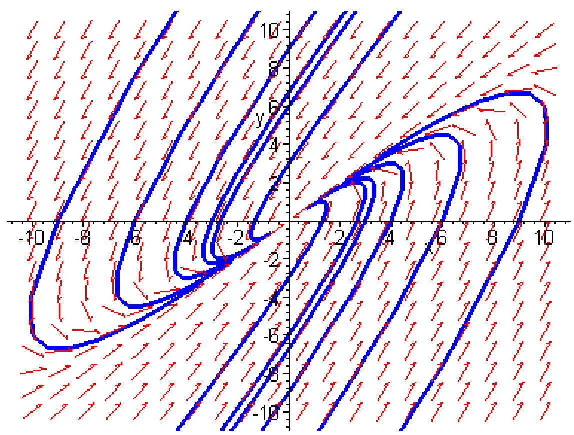
Section 7.5

2. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$, and substituting into the ODE, we obtain the algebraic equations

$$\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 3r + 2 = 0$. The roots of the characteristic equation are $r_1 = -1$ and $r_2 = -2$. For $r = -1$, the two equations reduce to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -2$ results in the single equation $3\,\xi_1 = 2\,\xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2\,,3)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}.$$

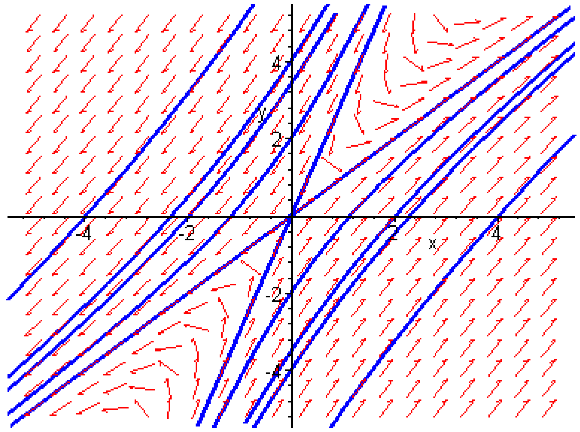


3. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -1$ results in the single equation $3\,\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}.$$



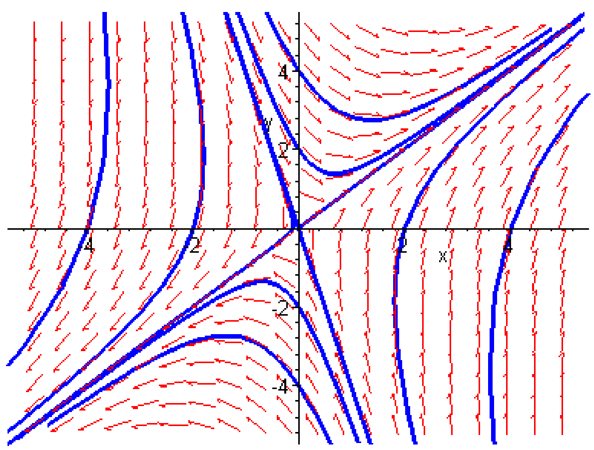
The system has an *unstable* eigendirection along $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Unless $c_1 = 0$, all solutions will diverge.

4. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + r - 6 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -3$. For $r = 2$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -3$ results in the single equation $4\,\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,-4)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$



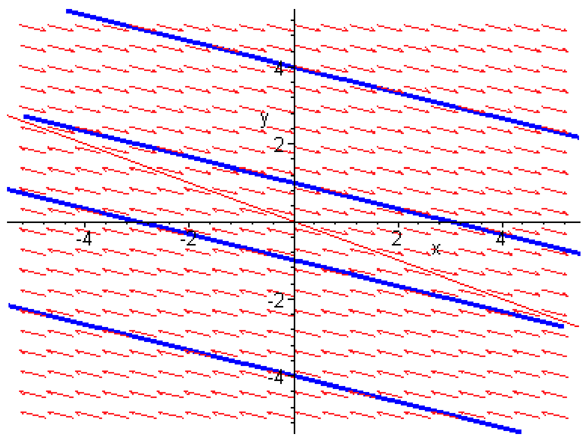
The system has an *unstable* eigendirection along $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Unless $c_1 = 0$, all solutions will diverge.

