Section 10.4

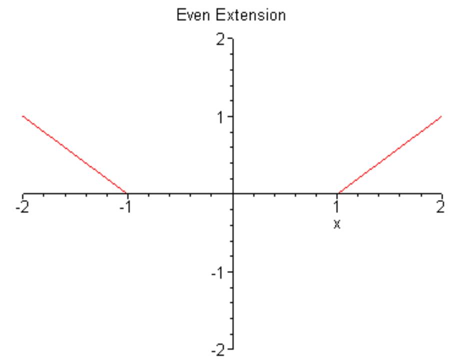
1. Since the function contains only odd powers of x, the function is *odd*.

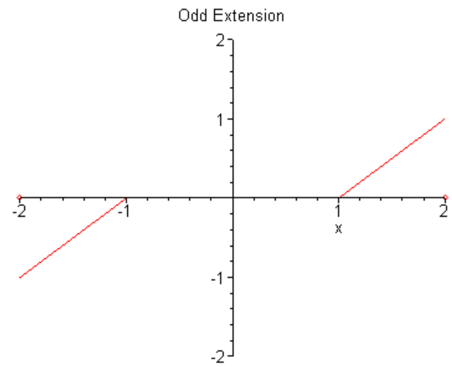
2. Since the function contains both odd and even powers of x, the function is *neither* even nor odd.

4. We have $sec\, x = 1/cos\, x$. Since the *quotient* of two even functions is even, the function is *even*.

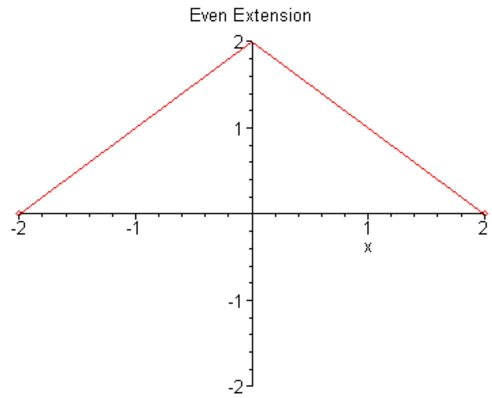
5. We can write $|x|^3 = |x| \cdot |x|^2 = |x| \cdot x^2$. Since both factors are even, it follows that the function is *even*.

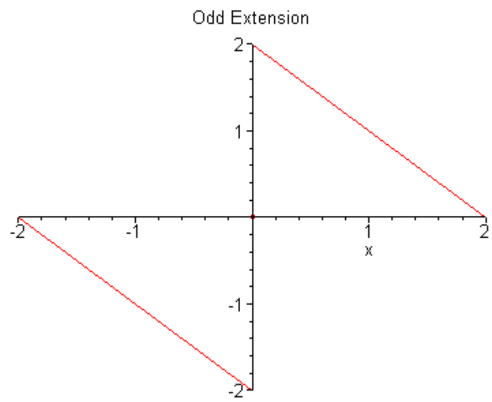
8. $L = 2$.



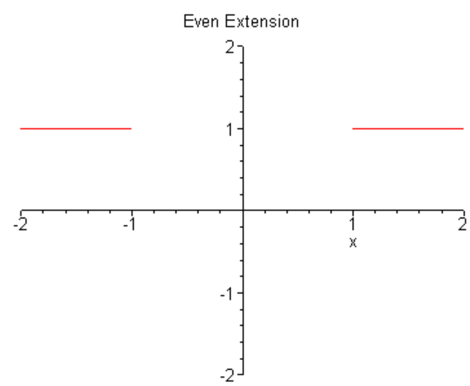


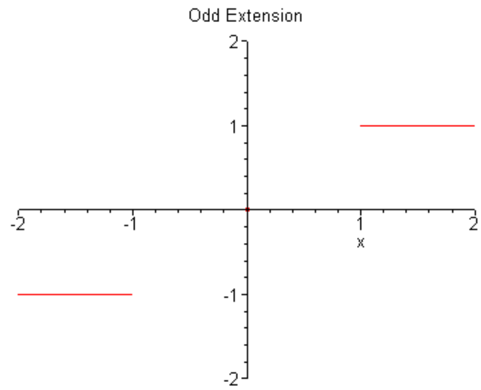
9. $L = 2$.


Even Extension
2
1
-2
-1
1
2
x
-1
-2

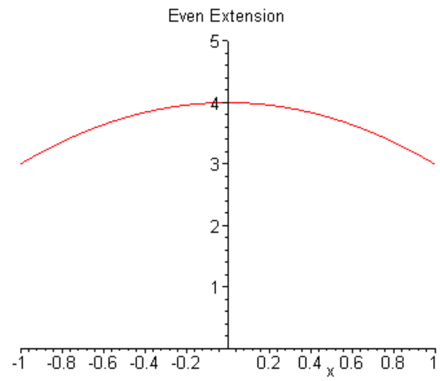

Odd Extension
2
1
-2
-1
1
2
x
-1
-2

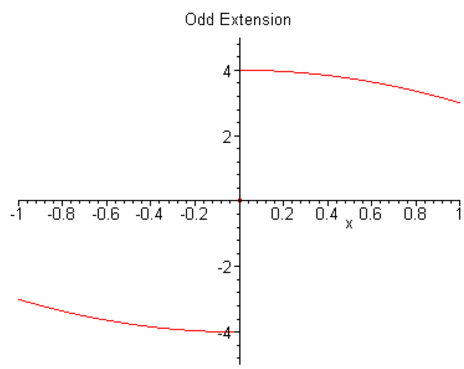
11. $L = 2$.


