Section 11.2

2. Based on the boundary conditions, $\lambda > 0$. The general solution of the ODE is

$$y(x) = c_1 cos\sqrt{\lambda}x + c_2 sin\sqrt{\lambda}x .$$

The boundary condition $y'(0) = 0$ requires that $c_2 = 0$. Imposing the second boundary condition, we find that $c_1 cos\sqrt{\lambda} = 0$. So for a nontrivial solution, $\sqrt{\lambda} = (2n-1)\pi/2$, $n = 1, 2, \cdots$. Therefore the eigenfunctions are given by

$$\phi_n(x) = k_n cos\frac{(2n-1)\pi x}{2} .$$

In this problem, $r(x) = 1$, and the normalization condition is

$$k_n^2 \int_0^1 \left[cos\frac{(2n-1)\pi x}{2}\right]^2 dx = 1 .$$

It follows that $k_n^2 = 2$. Therefore the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\, cos\frac{(2n-1)\pi x}{2} , \quad n = 1, 2, \cdots .$$

3. Based on the boundary conditions, $\lambda \geq 0$. For $\lambda = 0$, the eigenfunction is
$\phi_0(x) = k_0$.
Set $k_0 = 1$. With $\lambda > 0$, the general solution of the ODE is

$$y(x) = c_1 cos\sqrt{\lambda}x + c_2 sin\sqrt{\lambda}x .$$

Invoking the boundary conditions, we require that $c_2 = 0$ and $c_1\sqrt{\lambda}\, sin\sqrt{\lambda} = 0$. Since
$\lambda > 0$, the eigenvalues are $\lambda_n = n^2\pi^2,\ n = 1, 2, \cdots$, with corresponding eigenfunctions

$$\phi_n(x) = k_n cos\, n\pi x .$$

The normalization condition is

$$k_n^2 \int_0^1 cos^2 n\pi x\, dx = 1 .$$

It follows that $k_n^2 = 2$. Therefore the normalized eigenfunctions are

$$\phi_0(x) = 1 , \text{ and } \phi_n(x) = \sqrt{2}\, cos\, n\pi x , \quad n = 1, 2, \cdots .$$

4. From Prob. 8 in Section 11.1, the eigenfunctions are $\phi_n(x) = k_n cos\, \sqrt{\lambda_n}\, x$, in which
$cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. The normalization condition is

$$k_n^2 \int_0^1 cos^2 \sqrt{\lambda_n}\, x\, dx = 1\,.$$

First note that

$$\int_0^1 cos^2 \sqrt{\lambda_n}\, x\, dx = \frac{cos\sqrt{\lambda_n}\; sin\sqrt{\lambda_n} + \sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\,.$$

Based on the determinantal equation,

$$\begin{aligned} \frac{cos\sqrt{\lambda_n}\; sin\sqrt{\lambda_n} + \sqrt{\lambda_n}}{2\sqrt{\lambda_n}} &= \frac{1 + sin^2\sqrt{\lambda_n}}{2} \\ &= \frac{3 - cos\, 2\sqrt{\lambda_n}}{4}\,. \end{aligned}$$

Therefore

$$k_n^2 = \frac{4}{3 - cos\, 2\sqrt{\lambda_n}}$$

and the normalized eigenfunctions are given by

$$\phi_n(x) = \frac{2\, cos\sqrt{\lambda_n}\, x}{\sqrt{3 - cos\, 2\sqrt{\lambda_n}}}\,.$$

6. As shown in Prob. 1, the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\, sin\frac{(2n-1)\pi x}{2}\,, \quad n = 1, 2, \cdots.$$

Based on Eq. (34), with $r(x) = 1$, the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \sqrt{2}\int_0^1 sin\frac{(2m-1)\pi x}{2}dx \\ &= \frac{2\sqrt{2}}{(2m-1)\pi}\,. \end{aligned}$$

Therefore we obtain the formal expansion

$$1 = \frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}sin\frac{(2n-1)\pi x}{2}\,.$$

8. We consider the normalized eigenfunctions

$$\phi_n(x) = \sqrt{2}\, sin\frac{(2n-1)\pi x}{2}\,, \quad n = 1, 2, \cdots.$$

Based on Eq. (34), with $r(x) = 1$, the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \sqrt{2}\int_0^{1/2} sin\frac{(2m-1)\pi x}{2}dx \\ &= \frac{2\sqrt{2}}{(2m-1)\pi}\left[1 - cos\frac{(2m-1)\pi}{4}\right]. \end{aligned}$$

Therefore we obtain the formal expansion

$$f(x) = \frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\left[1 - cos\frac{(2n-1)\pi}{4}\right] sin\frac{(2n-1)\pi x}{2}\,.$$

