**Section 9.6**

2. We consider the function $V(x,y) = a\,x^2 + c\,y^2$. The rate of change of $V$ *along any trajectory* is

$$\begin{aligned}\dot{V} &= V_x\frac{dx}{dt} + V_y\frac{dy}{dt}\\ &= 2ax\Big(-\frac{1}{2}x^3 + 2xy^2\Big) + 2cy\big(-y^3\big)\\ &= -ax^4 + 4ax^2y^2 - 2cy^4.\end{aligned}$$

Let $u = x^2$, $v = y^2$, $\alpha = -a$, $\beta = 4a$, and $\gamma = -2c$. We then have

$$-ax^4 + 4ax^2y^2 - 2cy^4 = \alpha\,u^2 + \beta\,uv + \gamma\,v^2.$$

If $a > 0$ and $c > 0$, then $V(x,y)$ is *positive definite*. Furthermore, $\alpha < 0$. Recall that Theorem 9.6.4 asserts that if $4\alpha\gamma - \beta^2 = 8ac - 16\,a^2 > 0$, then the function

$$\alpha\,u^2 + \beta\,uv + \gamma\,v^2$$

is *negative definite*. Hence if $c > 2a$, then $\dot{V}(x,y)$ is *negative definite*. One such example is $V(x,y) = x^2 + 3\,y^2$. It follows from Theorem 9.6.1 that the origin is an asymptotically stable critical point.

4. Given $V(x,y) = a\,x^2 + c\,y^2$, the rate of change of $V$ *along any trajectory* is

$$\begin{aligned}\dot{V} &= V_x\frac{dx}{dt} + V_y\frac{dy}{dt}\\ &= 2ax\big(x^3 - y^3\big) + 2cy\big(2xy^2 + 4x^2y + 2y^3\big)\\ &= 2a\,x^4 + (4c - 2a)xy^3 + 8c\,x^2y^2 + 4c\,y^4.\end{aligned}$$

Setting $a = 2c$,

$$\begin{aligned}\dot{V} &= 4c\,x^4 + 8c\,x^2y^2 + 4c\,y^4\\ &\geq 4c\,x^4 + 4c\,y^4.\end{aligned}$$

As long as $a = 2c > 0$, the function $V(x,y)$ is *positive definite* and $\dot{V}(x,y)$ is also *positive definite.* It follows from Theorem 9.6.2 that $(0\,,0)$ is an unstable critical point.

5. Given $V(x,y) = c(x^2 + y^2)$, the rate of change of $V$ *along any trajectory* is

$$\begin{aligned}\dot{V} &= V_x\frac{dx}{dt} + V_y\frac{dy}{dt}\\ &= 2c\,x[y - xf(x,y)] + 2cy[-x - yf(x,y)]\\ &= -2c\big(x^2 + y^2\big)f(x,y)\,.\end{aligned}$$

If $c > 0$, then $V(x,y)$ is *positive definite*. Furthermore, if $f(x,y)$ is *positive* in some neighborhood of the origin, then $\dot{V}(x,y)$ is *negative definite*. Theorem 9.6.1 asserts that the origin is an asymptotically stable critical point.

On the other hand, if $f(x,y)$ is *negative* in some neighborhood of the origin, then $V(x,y)$
and $\dot{V}(x,y)$ are both *positive definite*. It follows from Theorem 9.6.2 that the origin is an unstable critical point.

9$(a)$. Letting $x = u$ and $y = u'$, we obtain the system of equations

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -g(x) - y\,. \end{aligned}$$

Since $g(0) = 0$, it is evident that $(0\,,0)$ is a critical point of the system. Consider the function

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)ds\,.$$

It is clear that $V(0,0) = 0$. Since $g(u)$ is an *odd* function in a neighborhood of $u = 0$,

$$\int_0^x g(s)ds > 0 \text{ for } x > 0\,,$$

and

$$\int_0^x g(s)ds = -\int_x^0 g(s)ds > 0 \text{ for } x < 0\,.$$

Therefore $V(x,y)$ is *positive definite*.

The rate of change of $V$ along any trajectory is

$$\begin{aligned} \dot{V} &= V_x\frac{dx}{dt} + V_y\frac{dy}{dt} \\ &= g(x)\cdot(y) + y[-g(x) - y] \\ &= -y^2\,. \end{aligned}$$

It follows that $\dot{V}(x,y)$ is only *negative semidefinite*. Hence the origin is a *stable* critical point.

$(b)$. Given

$$V(x,y) = \frac{1}{2}y^2 + \frac{1}{2}y\,sin(x) + \int_0^x sin(s)ds\,,$$

It is easy to see that $V(0\,,0) = 0$. The rate of change of $V$ along any trajectory is

$$\begin{aligned}\dot{V} &= V_x\frac{dx}{dt} + V_y\frac{dy}{dt} \\ &= \left[sin\,x + \frac{y}{2}cos\,x\right](y) + \left[y + \frac{1}{2}sin\,x\right][-\,sin\,x - y] \\ &= \frac{1}{2}y^2\,cos\,x - \frac{1}{2}sin^2 x - \frac{y}{2}sin\,x - y^2\,.\end{aligned}$$

For $-\pi/2 < x < \pi/2$ , we can write $sin\,x = x - \alpha\,x^3/6$ and $cos\,x = 1 - \beta\,x^2/2$ , in which $\alpha = \alpha(x)\,,\beta = \beta(x)$ . Note that $0 < \alpha\,,\beta < 1$ . Then

$$\dot{V}(x\,,y) = \frac{y^2}{2}\left(1 - \frac{\beta\,x^2}{2}\right) - \frac{1}{2}\left(x - \frac{\alpha\,x^3}{6}\right)^2 - \frac{y}{2}\left(x - \frac{\alpha\,x^3}{6}\right) - y^2\ .$$