Even Extension
2
1
-2
-1
1
2
x
-1
-2



12. $L = 1$.





16. $L = 2$. For an *odd* extension of the function, the cosine coefficients are *zero*. The sine coefficients are given by

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L}dx \\
&= \int_0^1 x\, sin\frac{n\pi x}{2}dx + \int_1^2 sin\frac{n\pi x}{2}dx \\
&= 2\frac{2\, sin\frac{n\pi}{2} - n\pi\, cos\, n\pi}{n^2\pi^2}\,.
\end{aligned}
$$

Observe that

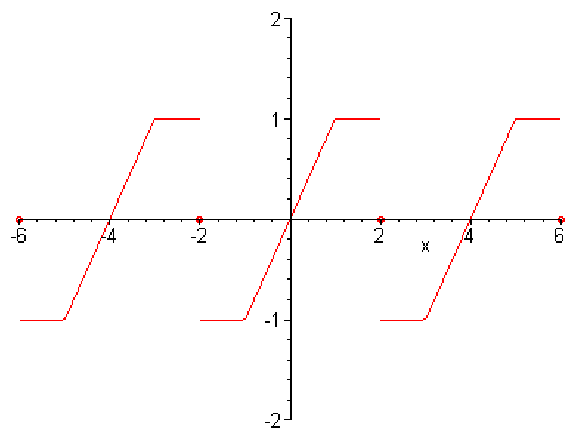
$$
sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0\,, & n = 2k \\ (-1)^{k+1}\,, & n = 2k-1 \end{cases} \quad , k = 1, 2, \cdots.
$$

Likewise,

$$
cos\, n\pi = \begin{cases} 1\,, & n = 2k \\ -1\,, & n = 2k-1 \end{cases} \quad , k = 1, 2, \cdots.
$$

Therefore the Fourier sine series of the specified function is

$$
f(x) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\, sin\, n\pi x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1} + (2n-1)\pi}{(2n-1)^2}\, sin\frac{(2n-1)\pi x}{2}.
$$



17. $L = \pi$. For an *even* extension of the function, the sine coefficients are *zero*. The cosine coefficients are given by

$$
\begin{aligned}
a_0 &= \frac{2}{L}\int_0^L f(x)dx \\
&= \frac{2}{\pi}\int_0^{\pi}(1)dx \\
&= 2\,,
\end{aligned}
$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{2}{\pi}\int_0^\pi (1)cos\ nx\,dx \\ &= 0\,. \end{aligned}$$

The even extension of the given function is a *constant* function. As expected, the Fourier cosine series is

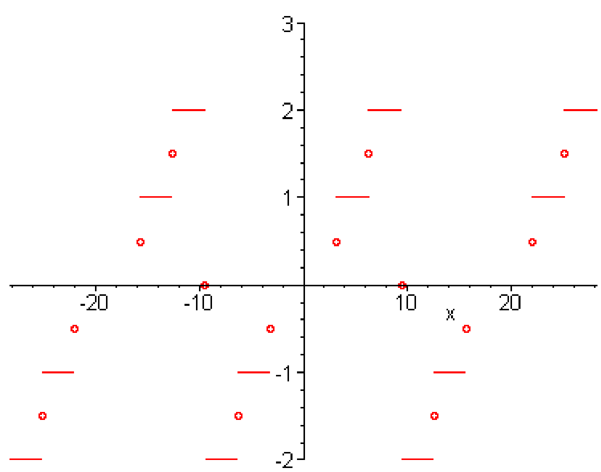
$$f(x) = \frac{a_0}{2} = 1\,.$$

19. $L = 3\pi$. For an *odd* extension of the function, the cosine coefficients are *zero*. The sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{2}{3\pi}\int_\pi^{2\pi} sin\frac{nx}{3}dx + \frac{2}{3\pi}\int_{2\pi}^{3\pi} 2\,sin\frac{nx}{3}dx \\ &= -\,2\frac{2\cos n\pi - cos\frac{n\pi}{3} - cos\frac{2n\pi}{3}}{n\pi}\,. \end{aligned}$$

Therefore the Fourier sine series of the specified function is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[cos\frac{n\pi}{3} + cos\frac{2n\pi}{3} - 2\cos n\pi\right] sin\frac{nx}{3}.$$



21. Extend the function over the interval $[-L\,, L]$ as

$$f(x) = \begin{cases} x + L\,, & -L \le x < 0 \\ L - x\,, & 0 \le x \le L\,. \end{cases}$$

Since the extended function is *even*, the sine coefficients are *zero*. The cosine coefficients
are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{2}{L}\int_0^L (L-x)dx \\ &= L\,, \end{aligned}$$