8. Setting $\mathbf{x} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3-r & 6 \\ -1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = 0$. With $r = 1$, the system of equations reduces to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3\,,\,-1)^T$. For the case $r = 0$, the system is equivalent to the equation $\xi_1 + 2\,\xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (2\,,\,-1)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^t + c_2\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$



The *entire line* along the eigendirection $\boldsymbol{\xi}^{(2)} = (2\,,\,-1)^T$ consists of equilibrium points. All other solutions diverge. The direction field changes across the line $x_1 + 2x_2 = 0$. Eliminating the exponential terms in the solution, the trajectories are given by

$$x_1 + 3\,x_2 = -\,c_2\,.$$

10. The characteristic equation is given by

$$\begin{vmatrix} 2-r & 2+i \\ -1 & -1-i-r \end{vmatrix} = r^2 - (1-i)r - i = 0\,.$$

The equation has *complex* roots $r_1 = 1$ and $r_2 = -i$. For $r = 1$, the components of the solution vector must satisfy $\xi_1 + (2+i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i\,,\,-1)^T$. Substitution of $r = -i$ results in the single equation $\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,\,-1)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 2+i \\ -1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-it}.$$

11. Setting $\mathbf{x} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^3 - 4r^2 - r + 4 = 0$. The roots of the characteristic equation are $r_1 = 4$, $r_2 = 1$ and $r_3 = -1$. Setting $r = 4$, we have

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,1\,,1)^T$. Setting $\lambda = 1$, the *reduced* system of equations is

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,\,-2\,,1)^T$. Finally, setting $\lambda = -1$, the *reduced* system of equations is

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (1\,,0\,,\,-1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}.$$

12. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 2 & 4 \\ 2 & -r & 2 \\ 4 & 2 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 6r^2 - 15r - 8 = 0$, with roots $r_1 = 8$, $r_2 = -1$ and $r_3 = -1$. Setting $r = r_1 = 8$, we have

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ 2\,\xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (2\,,1\,,2)^T$. Setting $r = -1$, the system of equations is reduced to the *single* equation

$$2\,\xi_1 + \xi_2 + 2\,\xi_3 = 0\,.$$

Two independent solutions are obtained as

$$\boldsymbol{\xi}^{(2)} = (1\,,\,-2\,,0)^T \text{ and } \boldsymbol{\xi}^{(3)} = (0\,,\,-2\,,1)^T.$$

Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t}.$$

13. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -8 & -5 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^3 + r^2 - 4r - 4 = 0$. The roots of the characteristic equation are $r_1 = 2\,, r_2 = -2$ and $r_3 = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 &= 0 \\ \xi_2 + \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (0\,,1\,,\,-1)^T$. Setting $\lambda = -1$, the *reduced* system of equations is

$$\begin{aligned} 2\,\xi_1 + 3\,\xi_3 &= 0 \\ \xi_2 - 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (3\,,\,-4\,,\,-2)^T$. Finally, setting $\lambda = -2$, the *reduced* system of equations is

$$\begin{aligned} 7\,\xi_1 + 4\,\xi_3 &= 0 \\ 7\xi_2 - 5\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (4\,,\,-5\,,\,-7)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} e^{-2t}.$$

15. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + 3\,c_2 &= -1\,. \end{aligned}$$

Hence $c_1 = 7/2$ and $c_2 = -3/2$, and the solution of the IVP is

$$\mathbf{x} = \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

17. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 0 & 2-r & 2 \\ -1 & 1 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^3 - 6r^2 + 11r - 6 = 0$. The roots of the characteristic equation are $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. Setting $r = 1$, we have

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (0\,,\,-2\,,1)^T$. Setting $\lambda = 2$, the *reduced* system of equations is

$$\begin{aligned} \xi_1 - \xi_2 &= 0 \\ \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,1\,,0)^T$. Finally, upon setting $\lambda = 3$, the *reduced* system of equations is

$$\begin{aligned} \xi_1 - 2\,\xi_3 &= 0 \\ \xi_2 - 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (2\,,2\,,1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^{3t}.$$