9. The normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\, sin\frac{(2n-1)\pi x}{2}\,, \quad n = 1, 2, \cdots.$$

Based on Eq. (34), with $r(x) = 1$, the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \sqrt{2}\int_0^{1/2} 2x\, sin\frac{(2m-1)\pi x}{2}dx + \sqrt{2}\int_{1/2}^1 sin\frac{(2m-1)\pi x}{2}dx \\ &= \frac{8}{(2m-1)^2\pi^2}\left[sin\frac{m\pi}{2} - cos\frac{m\pi}{2}\right]. \end{aligned}$$

Therefore the formal expansion of the given function is

$$f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{sin\frac{n\pi}{2} - cos\frac{n\pi}{2}}{(2n-1)^2} sin\frac{(2n-1)\pi x}{2}\,.$$

11. From Prob. 4, the normalized eigenfunctions are given by

$$\phi_n(x) = \frac{2\, cos\sqrt{\lambda_n}\, x}{\sqrt{3 - cos\, 2\sqrt{\lambda_n}}}\,,$$

in which the eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. Based on Eq. (34), the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \frac{2}{\sqrt{3 - cos\, 2\sqrt{\lambda_m}}} \int_0^1 x\, cos\, \sqrt{\lambda_m}\, x dx \\ &= \frac{\sqrt{2}\left(2\, cos\, \sqrt{\lambda_m} - 1\right)}{\lambda_m \alpha_m}, \end{aligned}$$

in which $\alpha_m = \sqrt{1 + sin^2\sqrt{\lambda_m}}$.

12. The normalized eigenfunctions are given by

$$\phi_n(x) = \frac{2\, cos\, \sqrt{\lambda_n}\, x}{\sqrt{3 - cos\, 2\sqrt{\lambda_n}}},$$

in which the eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. Based on Eq. (34), the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \frac{2}{\sqrt{3 - cos\, 2\sqrt{\lambda_m}}} \int_0^1 (1-x) cos\, \sqrt{\lambda_m}\, x dx \\ &= \frac{\sqrt{2}\left(1 - cos\, \sqrt{\lambda_m}\right)}{\lambda_m \alpha_m}, \end{aligned}$$

in which $\alpha_m = \sqrt{1 + sin^2\sqrt{\lambda_m}}$.

13. We consider the normalized eigenfunctions

$$\phi_n(x) = \frac{2\, cos\, \sqrt{\lambda_n}\, x}{\sqrt{3 - cos\, 2\sqrt{\lambda_n}}},$$

in which the eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. The coefficients in the eigenfunction expansion are given by

$$
\begin{aligned}
c_n &= \int_0^1 f(x)\phi_n(x)dx \\
&= \frac{2}{\sqrt{3 - cos\, 2\sqrt{\lambda_n}}} \int_0^{1/2} cos\, \sqrt{\lambda_n}\, x dx \\
&= \frac{\sqrt{2}\, sin\big(\sqrt{\lambda_n}/2\big)}{\sqrt{\lambda_n}\, \alpha_n}\,,
\end{aligned}
$$

in which $\alpha_n = \sqrt{1 + sin^2\sqrt{\lambda_n}}$.

15. The differential equation can be written as

$$\big[(1+x^2)y'\big]' + y = 0\,,$$

with $p(x) = -1 - x^2$ and $q(x) = 1$. The boundary conditions are homogeneous and *separated*. Hence the BVP is *self-adjoint*.

16. Since the boundary conditions are *not* separated, the inner product is computed: Given u and v, sufficiently smooth and satisfying the boundary conditions,

$$
\begin{aligned}
(L[u], v) &= \int_0^1 [u''v + uv]dx \\
&= u'v\,\Big|_0^1 - \int_0^1 [u'v' + uv]dx \\
&= [u'v - uv']\,\Big|_0^1 + (u, L[v]).
\end{aligned}
$$

Based on the given boundary conditions,

$$
\begin{aligned}
u'(1)v(1) - u'(0)v(0) &= u(0)v(1) + 2u(1)v(0) \\
-u(1)v'(1) + u(0)v'(0) &= -u(1)v(0) - 2u(0)v(1)\,.
\end{aligned}
$$

Since

$$[u'v - uv']\,\Big|_0^1 = u(1)v(0) - u(0)v(1)\,,$$

the BVP is *not* self-adjoint.

18. The differential equation can be written as

$$-[y']' = \lambda y\,,$$

with $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. The boundary conditions are homogeneous and *separated*. Hence the BVP is *self-adjoint*.