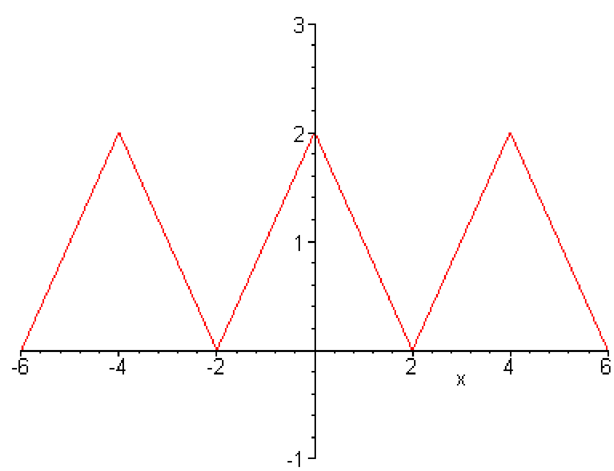
and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\int_0^L (L-x)cos\frac{n\pi x}{L}\,dx \\ &= 2L\frac{1-cos\,n\pi}{n^2\pi^2}\,. \end{aligned}$$

Therefore the Fourier cosine series of the extended function is

$$f(x) = \frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,cos\frac{(2n-1)\pi x}{L}.$$

In order to compare the result with Example 1 of Section 10.2, set $L = 2$. The cosine series converges to the function graphed below:



This function is a *shift* of the function in Example 1 of Section 10.2.

22. Extend the function over the interval $[-L, L]$ as

$$f(x) = \begin{cases} -x - L, & -L \le x < 0 \\ L - x, & 0 < x \le L, \end{cases}$$

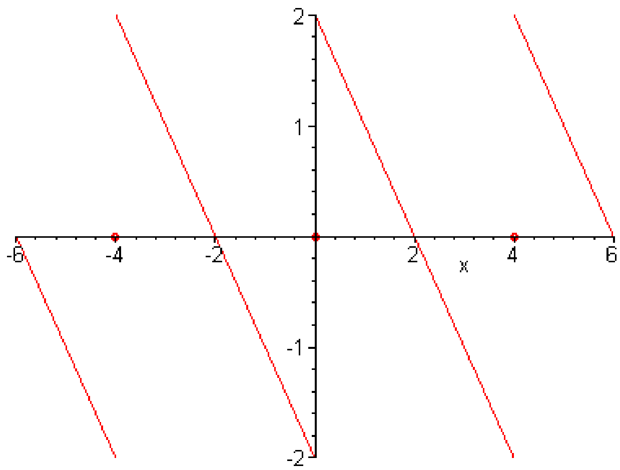
with $f(0) = 0$. Since the extended function is *odd*, the cosine coefficients are *zero*. The sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= \frac{2}{L}\int_0^L (L - x) sin\frac{n\pi x}{L}\, dx \\ &= \frac{2L}{n\pi}\,. \end{aligned}$$

Therefore the Fourier cosine series of the extended function is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\, sin\frac{n\pi x}{L}.$$

Setting $L = 2$, for example, the series converges to the function graphed below:



$23(a)$. $L = 2\pi$. For an *even* extension of the function, the sine coefficients are *zero*. The cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{1}{\pi}\int_0^\pi x\,dx \\ &= \pi/2\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{\pi}\int_0^\pi x\,cos\frac{nx}{2}\,dx \\ &= 2\frac{2\,cos\left(\frac{n\pi}{2}\right) + n\pi sin\left(\frac{n\pi}{2}\right) - 2}{n^2\pi}\,. \end{aligned}$$

Therefore the Fourier cosine series of the given function is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\pi}{n}sin\frac{n\pi}{2} + \frac{2}{n^2}\left(cos\frac{n\pi}{2} - 1\right)\right]cos\frac{nx}{2}.$$

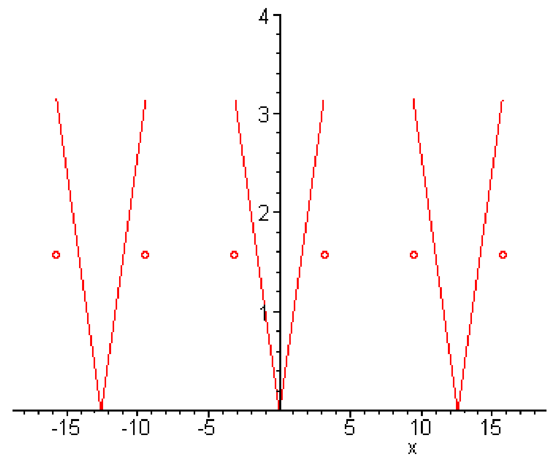
Observe that

$$sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0\,, & n = 2k \\ (-1)^{k+1}\,, & n = 2k-1 \end{cases}\quad, k = 1, 2, \cdots.$$

