Section 11.4

1. Let $\phi_n(x) = J_0\left(\sqrt{\lambda_n}\, x\right)$ be the eigenfunctions of the singular problem

$$\begin{gathered} -\,(x\,y')' = \lambda x y\,, \quad 0 < x < 1\,, \\ y\,, y' \text{ bounded as } x \to 0\,,\; y(1) = 0\,. \end{gathered}$$

Let $\phi(x)$ be a solution of the given BVP, and set

$$\phi(x) = \sum_{n=0}^{\infty} b_n \phi_n(x)\,. \tag{$*$}$$

Then

$$\begin{aligned} -\,(x\,\phi')' &= \mu\, x\phi + f(x) \\ &= \mu\, x\phi + x\,\frac{f(x)}{x}\,. \end{aligned}$$

Substituting $(*)$, we obtain

$$\sum_{n=0}^{\infty} b_n \lambda_n x\,\phi_n(x) = \mu x \sum_{n=0}^{\infty} b_n \phi_n(x) + x \sum_{n=0}^{\infty} c_n \phi_n(x)\,,$$

in which the c_n are the expansion coefficients of $f(x)/x$ for $x > 0$. That is,

$$\begin{aligned} c_n &= \frac{1}{\|\phi_n(x)\|^2} \int_0^1 x\,\frac{f(x)}{x}\,\phi_n(x)dx \\ &= \frac{1}{\|\phi_n(x)\|^2} \int_0^1 f(x)\phi_n(x)dx\,. \end{aligned}$$

It follows that if $x \neq 0$,

$$\sum_{n=0}^{\infty} [c_n - b_n(\lambda_n - \mu)]\phi_n(x) = 0\,.$$

As long as $\mu \neq \lambda_n$, linear independence of the eigenfunctions implies that

$$b_n = \frac{c_n}{\lambda_n - \mu}\,, \quad n = 1, 2, \cdots.$$

Therefore a formal solution is given by

$$\phi(x) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n - \mu}\, J_0\left(\sqrt{\lambda_n}\, x\right),$$

in which $\sqrt{\lambda_n}$ are the positive roots of $J_0(x) = 0$.

$3(a)$. Setting $t = \sqrt{\lambda}\, x$, it follows that

$$\frac{dy}{dx} = \sqrt{\lambda}\,\frac{dy}{dt} \text{ and } \frac{d^2y}{dx^2} = \lambda\,\frac{d^2y}{dt^2}\,.$$

The given ODE can be expressed as

$$-\sqrt{\lambda}\,\frac{d}{dt}\left(\frac{t}{\sqrt{\lambda}}\sqrt{\lambda}\,\frac{dy}{dt}\right) + \frac{k^2\sqrt{\lambda}}{t} = \sqrt{\lambda}\,t\,y\,,$$

or

$$-\frac{d}{dt}\left(t\,\frac{dy}{dt}\right) + \frac{k^2}{t} = t\,y\,.$$

An equivalent form is given by

$$t^2\frac{dy}{dt} + t\,\frac{dy}{dt} + \left(t^2 - k^2\right)y = 0\,,$$

which is known as a Bessel equation of order k. A *bounded* solution is $J_k(t)$.

(b). $J_k\left(\sqrt{\lambda}\,x\right)$ satisfies the boundary condition at $x = 0$. Imposing the other boundary
condition, it is necessary that $J_k\left(\sqrt{\lambda}\right) = 0$. Therefore the eigenvalues are given by λ_n, $n = 1, 2\cdots$, where $\sqrt{\lambda_n}$ are the positive zeroes of $J_k(x)$. The eigenfunctions of the BVP are $\phi_n(x) = J_k\big(\sqrt{\lambda_n}\,x\big)$.

(c). The BVP is a *singular Sturm-Liouville* problem with

$$L[y] = -\left(x\,y'\right)' + \frac{k^2}{x}\,y \text{ and } r(x) = 1\,.$$

We note that

$$\begin{aligned}
\lambda_n\int_0^1 x\,\phi_n(x)\phi_m(x)dx &= \int_0^1 L[\phi_n]\,\phi_m(x)dx \\
&= \int_0^1 \phi_n(x)L[\phi_m]dx \\
&= \lambda_m\int_0^1 x\,\phi_n(x)\phi_m(x)dx\,.
\end{aligned}$$

Therefore

$$(\lambda_n - \lambda_m)\int_0^1 x\,\phi_n(x)\phi_m(x)dx = 0\,.$$

So for $n \neq m$, we have $\lambda_n \neq \lambda_m$ and

$$\int_0^1 x\,\phi_n(x)\phi_m(x)dx = 0\,.$$

(d). Consider the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)\,.$$

Multiplying both sides of equation by $x\,\phi_j(x)$ and integrating from 0 to 1, and using the orthogonality of the eigenfunction,

$$\begin{aligned}\int_0^1 x\,f(x)\phi_j(x)dx &= \sum_{n=0}^{\infty} a_n \int_0^1 x\,\phi_j(x)\phi_n(x)dx \\ &= a_j \int_0^1 x\,\phi_j(x)\phi_j(x)dx\,.\end{aligned}$$

