Section 5.5

1. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) + 4r + 2 \\ &= r^2 + 3r + 2\,. \end{aligned}$$

The roots are $r = -2\,,\ -1$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\, x^{-2} + c_2\, x^{-1}.$$

3. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) - 3r + 4 \\ &= r^2 - 4r + 4\,. \end{aligned}$$

The root is $r = 2$, with multiplicity *two*. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2\, ln|x|)\, x^2.$$

5. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) - r + 1 \\ &= r^2 - 2r + 1\,. \end{aligned}$$

The root is $r = 1$, with multiplicity *two*. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2\, ln|x|)\, x\,.$$

6. Substitution of $y = (x-1)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 7r + 12\,.$$

The roots are $r = -3\,,\ -4$. Hence the general solution, for $x \neq 1$, is

$$y = c_1\,(x-1)^{-3} + c_2(x-1)^{-4}.$$

7. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 5r - 1\,.$$

The roots are $r = -\left(5 \pm \sqrt{29}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{-\left(5+\sqrt{29}\right)/2} + c_2|x|^{-\left(5-\sqrt{29}\right)/2}.$$

8. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 3r + 3\,.$$

The roots are complex, with $r = \left(3 \pm i\sqrt{3}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{3/2} cos\left(\frac{\sqrt{3}}{2}\, ln|x|\right) + c_2 |x|^{3/2} sin\left(\frac{\sqrt{3}}{2}\, ln|x|\right).$$

10. Substitution of $y = (x-2)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 4r + 8\,.$$

The roots are complex, with $r = -2 \pm 2i$. Hence the general solution, for $x \neq 2$, is

$$y = c_1 (x-2)^{-2} cos(2\, ln|x-2|) + c_2 (x-2)^{-2} sin(2\, ln|x-2|).$$

11. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + r + 4\,.$$

The roots are complex, with $r = -\left(1 \pm i\sqrt{15}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{-1/2} cos\left(\frac{\sqrt{15}}{2}\, ln|x|\right) + c_2 |x|^{-1/2} sin\left(\frac{\sqrt{15}}{2}\, ln|x|\right).$$

12. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 5r + 4\,.$$

The roots are $r = 1\,, 4$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x + c_2 x^4.$$

14. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = 4r^2 + 4r + 17\,.$$

The roots are complex, with $r = -1/2 \pm 2i$. Hence the general solution, for $x > 0$, is

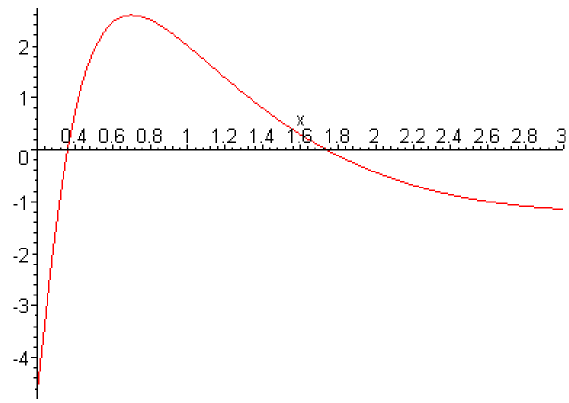
$$y = c_1 x^{-1/2} cos(2\, ln\, x) + c_2 x^{-1/2} sin(2\, ln\, x).$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2$$
$$-\frac{1}{2}c_1 + 2c_2 = -3$$

Hence the solution of the initial value problem is

$$y(x) = 2\,x^{-1/2}cos(2\,ln\,x) - x^{-1/2}sin(2\,ln\,x).$$



As $x \to 0^+$, the solution decreases without bound.

15. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 4r + 4\,.$$

The root is $r = 2$, with multiplicity *two*. Hence the general solution, for $x < 0$, is

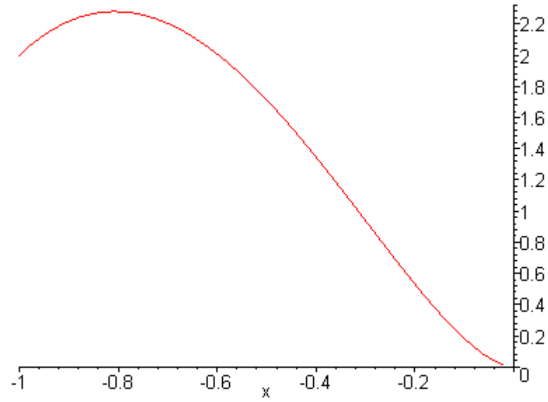
$$y = (c_1 + c_2\,ln\,|x|)\,x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2$$
$$-2c_1 - c_2 = 3$$

Hence the solution of the initial value problem is

$$y(x) = (2 - 7\,ln\,|x|)\,x^2.$$



We find that $y(x) \to 0$ as $x \to 0^-$.

18. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r + \beta = 0$. The roots are

$$r = \frac{1 \pm \sqrt{1 - 4\beta}}{2}.$$

If $\beta > 1/4$, the roots are complex, with $r_{1,2} = \left(1 \pm i\sqrt{4\beta - 1}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2} cos\left(\frac{1}{2}\sqrt{4\beta - 1}\ ln|x|\right) + c_2 |x|^{1/2} sin\left(\frac{1}{2}\sqrt{4\beta - 1}\ ln|x|\right).$$

Since the trigonometric factors are *bounded*, $y(x) \to 0$ as $x \to 0$. If $\beta = 1/4$, the roots are *equal*, and

$$y = c_1 |x|^{1/2} + c_2 |x|^{1/2}\, ln\,|x|\,.$$

Since $\lim_{x \to 0} \sqrt{|x|}\, ln\,|x| = 0$, $y(x) \to 0$ as $x \to 0$. If $\beta < 1/4$, the roots are real, with $r_{1,2} = \left(1 \pm \sqrt{1 - 4\beta}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2 + \sqrt{1 - 4\beta}/2} + c_2 |x|^{1/2 - \sqrt{1 - 4\beta}/2}.$$

Evidently, solutions approach *zero* as long as $1/2 - \sqrt{1 - 4\beta}/2 > 0$. That is,

$$0 < \beta < 1/4\,.$$

Hence *all* solutions approach *zero*, for $\beta > 0$.

19. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r - 2 = 0$. The roots are $r = -1\,, 2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-1} + c_2\, x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_2 = 1$$
$$-\,c_1 + 2c_2 = \gamma$$

Hence the solution of the initial value problem is

$$y(x) = \frac{2-\gamma}{3}x^{-1} + \frac{1+\gamma}{3}\,x^2.$$

The solution is *bounded*, as $x \to 0$, if $\gamma = 2$.

20. Substitution of $y = x^r$ results in the quadratic equation $r^2 + (\alpha - 1)r + 5/2 = 0$. Formally, the roots are given by

$$\begin{aligned} r &= \frac{1-\alpha \pm \sqrt{\alpha^2 - 2\alpha - 9}}{2} \\ &= \frac{1-\alpha \pm \sqrt{\left(\alpha - 1 - \sqrt{10}\right)\left(\alpha - 1 + \sqrt{10}\right)}}{2}\,. \end{aligned}$$

(i) The roots $r_{1,2}$ will be *complex*, if $|1-\alpha| < \sqrt{10}$. For solutions to approach *zero*, as $x \to \infty$, we need $-\sqrt{10} < 1 - \alpha < 0$.
(ii) The roots will be *equal*, if $|1-\alpha| = \sqrt{10}$. In this case, all solutions approach *zero* as long as $1 - \alpha = -\sqrt{10}$.
(iii) The roots will be real and *distinct*, if $|1-\alpha| > \sqrt{10}$. It follows that

$$r_{max} = \frac{1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9}}{2}\,.$$

For solutions to approach *zero*, we need $1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9} < 0$. That is, $1 - \alpha < -\sqrt{10}$.

Hence all solutions approach *zero*, as $x \to \infty$, as long as $\alpha > 1$.

23(a). Given that $x = e^z$, $y(x) = y(e^z) = w(z)$. By the chain rule,

$$\frac{dy}{dx} = \frac{d}{dx}w(z) = \frac{dw}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dw}{dz}\,.$$

Similarly,

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{dw}{dz}\right] &= -\frac{1}{x^2}\frac{dw}{dz} + \frac{1}{x}\frac{d^2w}{dz^2}\frac{dz}{dx} \\ &= -\frac{1}{x^2}\frac{dw}{dz} + \frac{1}{x^2}\frac{d^2w}{dz^2}\,. \end{aligned}$$

(b). Direct substitution results in

$$x^2\left[\frac{1}{x^2}\,\frac{d^2w}{dz^2}-\frac{1}{x^2}\,\frac{dw}{dz}\right]+\alpha\,x\left[\frac{1}{x}\,\frac{dw}{dz}\right]+\beta\,w=0\,,$$

that is,

$$\frac{d^2w}{dz^2}+(\alpha-1)\frac{dw}{dz}+\beta\,w=0\,.$$

The associated *characteristic equation* is $r^2+(\alpha-1)r+\beta=0$. Since $z=ln\,x$, it follows that $y(x)=w(ln\,x)$.