Invoking the initial conditions, the coefficients must satisfy the equations

$$\begin{aligned} c_2 + 2\,c_3 &= 2 \\ -2\,c_1 + c_2 + 2\,c_3 &= 0 \\ c_1 + c_3 &= 1\,. \end{aligned}$$

It follows that $c_1 = 1$, $c_2 = 2$ and $c_3 = 0$. Hence the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

18. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 0 & -1 \\ 2 & -r & 0 \\ -1 & 2 & 4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 4r^2 - r + 4 = 0$, with roots $r_1 = -1$, $r_2 = 1$ and $r_3 = 4$. Setting $r = r_1 = -1$, we have

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,\,-2\,,1)^T$. Setting $r = 1$, the system reduces to the equations

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2\,,\,-1)^T$. Finally, upon setting $r = 4$, the system is equivalent to the equations

$$\begin{aligned} 4\,\xi_1 + \xi_3 &= 0 \\ 8\,\xi_2 + \xi_3 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(3)} = (2\,,1\,,\,-8)^T$. Hence the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}e^{t} + c_3\begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}e^{4t}.$$

Invoking the initial conditions,

$$\begin{aligned} c_1 + c_2 + 2\,c_3 &= 7 \\ -2\,c_1 + 2\,c_2 + c_3 &= 5 \\ c_1 - c_2 - 8\,c_3 &= 5\,. \end{aligned}$$

It follows that $c_1 = 3$, $c_2 = 6$ and $c_3 = -1$. Hence the solution of the IVP is

$$\mathbf{x} = 3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}e^{-t} + 6\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}e^{t} - \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}e^{4t}.$$

19. Set $\mathbf{x} = \boldsymbol{\xi}\,t^r$. Substitution into the system of differential equations results in

$$t \cdot r t^{r-1}\boldsymbol{\xi} \;= \mathbf{A}\,\boldsymbol{\xi}\,t^r,$$

which upon simplification yields is, $\mathbf{A}\,\boldsymbol{\xi} - r\boldsymbol{\xi} = \mathbf{0}$. Hence the vector $\boldsymbol{\xi}$ and constant r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$.

21. Setting $\mathbf{x} = \boldsymbol{\xi}\,t^r$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 2t^6$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

22. As shown in Prob. 19, solution of the ODE requires analysis of the equations

$$\begin{pmatrix} 4-r & -3 \\ 8 & -6-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 2r = 0$. The roots of the characteristic equation are $r_1 = 0$ and $r_2 = -2$. For $r = 0$, the system of equations reduces to $4\,\xi_1 = 3\,\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3\,,4)^T$. Setting $r = -2$ results in the single equation $2\,\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 2\,t^{-2}$. These solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

23. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r - 2 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -1$. Setting $r = 2$, the system of equations reduces to $\xi_1 - 2\,\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2\,,1)^T$.

With $r = -1$, the system is equivalent to the equation $2\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 2)^T$. It follows that

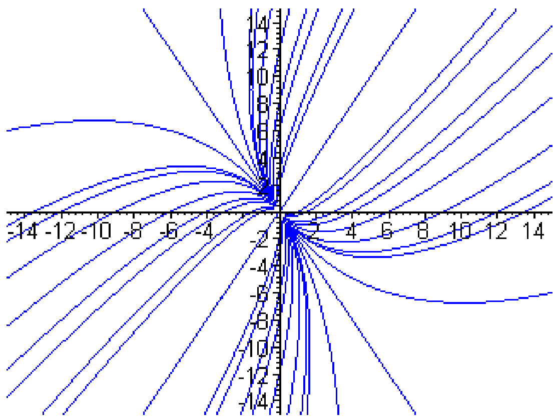
$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 3t$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

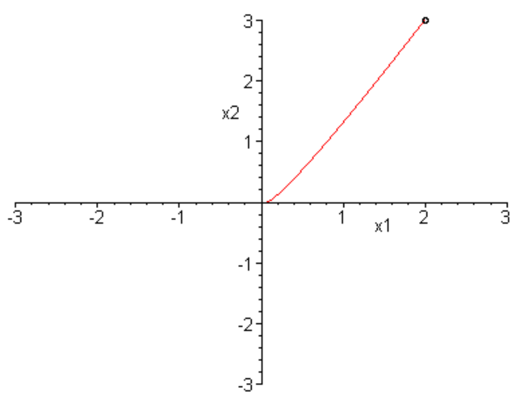
$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}.$$

