligned} W(f\,g\,, f\,h) &= (fg)(fh)' - (fg)'(fh) \\ &= (fg)(f'h + fh') - (f'g + fg')(fh) \\ &= f^2\,W(g\,, h)\,. \end{aligned}$$

25. Since y_1 and y_2 are solutions, they are differentiable. The hypothesis can thus be restated as $y_1'(t_0) = y_2'(t_0) = 0$ at some point t_0 in the interval of definition. This implies that $W(y_1\,, y_2)(t_0) = 0$. But $W(y_1\,, y_2)(t_0) = c\,exp\big(-\int p(t)dt\big)$, which *cannot* be equal to zero, unless $c = 0$. Hence $W(y_1\,, y_2) \equiv 0$, which is ruled out for a fundamental set of solutions.