(c). If the roots $r_{1,2}$ are real and *distinct*, then

$$\begin{aligned} y &= c_1e^{r_1z}+c_2e^{r_2z} \\ &= c_1x^{r_1}+c_2x^{r_2}. \end{aligned}$$

(d). If the roots $r_{1,2}$ are real and *equal*, then

$$\begin{aligned} y &= c_1e^{r_1z}+c_2z\,e^{r_1z} \\ &= c_1x^{r_1}+c_2x^{r_1}ln\,x\,. \end{aligned}$$

(e). If the roots are *complex conjugates*, then $r=\lambda\pm i\mu$, and

$$\begin{aligned} y &= e^{\lambda z}(c_1cos\,\mu z+c_2\,sin\,\mu z) \\ &= x^{\lambda}[c_1cos(\mu\,ln\,x)+c_2\,sin(\mu\,ln\,x)]. \end{aligned}$$

24. Based on Prob. 23, the change of variable $x=e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2}-\frac{dw}{dz}-2w=0\,.$$

The associated *characteristic equation* is $r^2-r-2=0$, with roots $r=-1\,,2$. Hence $w(z)=c_1e^{-z}+c_2e^{2z}$, and $y(x)=c_1x^{-1}+c_2\,x^2$.

26. The change of variable $x=e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2}+6\frac{dw}{dz}+5w=e^z.$$

The associated *characteristic equation* is $r^2+6\,r+5=0$, with roots $r=-5\,,\,-1$. Hence $w_c(z)=c_1e^{-z}+c_2e^{-5z}$. Since the right hand side is *not* a solution of the homogeneous equation, we can use the *method of undetermined coefficients* to show that a particular solution is $W=e^z/12$. Therefore the general solution is given by $w(z)=c_1e^{-z}+c_2e^{-5z}+e^z/12$, that is, $y(x)=c_1x^{-1}+c_2\,x^{-5}+x/12$.

27. The change of variable $x=e^z$ transforms the given ODE into

$$\frac{d^2w}{dz^2} - 3\frac{dw}{dz} + 2w = 3e^{2z} + 2z\,.$$

The associated *characteristic equation* is $r^2 - 3\,r + 2 = 0$, with roots $r = 1\,,2$. Hence $w_c(z) = c_1 e^z + c_2 e^{2z}$. Using the *method of undetermined coefficients*, let $W = Ae^{2z} + Bze^{2z} + Cz + D$. It follows that the general solution is given by $w(z) = c_1 e^z + c_2 e^{2z} + 3ze^{2z} + z + 3/2$, that is,

$$y(x) = c_1 x + c_2\, x^2 + 3\,x^2 ln\, x + ln\, x + 3/2\,.$$

28. The change of variable $x = e^z$ transforms the given ODE into

$$\frac{d^2w}{dz^2} + 4w = sin\, z\,.$$

The solution of the homogeneous equation is $w_c(z) = c_1 cos\, 2z + c_2 sin\, 2z$. The right hand side is *not* a solution of the homogeneous equation. We can use the *method of undetermined coefficients* to show that a particular solution is $W = \frac{1}{3} sin\, z$. Hence the general solution is given by $w(z) = c_1 cos\, 2z + c_2 sin\, 2z + \frac{1}{3} sin\, z$, that is, $y(x) = c_1 cos(2\, ln\, x) + c_2 sin(2\, ln\, x) + \frac{1}{3} sin(ln\, x)$.

29. After dividing the equation by 3, the change of variable $x = e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2} + 3\frac{dw}{dz} + 3w = 0\,.$$

The associated *characteristic equation* is $r^2 + 3\,r + 3 = 0$, with complex roots $r = -\left(3 \pm i\sqrt{3}\right)/2$. Hence the general solution is

$$w(z) = e^{-3z/2}\left[c_1 cos\left(\sqrt{3}\; z/2\right) + c_2 sin\left(\sqrt{3}\; z/2\right)\right],$$

and therefore

$$y(x) = x^{-3/2}\left[c_1 cos\left(\frac{\sqrt{3}}{2}\, ln\, x\right) + c_2 sin\left(\frac{\sqrt{3}}{2}\, ln\, x\right)\right].$$

30. Let $x < 0$. Setting $y = (-x)^r$, successive differentiation gives $y' = -r(-x)^{r-1}$ and $y'' = r(r-1)(-x)^{r-2}$. It follows that

$$L[(-x)^r] = r(r-1)x^2(-x)^{r-2} - \alpha\, r\, x(-x)^{r-1} + \beta(-x)^r.$$

Since $x^2 = (-x)^2$, we find that

$$
\begin{aligned}
L[(-x)^r] &= r(r-1)(-x)^r + \alpha\, r(-x)^r + \beta(-x)^r \\
&= (-x)^r[r(r-1) + \alpha\, r + \beta].
\end{aligned}
$$

Given that r_1 and r_2 are roots of $F(r) = r(r-1) + \alpha\, r + \beta$, we have $L[(-x)^{r_i}] = 0$. Therefore $y_1 = (-x)^{r_1}$ and $y_2 = (-x)^{r_2}$ are *linearly independent* solutions of the differential equation, $L[y] = 0$, for $x < 0$, as long as $r_1 \neq r_2$.