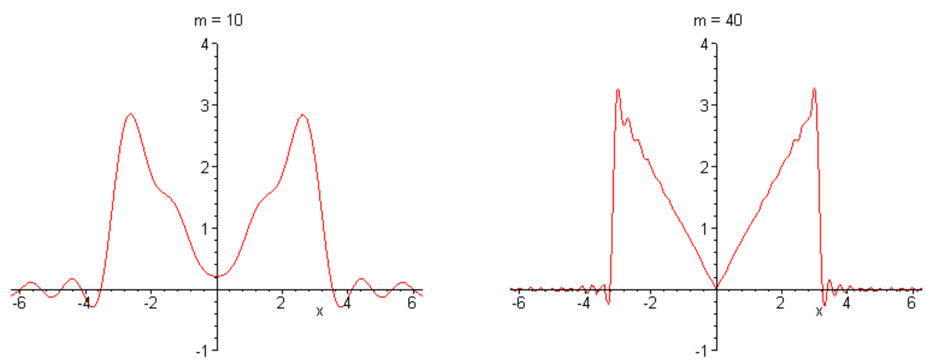
Likewise,

$$cos\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^k, & n = 2k \\ 0\,, & n = 2k-1 \end{cases} \quad , k = 1, 2, \cdots.$$

(b).



(c).



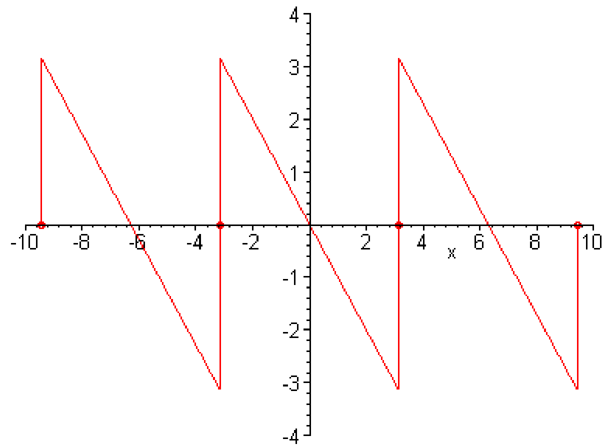
24(a). $L = \pi$. For an *odd* extension of the function, the cosine coefficients are *zero*. Note that $f(x) = -x$ on $0 \le x < \pi$. The sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= -\frac{2}{\pi}\int_0^{\pi} x\, sin\, nx\, dx \\ &= \frac{2\, cos\, n\pi}{n}. \end{aligned}$$

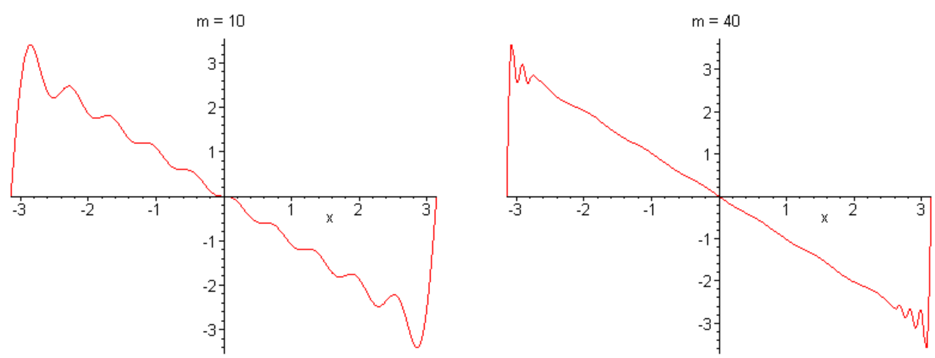
Therefore the Fourier sine series of the given function is

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} sin\, nx.$$

(b).



(c).



$26(a)$. $L = 4$. For an *even* extension of the function, the sine coefficients are *zero*. The cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{1}{2}\int_0^4 \left(x^2 - 2x\right)dx \\ &= 8/3\,, \end{aligned}$$

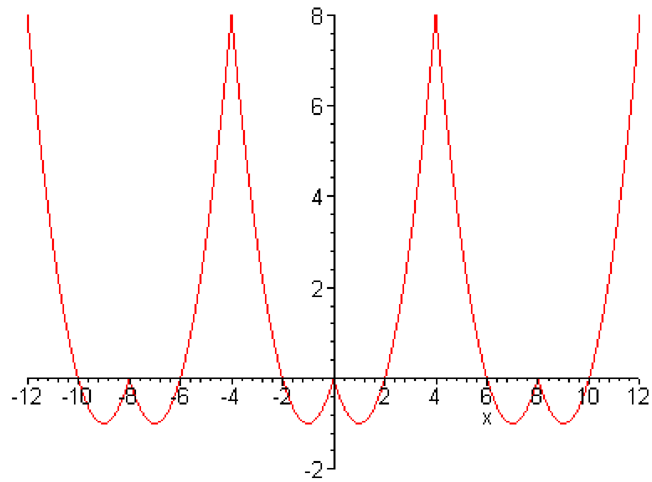
and for $n > 0$,

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x) cos \frac{n\pi x}{L} dx \\
&= \frac{1}{2}\int_0^4 (x^2 - 2x) cos \frac{n\pi x}{4}\, dx \\
&= 16\, \frac{1 + 3\, cos\, n\pi}{n^2\pi^2}\,.
\end{aligned}
$$