19. If $a_2 = 0$, then

$$u'(1)v(1) - u(1)v'(1) = -\frac{b_2}{b_1}u'(1)v'(1) + \frac{b_2}{b_1}u'(1)v'(1) = 0\,,$$

and since $u(0) = v(0) = 0$,

$$u'(0)v(0) - u(0)v'(0) = 0\,.$$

If $b_2 = 0$, then $u(1) = v(1) = 0$ implies that

$$u'(1)v(1) - u(1)v'(1) = 0\,.$$

Furthermore,

$$u'(0)v(0) - u(0)v'(0) = -\frac{a_2}{a_1}u'(0)v'(0) + \frac{a_2}{a_1}u'(0)v'(0) = 0\,.$$

Clearly, the results are also true if $a_2 = b_2 = 0$.

20. Suppose that $\phi_1(x)$ and $\phi_2(x)$ are linearly independent eigenfunctions associated with an eigenvalue λ. The Wronskian is given by

$$W(\phi_1\,,\phi_2)(x) = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x).$$

Each of the eigenfunctions satisfies the boundary condition $a_1y(0) + a_2y'(0) = 0$. If either $a_1 = 0$ or $a_2 = 0$, then clearly $W(\phi_1\,,\phi_2)(0) = 0$. On the other hand, if a_2 is *not* equal to zero, then

$$\begin{aligned} W(\phi_1\,,\phi_2)(0) &= \phi_1(0)\phi_2'(0) - \phi_2(0)\phi_1'(0) \\ &= -\frac{a_1}{a_2}\phi_1(0)\phi_2(0) + \frac{a_1}{a_2}\phi_2(0)\phi_1(0) \\ &= 0\,. \end{aligned}$$

By Theorem 3.3.2, $W(\phi_1\,,\phi_2)(x) = 0$ for all $0 \le x \le 1$. Based on Theorem 3.3.3, $\phi_1(x)$ and $\phi_2(x)$ must be linearly *dependent*. Hence λ must be a simple eigenvalue.

22. We consider the operator

$$L[y] = -[p(x)y']' + q(x)y$$

on the interval $0 < x < 1$, together with the boundary conditions

$$a_1y(0) + a_2y'(0) = 0\;,\quad b_1y(1) + b_2y'(1) = 0\,.$$

Let $u = \phi + i\psi$ and $v = \xi + i\eta$. If u and v both satisfy the boundary conditions, then the real and imaginary parts also satisfy the same boundary conditions. Using the inner product

$$(u\,,v) = \int_0^1 u(x)\overline{v}(x)dx\,,$$

$$\begin{aligned}(L[u]\,,v) &= \int_0^1 \big[-[p(x)u']'\overline{v} + q(x)u\overline{v}\big]dx \\ &= \int_0^1 \big\{-[p(x)(\phi' + i\psi')]'\overline{v} + q(x)u\overline{v}\big\}dx \\ &= -p(x)(\phi' + i\psi')\overline{v}\Big|_0^1 + \int_0^1 \{p(x)(\phi' + i\psi')\,\overline{v}' + q(x)u\overline{v}\}dx\,.\end{aligned}$$

Integrating by parts, again,

$$\int_0^1 \{p(x)(\phi' + i\psi')\overline{v}'\}dx = (\phi + i\psi)p(x)\overline{v}'\Big|_0^1 - \int_0^1 \big\{[p(x)\overline{v}']'u\big\}dx\,.$$

Collecting the boundary terms,

$$p(x)[(\phi' + i\psi')\overline{v} - (\phi + i\psi)\overline{v}']\Big|_0^1 = p(x)[(\phi' + i\psi')(\xi - i\eta) - (\phi + i\psi)(\xi' - i\eta')]\Big|_0^1.$$

The *real* part is given by

$$\begin{aligned}p(x)[(\phi'\xi + \psi'\eta) - (\phi\xi' + \psi\eta')]\Big|_0^1 &= p(x)[(\phi'\xi - \phi\xi') + (\psi'\eta - \psi\eta')]\Big|_0^1 \\ &= p(x)[\phi'\xi - \phi\xi']\Big|_0^1 + p(x)[\psi'\eta - \psi\eta']\Big|_0^1.\end{aligned}$$

Since ϕ, ψ, ξ and η satisfy the boundary conditions, it follows that

$$p(x)[(\phi'\xi + \psi'\eta) - (\phi\xi' + \psi\eta')]\Big|_0^1 = 0\,.$$

Similarly, the *imaginary* part also vanishes. That is,

$$p(x)[(\psi'\xi - \psi\xi') - (\phi'\eta - \phi\eta')]\Big|_0^1 = 0\,.$$

Therefore

$$\begin{aligned}(L[u]\,,v) &= \int_0^1 \big\{-[p(x)\overline{v}']'u + q(x)u\overline{v}\big\}dx \\ &= (L[\overline{v}]\,,\overline{u}) \\ &= \overline{(\overline{u}\,,L[\overline{v}])}\,.\end{aligned}$$

The result follows from the fact that $\overline{(\overline{u}\,,L[\overline{v}])} = (u\,,L[v])$.