24(a). The general solution is

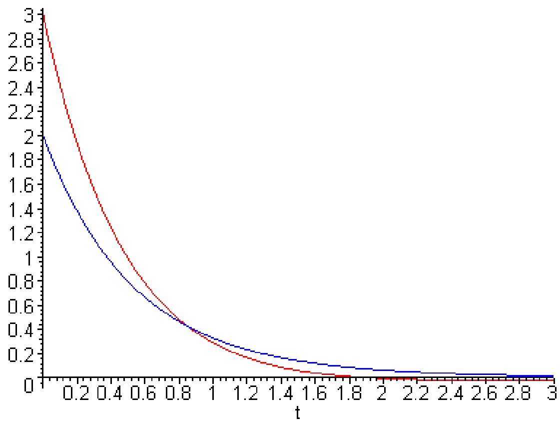
$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$



(b).



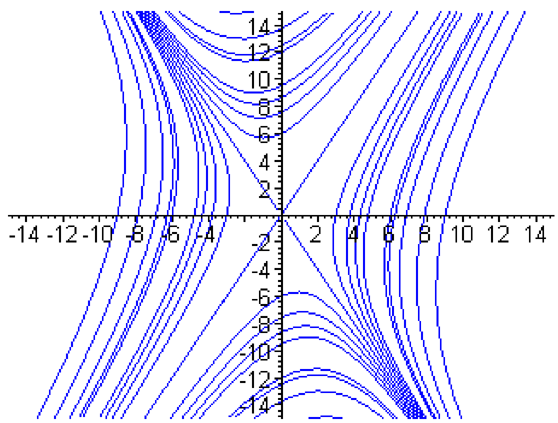
(c).



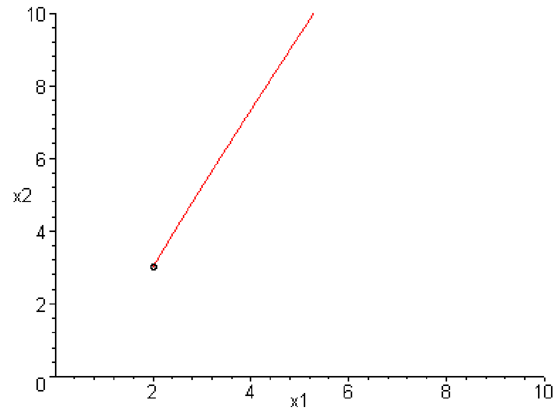
$26(a)$. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}.$$

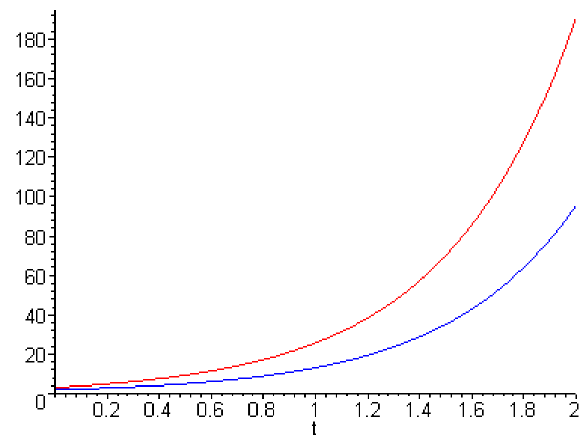
(b).



(b).



(c).



$28(a)$. We note that $(\mathbf{A} - r_i\mathbf{I})\boldsymbol{\xi}^{(i)} = \mathbf{0}$, for $i = 1, 2$.

(b). It follows that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{A}\,\boldsymbol{\xi}^{(1)} - r_2\boldsymbol{\xi}^{(1)} = r_1\boldsymbol{\xi}^{(1)} - r_2\boldsymbol{\xi}^{(1)}$.