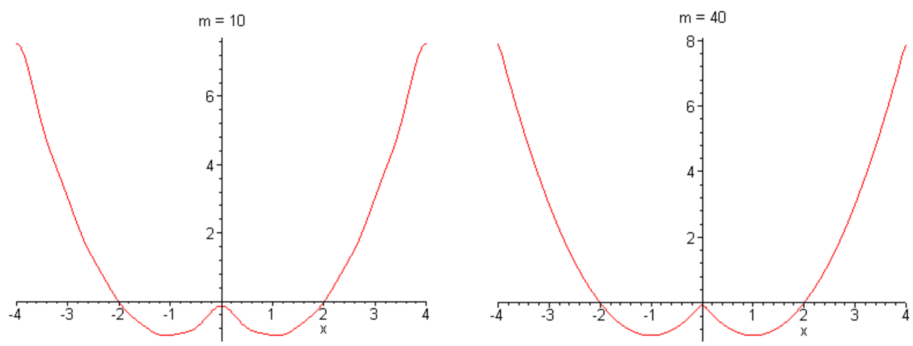
Therefore the Fourier cosine series of the given function is

$$
f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1 + 3(-1)^n}{n^2} cos \frac{n\pi x}{4}.
$$

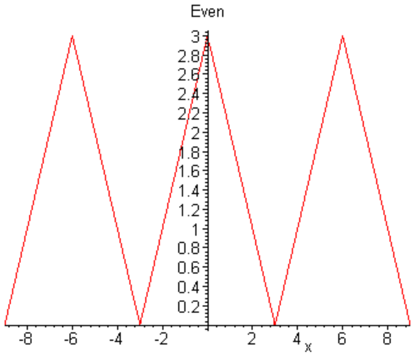
(b).

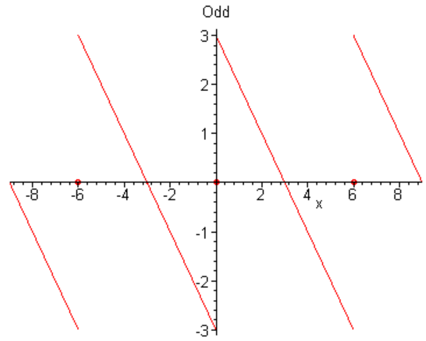


(c).



$27(a)$.





(b). $L = 3$. For an *even* extension of the function, the cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{2}{3}\int_0^3 (3-x)dx \\ &= 3\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{2}{3}\int_0^3 (3-x)cos\frac{n\pi x}{3}\,dx \\ &= 6\,\frac{1-cos\,n\pi}{n^2\pi^2}\,. \end{aligned}$$

Therefore the Fourier cosine series of the given function is

$$g(x) = \frac{3}{2} + \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}cos\frac{n\pi x}{3}.$$

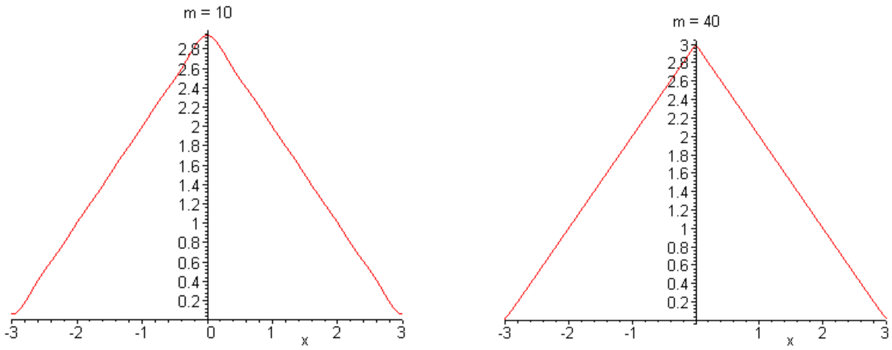
For an *odd* extension of the function, the sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{2}{3}\int_0^3 (3-x)sin\frac{n\pi x}{3}\,dx \\ &= \frac{6}{n\pi}\,. \end{aligned}$$

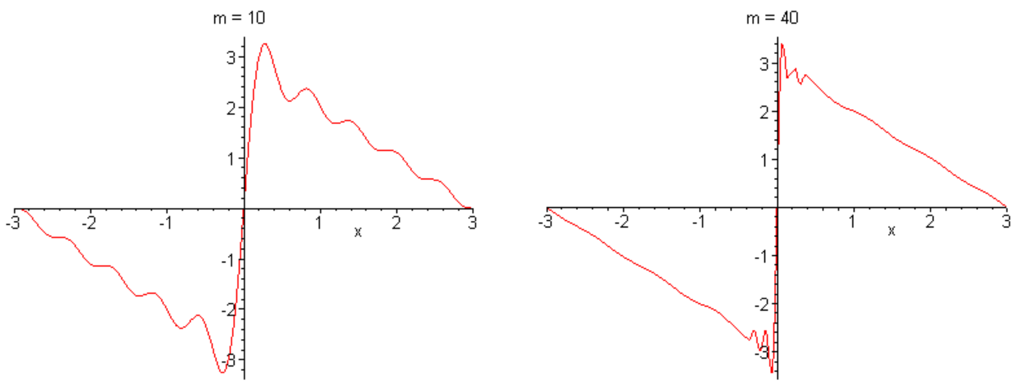
Therefore the Fourier sine series of the given function is

$$h(x) = \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} sin \frac{n\pi x}{3}.$$