Using polar coordinates,

$$\begin{aligned}\dot{V}(r\,,\theta) &= -\frac{r^2}{2}\left[1 + sin\,\theta\,cos\,\theta + h(r\,,\theta)\right] \\ &= -\frac{r^2}{2}\left[1 + \frac{1}{2}sin\,2\theta + h(r\,,\theta)\right].\end{aligned}$$

It is easy to show that

$$|h(r\,,\theta)| \le \frac{1}{2}r^2 + \frac{1}{72}r^4.$$

So if $r$ is *sufficiently small*, then $|h(r\,,\theta)| < 1/2$ and $\left|\frac{1}{2}sin\,2\theta + h(r\,,\theta)\right| < 1$ . Hence $\dot{V}(x\,,y)$ is *negative definite.*

Now we show that $V(x,y)$ is *positive definite*. Since $g(u) = sin\,u$ ,

$$V(x,y) = \frac{1}{2}y^2 + \frac{1}{2}y\,sin(x) + 1 - cos\,x\,.$$

This time we set

$$cos\,x = 1 - \frac{x^2}{2} + \gamma\,\frac{x^4}{24}\,.$$

Note that $0 < \gamma < 1$ for $-\pi/2 < x < \pi/2$ . Converting to polar coordinates,

$$\begin{aligned}V(r,\theta) &= \frac{r^2}{2}\left[1 + sin\,\theta\,cos\,\theta - \frac{r^2}{12}sin\,\theta\,cos^3\theta - \gamma\,\frac{r^2}{24}cos^4\theta\right] \\ &= \frac{r^2}{2}\left[1 + \frac{1}{2}sin\,2\theta - \frac{r^2}{12}sin\,\theta\,cos^3\theta - \gamma\,\frac{r^2}{24}cos^4\theta\right].\end{aligned}$$

Now

$$-\frac{r^2}{12}sin\,\theta\,cos^3\theta - \gamma\,\frac{r^2}{24}cos^4\theta > -\frac{1}{8} \text{ for } r < 1\,.$$

It follows that when $r > 0$,

$$V(r,\theta) > \frac{r^2}{2}\left[\frac{7}{8} + \frac{1}{2}sin\,2\theta\right] \geq \frac{3\,r^2}{16} > 0\,.$$

Therefore $V(x,y)$ is indeed *positive definite*, and by Theorem 9.6.1, the origin is an asymptotically stable critical point.

$12(a)$. We consider the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $V(x,y) = Ax^2 + Bxy + Cy^2$, in which

$$\begin{aligned} A &= -\frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta} \\ B &= \frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta} \\ C &= -\frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}\,, \end{aligned}$$

and $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$. Based on the hypothesis, the coefficients $A$ and $B$ are negative. Therefore, except for the origin, $V(x,y)$ is *negative* on each of the coordinate axes. Along each trajectory,

$$\begin{aligned} \dot{V} &= (2Ax + By)(a_{11}\,x + a_{12}\,y) + (2Cy + Bx)(a_{21}\,x + a_{22}\,y) \\ &= -x^2 - y^2. \end{aligned}$$

Hence $\dot{V}(x,y)$ is *negative definite*. Theorem 9.6.2 asserts that the origin is an *unstable* critical point.

$(b)$. We now consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F_1(x\,,y) \\ G_1(x\,,y) \end{pmatrix},$$

in which $F_1(x\,,y)/r \to 0$ and $G_1(x\,,y)/r \to 0$ as $r \to 0$. Let

$$V(x,y) = Ax^2 + Bxy + Cy^2,$$

in which

$$A = \frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}$$
$$B = -\frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta}$$
$$C = \frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta},$$

and $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$. Based on the hypothesis, $A, B > 0$. Except for the origin, $V(x, y)$ is *positive* on each of the coordinate axes. Along each trajectory,

$$\dot{V} = x^2 + y^2 + (2Ax + By)F_1(x, y) + (2Cy + Bx)G_1(x, y).$$

Converting to polar coordinates, for $r \neq 0$,

$$\begin{aligned}\dot{V} &= r^2 + r(2A\cos\theta + B\sin\theta)\,F_1 + r(2C\sin\theta + B\cos\theta)\,G_1 \\ &= r^2 + r^2\left[(2A\cos\theta + B\sin\theta)\,\frac{F_1}{r} + (2C\sin\theta + B\cos\theta)\,\frac{G_1}{r}\right].\end{aligned}$$

Since the system is *almost linear*, there is an $R$ such that

$$\left|(2A\cos\theta + B\sin\theta)\,\frac{F_1}{r} + (2C\sin\theta + B\cos\theta)\,\frac{G_1}{r}\right| < \frac{1}{2},$$

and hence

$$(2A\cos\theta + B\sin\theta)\,\frac{F_1}{r} + (2C\sin\theta + B\cos\theta)\,\frac{G_1}{r} > -\frac{1}{2}$$

for $r < R$. It follows that

$$\dot{V} > \frac{1}{2}r^2$$

as long as $0 < r < R$. Hence $\dot{V}$ is *positive definite* on the domain

$$D = \{(x, y) \mid x^2 + y^2 < R^2\}.$$

By Theorem 9.6.2, the origin is an *unstable* critical point.