Therefore

$$a_j = \int_0^1 x\,f(x)\phi_j(x)dx \Big/ \int_0^1 x[\phi_j(x)]^2 dx\,,\ j = 1, 2, \cdots.$$

(e). Let $\phi(x)$ be a solution of the given BVP, and set

$$\phi(x) = \sum_{n=0}^{\infty} b_n \phi_n(x)\,, \tag{$*$}$$

where $\phi_n(x) = J_k\big(\sqrt{\lambda_n}\,x\big)$. Then

$$\begin{aligned}L[\phi] &= \mu\,x\phi + f(x) \\ &= \mu\,x\phi + x\,\frac{f(x)}{x}\,.\end{aligned}$$

Substituting $(*)$, we obtain

$$\sum_{n=0}^{\infty} b_n \lambda_n x\,\phi_n(x) = \mu x \sum_{n=0}^{\infty} b_n \phi_n(x) + x \sum_{n=0}^{\infty} c_n \phi_n(x)\,,$$

in which the c_n are the expansion coefficients of $f(x)/x$ for $x > 0$. That is,

$$\begin{aligned}c_n &= \frac{1}{\|\phi_n(x)\|^2} \int_0^1 x\,\frac{f(x)}{x}\phi_n(x)dx \\ &= \frac{1}{\|J_k\big(\sqrt{\lambda_n}\,x\big)\|^2} \int_0^1 f(x) J_k\Big(\sqrt{\lambda_n}\,x\Big)dx\,.\end{aligned}$$

It follows that if $x \neq 0$,

$$\sum_{n=0}^{\infty}[c_n - b_n(\lambda_n - \mu)]J_k\left(\sqrt{\lambda_n}\, x\right) = 0\,.$$

As long as $\mu \neq \lambda_n$, linear independence of the eigenfunctions implies that

$$b_n = \frac{c_n}{\lambda_n - \mu}\,, \quad n = 1, 2, \cdots.$$

Therefore a formal solution is given by

$$\phi(x) = \sum_{n=0}^{\infty}\frac{c_n}{\lambda_n - \mu}\, J_k\left(\sqrt{\lambda_n}\, x\right).$$

$5(a)$. Setting $\lambda = \alpha^2$ in Prob. 15 of Section 11.1, the *Chebyshev equation* can also be written as

$$-\left[\sqrt{1 - x^2}\, y'\right]' = \frac{\lambda}{\sqrt{1 - x^2}}\, y\,.$$

Note that

$$p(x) = \sqrt{1 - x^2}\,,\ q(x) = 0\,, \text{and } r(x) = 1/\sqrt{1 - x^2}\,,$$

hence both boundary points are singular.

(b). Observe that $p(1 - \varepsilon) = \sqrt{2\varepsilon - \varepsilon^2}$ and $p(-1 + \varepsilon) = \sqrt{2\varepsilon - \varepsilon^2}$. It follows that if $u(x)$ and $v(x)$ satisfy the boundary conditions (iii), then

$$\lim_{\varepsilon \to 0^+} p(1 - \varepsilon)[u'(1 - \varepsilon)v(1 - \varepsilon) - u(1 - \varepsilon)v'(1 - \varepsilon)] = 0$$

and

$$\lim_{\varepsilon \to 0^+} p(-1 + \varepsilon)[u'(-1 + \varepsilon)v(-1 + \varepsilon) - u(-1 + \varepsilon)v'(-1 + \varepsilon)] = 0\,.$$

Therefore Eq. (17) is satisfied and the boundary value problem is *self-adjoint*.

(c). For $n \neq 0$,

$$\begin{aligned} n^2\int_{-1}^{1}\frac{T_0(x)\, T_n(x)}{\sqrt{1 - x^2}}dx &= \int_{-1}^{1} T_0(x)\, L[T_n]dx \\ &= \int_{-1}^{1} L[T_0]\, T_n(x)dx \\ &= 0\,, \end{aligned}$$

since $L[T_0] = 0 \cdot T_0 = 0$. Otherwise,

$$
\begin{aligned}
n^2\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx &= \int_{-1}^{1}L[T_n]\,T_m(x)dx \\
&= \int_{-1}^{1}T_n(x)L[T_m]dx \\
&= m^2\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx\,.
\end{aligned}
$$

Therefore

$$\left(n^2-m^2\right)\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx = 0\,.$$

So for $n \neq m$,

$$\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx = 0\,.$$