(c). For the *even* extension:

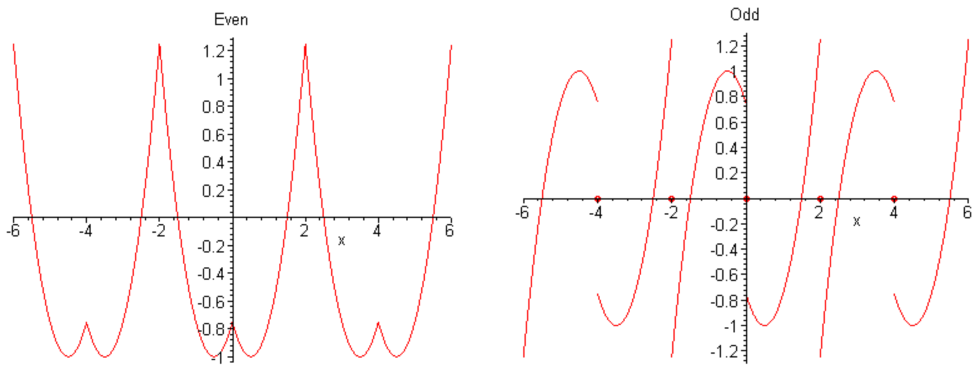


For the *odd* extension:



(d). Since the *even* extension is *continuous*, the series converges uniformly. On the other
hand, the *odd* extension is *discontinuous*. Gibbs' phenomenon results in a finite error for all values of n .

$29(a)$.



(b). $L = 2$. For an *even* extension of the function, the cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \int_0^2 \left[\frac{4x^2 - 4x - 3}{4}\right] dx \\ &= -5/6\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x) cos\frac{n\pi x}{L} dx \\ &= \int_0^2 \left[\frac{4x^2 - 4x - 3}{4}\right] cos\frac{n\pi x}{2}\, dx \\ &= 4\,\frac{1 + 3\,cos\,n\pi}{n^2\pi^2}\,. \end{aligned}$$

Therefore the Fourier cosine series of the given function is

$$g(x) = -\frac{5}{12} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1 + 3(-1)^n}{n^2} cos\frac{n\pi x}{2}.$$

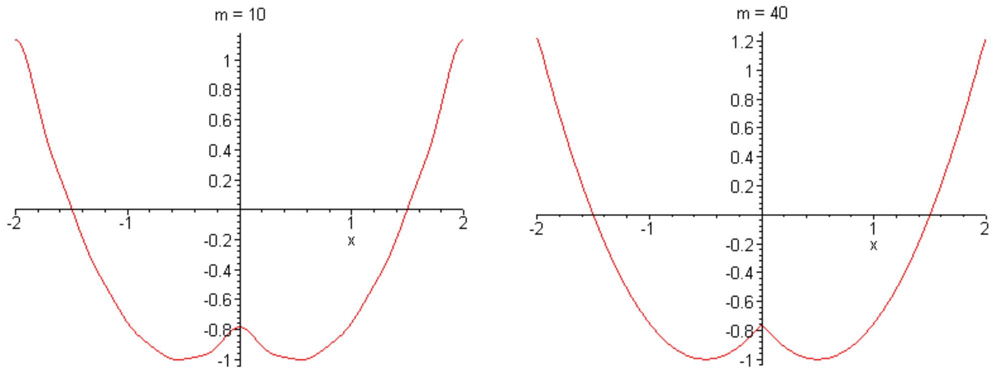
For an *odd* extension of the function, the sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= \int_0^2 \left[\frac{4x^2 - 4x - 3}{4}\right] sin\frac{n\pi x}{2}\, dx \\ &= -\frac{32 + 3n^2\pi^2 + 5n^2\pi^2 cos\,n\pi - 32\,cos\,n\pi}{2\,n^3\pi^3}\,. \end{aligned}$$