(c). Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly *dependent*. Then there exist constants c_1 and c_2, not both zero, such that $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$. Assume that $c_1 \neq 0$. It is clear that $(\mathbf{A} - r_2\mathbf{I})\big(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)}\big) = \mathbf{0}$. On the other hand,

$$\begin{aligned}(\mathbf{A} - r_2\mathbf{I})\big(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)}\big) &= c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + \mathbf{0} \\ &= c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}.\end{aligned}$$

Since $r_1 \neq r_2$, we must have $c_1 = 0$, which leads to a contradiction.

(d). Note that $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(2)} = (r_2 - r_1)\boldsymbol{\xi}^{(2)}$.

(e). Let $n = 3$, with $r_1 \neq r_2 \neq r_3$. Suppose that $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$and $\boldsymbol{\xi}^{(3)}$ are indeed linearly *dependent*. Then there exist constants c_1 , c_2 and c_3 , not all zero, such that

$$c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)} = \mathbf{0}\,.$$

Assume that $c_1 \neq 0$. It is clear that $(\mathbf{A} - r_2\mathbf{I})\big(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)}\big) = \mathbf{0}$. On the other hand,

$$(\mathbf{A} - r_2\mathbf{I})\big(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)}\big) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + c_3(r_3 - r_2)\boldsymbol{\xi}^{(3)}.$$

It follows that $c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + c_3(r_3 - r_2)\boldsymbol{\xi}^{(3)} = \mathbf{0}$. Based on the result of Part (a), which is actually not dependent on the value of n , the vectors $\boldsymbol{\xi}^{(1)}$and $\boldsymbol{\xi}^{(3)}$ are linearly *independent*. Hence we must have $c_1(r_1 - r_2) = c_3(r_3 - r_2) = 0$, which leads to a contradiction.

29(a). Let $x_1 = y$ and $x_2 = y'$. It follows that $x_1' = x_2$ and

$$\begin{aligned} x_2' &= y'' \\ &= -\frac{1}{a}(c\,y + b\,y'). \end{aligned}$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{1}{a}(c\,x_1 + b\,x_2)\,. \end{aligned}$$

(b). The coefficient matrix is given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}.$$

Setting $\mathbf{x} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -r & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have

$$det(\mathbf{A} - r\,\mathbf{I}) = r^2 + \frac{b}{a}r + \frac{c}{a} = 0\,.$$

Multiplying both sides of the equation by a , we obtain $a\,r^2 + b\,r + c = 0$.

30. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1 - r & 1 \\ 4 & -2 - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = 0$. The characteristic equation is

$80\,r^2 + 24\,r + 1 = 0$, with roots $r_1 = -1/4$ and $r_2 = -1/20$. With $r = -1/4$, the system of equations reduces to $2\,\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,\,-2)^T$. Substitution of $r = -1/20$ results in the equation $2\,\xi_1 - 3\,\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (3\,,2)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix}1\\-2\end{pmatrix}e^{-t/4} + c_2\begin{pmatrix}3\\2\end{pmatrix}e^{-t/20}.$$

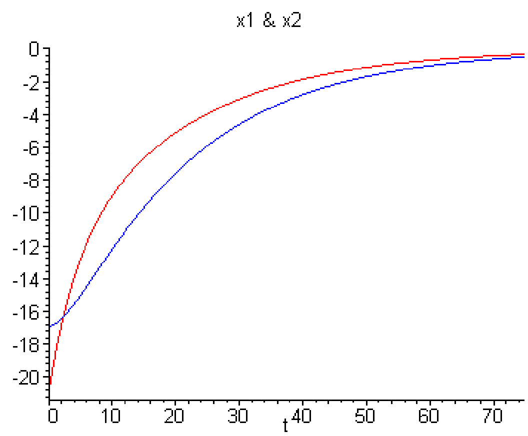
Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + 3\,c_2 &= -17 \\ -2\,c_1 + 2\,c_2 &= -21\,. \end{aligned}$$

Hence $c_1 = 29/8$ and $c_2 = -55/8$, and the solution of the IVP is

$$\mathbf{x} = \frac{29}{8}\begin{pmatrix}1\\-2\end{pmatrix}e^{-t/4} - \frac{55}{8}\begin{pmatrix}3\\2\end{pmatrix}e^{-t/20}.$$

(b).