24. Based on the physical problem, $\lambda = P/EI > 0$. Let $\lambda = \mu^2$. The characteristic equation is $r^4 + \mu^2 r^2 = 0$, with roots $r_{1,2} = 0$, $r_3 = -\mu i$ and $r_4 = \mu i$. Hence the general solution is

$$y(x) = c_1 + c_2\,x + c_3 cos\,\mu x + c_4 sin\,\mu x\,.$$

(a). Simply supported on both ends : $y(0) = y''(0) = 0\,;\; y(L) = y''(L) = 0\,.$
Invoking the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_3 &= 0 \\ c_3 cos\,\mu L + c_4 sin\,\mu L &= 0 \\ c_1 + c_2 L + c_3 cos\,\mu L + c_4 sin\,\mu L &= 0\,. \end{aligned}$$

The determinantal equation is

$$sin\,\mu L = 0\,.$$

The nonzero roots are $\mu_n = n\pi/L\,,\; n = 1, 2, \cdots$. Therefore the eigenfunctions are $\phi_n = sin\,\mu_n x$, with corresponding eigenvalues $\lambda_n = n^2\pi^2/L^2$. Hence the smallest eigenvalue is $\lambda_1 = \pi^2/L^2$.

(b). Simply supported : $y(0) = y''(0) = 0\,;$ clamped : $y(L) = y'(L) = 0\,.$
Invoking the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_3 &= 0 \\ c_2 - c_3\mu sin\,\mu L + c_4\mu cos\,\mu L &= 0 \\ c_1 + c_2 L + c_3\, cos\,\mu L + c_4\, sin\,\mu L &= 0\,. \end{aligned}$$

The determinantal equation is

$$\mu L\, cos\,\mu L - sin\,\mu L = 0\,.$$

It follows that the eigenfunctions are given by

$$\phi_n(x) = sin\sqrt{\lambda_n}\,x - \left(\sqrt{\lambda_n} cos\sqrt{\lambda_n}L\right)x\,,$$

and the eigenvalues satisfy the equation $L\sqrt{\lambda_n}\, cos\,\sqrt{\lambda_n}\,L - sin\,\sqrt{\lambda_n}\,L = 0\,.$
The smallest eigenvalue is estimated as $\lambda_1 \approx (4.4934)^2/L^2$.

(c). Clamped : $y(0) = y'(0) = 0\,;$ clamped : $y(L) = y'(L) = 0\,.$
Invoking the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 + \mu c_4 &= 0 \\ c_1 + c_2 L + c_3 cos\,\mu L + c_4 sin\,\mu L &= 0 \\ c_2 - c_3\,\mu sin\,\mu L + c_4\,\mu cos\,\mu L &= 0\,. \end{aligned}$$

The determinantal equation is

$$2 - 2cos\,\mu L = \mu L\, sin\,\mu L\,.$$

It follows that the eigenfunctions are given by

$$\phi_n(x) = 1 - cos\sqrt{\lambda_n}\, x\,,$$

and the eigenvalues satisfy the equation $2 - 2cos\,\sqrt{\lambda_n}L = \sqrt{\lambda_n}L\, sin\,\sqrt{\lambda_n}L$. The smallest eigenvalue is $\lambda_1 = (2\pi)^2/L^2$.

26. As shown is Prob. 25 , the general solution is

$$y(x) = c_1 + c_2\, x + c_3 cos\,\mu x + c_4 sin\,\mu x\,.$$

Imposing the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_2 &= 0 \\ c_1 + c_3 &= 0 \\ c_2 + \mu c_4 &= 0 \\ c_3 cos\,\mu L + c_4 sin\,\mu L &= 0\,. \end{aligned}$$

For a nontrivial solution, it is necessary that

$$cos\,\mu L = 0\,.$$

We find that $c_2 = c_4 = 0$, and hence the eigenfunctions are given by

$$\phi_n(x) = 1 - cos\sqrt{\lambda_n}\, x\,.$$

The corresponding eigenvalues are $\lambda_n = (2n-1)^2\pi^2/4L^2,\, n = 1, 2, \cdots$. The smallest eigenvalue is $\lambda_1 = \pi^2/4L^2$.