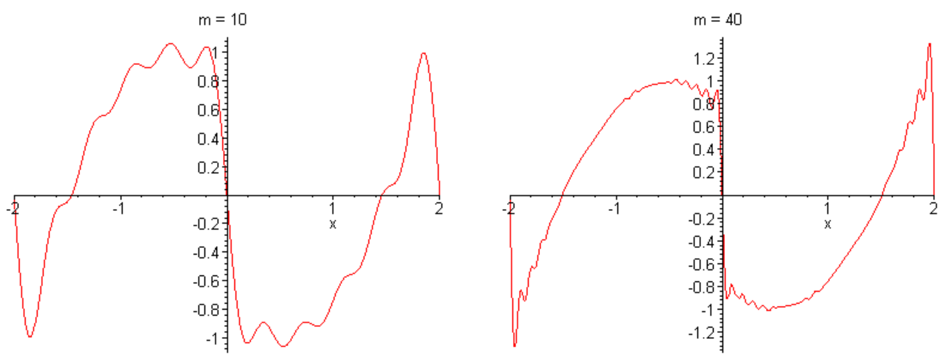
Therefore the Fourier sine series of the given function is

$$h(x) = -\frac{1}{2\pi^3}\sum_{n=1}^{\infty}\frac{32(1-\cos n\pi)+n^2\pi^2(3+5\cos n\pi)}{n^3}\sin\frac{n\pi x}{2}.$$

(c). For the *even* extension:

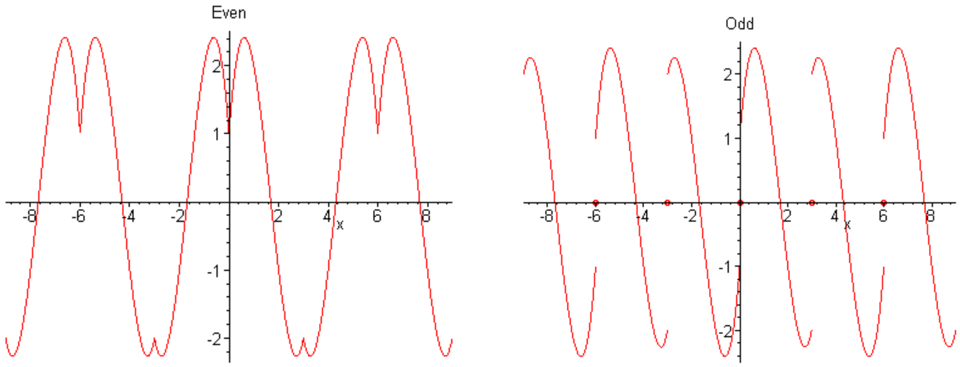


For the odd extension:



(d). Since the *even* extension is *continuous*, the series converges uniformly. On the other
hand, the *odd* extension is *discontinuous*. Gibbs' phenomenon results in a finite error for all values of n .

$30(a)$.



(b). $L = 3$. For an *even* extension of the function, the cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{2}{3}\int_0^3 \left(x^3 - 5x^2 + 5x + 1\right)dx \\ &= 1/2\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{2}{3}\int_0^3 \left(x^3 - 5x^2 + 5x + 1\right)cos\frac{n\pi x}{3}\,dx \\ &= 2\,\frac{162 - 15\,n^2\pi^2 + 6\,n^2\pi^2 cos\,n\pi - 162\,cos\,n\pi}{n^4\pi^4}\,. \end{aligned}$$

Therefore the Fourier cosine series of the given function is

$$g(x) = \frac{1}{4} + \frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{162(1 - \,cos\,n\pi) - 3\,n^2\pi^2(5 - 2\,cos\,n\pi)}{n^4}cos\frac{n\pi x}{3}.$$

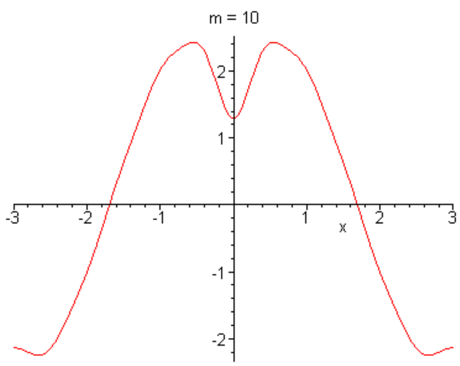
For an *odd* extension of the function, the sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{2}{3}\int_0^3 \left(x^3 - 5x^2 + 5x + 1\right)sin\frac{n\pi x}{3}\,dx \\ &= 2\,\frac{90 + \,n^2\pi^2 + 2\,n^2\pi^2 cos\,n\pi + 72\,cos\,n\pi}{n^3\pi^3}\,. \end{aligned}$$

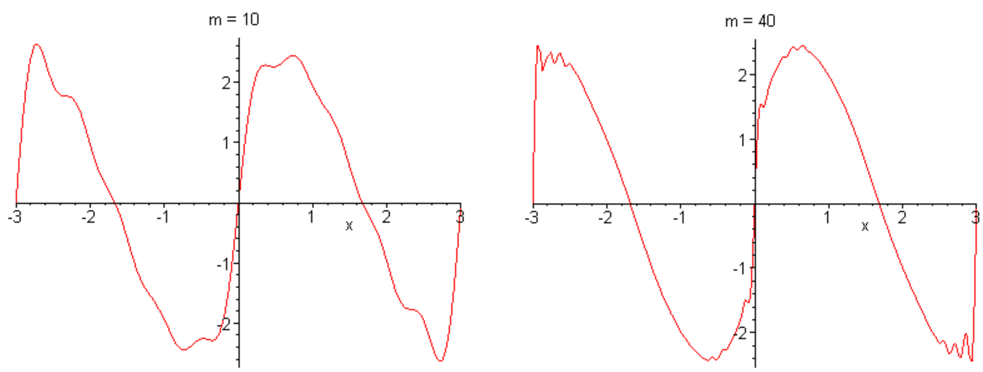
Therefore the Fourier sine series of the given function is

$$h(x) = \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{18(5+4\cos n\pi) + n^2\pi^2(1+2\cos n\pi)}{n^3} \sin\frac{n\pi x}{3}.$$

(c). For the *even* extension:



For the *odd* extension:



(d). Since the *even* extension is *continuous*, the series converges uniformly. On the other
hand, the *odd* extension is *discontinuous*. Gibbs' phenomenon results in a finite error for all values of n; particularly at $x = \pm 3$.