(c). Both functions are monotone increasing. It is easy to show that $-0.5 \le x_1(t) < 0$ and $-0.5 \le x_2(t) < 0$ provided that $t > T \approx 74.39$.

$31(a)$. For $\alpha = 1/2$, solution of the ODE requires that

$$\begin{pmatrix}-1-r & -1\\ -1/2 & -1-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

The characteristic equation is $2\,r^2 + 4\,r + 1 = 0$, with roots $r_1 = -1 + 1/\sqrt{2}$ and $r_2 = -1 - 1/\sqrt{2}$. With $r = -1 + 1/\sqrt{2}$, the system of equations reduces to $\sqrt{2}\,\xi_1 + 2\,\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = \left(-\sqrt{2}\,,1\right)^T$. Substitution of $r = -1 - 1/\sqrt{2}$ results in the equation $\sqrt{2}\,\xi_1 - 2\,\xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = \left(\sqrt{2}\,,1\right)^T$. The general solution is

$$\mathbf{x} = c_1\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{\left(-2+\sqrt{2}\right)t/2} + c_2\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{\left(-2-\sqrt{2}\right)t/2}.$$

The eigenvalues are distinct and both *negative*. The equilibrium point is a stable *node*.

(b). For $\alpha = 2$, the characteristic equation is given by $r^2 + 2\,r - 1 = 0$, with roots $r_1 = -1 + \sqrt{2}$ and $r_2 = -1 - \sqrt{2}$. With $r = -1 + \sqrt{2}$, the system of equations reduces to $\sqrt{2}\,\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = \left(1\,, -\sqrt{2}\right)^T$. Substitution of $r = -1 - \sqrt{2}$ results in the equation $\sqrt{2}\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = \left(1\,,\sqrt{2}\right)^T$. The general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{\left(-1+\sqrt{2}\right)t} + c_2\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{\left(-1-\sqrt{2}\right)t}.$$

The eigenvalues are of opposite sign, hence the equilibrium point is a *saddle point*.

32. The system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

Solution of the system requires analysis of the eigenvalue problem

$$\begin{pmatrix} -\frac{1}{2} - r & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 3\,r + 2$, with roots $r_1 = -1$ and $r_2 = -2$. With $r = -1$, the equations reduce to $\xi_1 - \xi_2 = 0$. A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Setting $r = -2$, the system reduces to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Hence the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t}.$$

(b). The eigenvalues are distinct and both *negative*. We find that the equilibrium point $(0\,,0)$ is a stable *node*. Hence all solutions converge to $(0\,,0)$.

$33(a)$. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -\frac{R_1}{L} - r & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is

$$r^2 + \left(\frac{L + CR_1R_2}{LCR_2} \right) r + \frac{R_1 + R_2}{LCR_2} = 0\,.$$

The eigenvectors are *real* and *distinct*, provided that the *discriminant* is positive. That is,

$$\left(\frac{L + CR_1R_2}{LCR_2} \right)^2 - 4 \left(\frac{R_1 + R_2}{LCR_2} \right) > 0\,,$$

which simplifies to the condition

$$\left(\frac{1}{CR_2} - \frac{R_1}{L} \right)^2 - \frac{4}{LC} > 0\,.$$

(b). The parameters in the ODE are all positive. Observe that the *sum* of the roots is

$$-\frac{L + CR_1R_2}{LCR_2} < 0\,.$$

Also, the *product* of the roots is

$$\frac{R_1 + R_2}{LCR_2} > 0\,.$$

It follows that *both* roots are negative. Hence the *equilibrium solution* $I = 0\,,\ V = 0$ represents a stable node, which attracts *all* solutions.

(c). If the condition in Part (a) is not satisfied, that is,

$$\left(\frac{1}{CR_2} - \frac{R_1}{L} \right)^2 - \frac{4}{LC} \leq 0\,,$$

then the *real part* of the eigenvalues is

$$Re(r_{1,2}) = -\frac{L + CR_1R_2}{2\,LCR_2}\,.$$

As long as the parameters are *all* positive, then the solutions will still converge to the equilibrium point $(0\,,0)$.