33. Let $f(x)$ be a differentiable *even* function. For any x in its domain,

$$f(-x+h) - f(-x) = f(x-h) - f(x).$$

It follows that

$$
\begin{aligned}
f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} \\
&= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h} \\
&= -\lim_{h \to 0} \frac{f(x-h) - f(x)}{(-h)} .
\end{aligned}
$$

Setting $h = -\delta$, we have

$$
\begin{aligned}
f'(-x) &= -\lim_{h \to 0} \frac{f(x+\delta) - f(x)}{\delta} \\
&= -\lim_{-\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} \\
&= -f'(x) .
\end{aligned}
$$

Therefore $f'(-x) = -f'(x)$.

If $f(x)$ is a differentiable *odd* function, for any x in its domain,

$$
f(-x+h) - f(-x) = -f(x-h) + f(x) .
$$

It follows that

$$
\begin{aligned}
f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} \\
&= \lim_{h \to 0} \frac{-f(x-h) + f(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x-h) - f(x)}{(-h)} .
\end{aligned}
$$

Setting $h = -\delta$, we have

$$
\begin{aligned}
f'(-x) &= \lim_{h \to 0} \frac{f(x+\delta) - f(x)}{\delta} \\
&= \lim_{-\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} \\
&= f'(x) .
\end{aligned}
$$

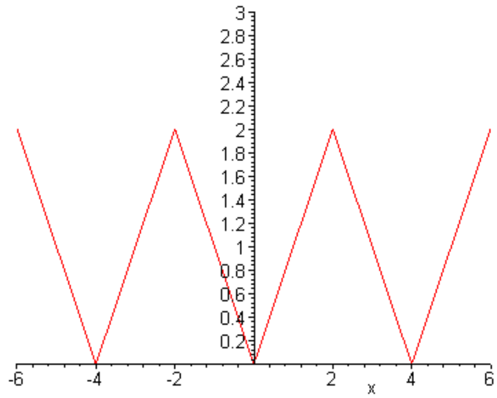
Therefore $f'(-x) = f'(x)$.

36. From Example 1 of Section 10.2, the function

$$
f(x) = \begin{cases} -x , & -2 \le x < 0 \\ x, & 0 \le x < 2 , \end{cases}
$$

$(L = 2)$ has a convergent Fourier series

$$f(x) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} cos \frac{(2n-1)\pi x}{2}.$$



Since $f(x)$ is continuous, the series converges everywhere. In particular, at $x = 0$, we have

$$0 = f(0) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

It follows immediately that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

40. Since one objective is to obtain a Fourier series containing only *cosine* terms, any extension of $f(x)$ should be an *even* function. Another objective is to derive a series containing only the terms

$$cos \frac{(2n-1)\pi x}{2L}, \ n = 1, 2, \cdots.$$

First note that the functions

$$cos \frac{n\pi x}{L}, \ n = 1, 2, \cdots$$

are *symmetric* about $x = L$. Indeed,

$$\begin{aligned} cos\frac{n\pi(2L-x)}{L} &= cos\Big(2n\pi - \frac{n\pi x}{L}\Big) \\ &= cos\Big(- \frac{n\pi x}{L}\Big) \\ &= cos\frac{n\pi x}{L} \,. \end{aligned}$$

It follows that if $f(x)$ is extended into $(L\,,2L)$ as an *antisymmetric* function about $x = L$,
that is, $f(2L-x) = -f(x)$ for $0 \le x \le 2L$, then

$$\int_0^{2L} f(x) cos\frac{n\pi x}{L}\,dx = 0\,.$$

This follows from the fact that the integrand is *antisymmetric* function about $x = L$. Now
extend the function $f(x)$ to obtain

$$\tilde{f}\,(x) = \begin{cases} f(x), & 0 \le x < L \\ -f(2L-x), & L < x < 2L\,. \end{cases}$$

Finally, extend the resulting function into $(-2L\,,0)$ as an *even* function, and then as a periodic function of period $4L$.

By construction, the Fourier series will contain only *cosine* terms. We first note that

$$\begin{aligned} a_0 &= \frac{2}{2L}\int_0^{2L} \tilde{f}\,(x)\,dx \\ &= \frac{1}{L}\int_0^{L} f(x)dx - \frac{1}{L}\int_L^{2L} f(2L-x)dx \\ &= \frac{1}{L}\int_0^{L} f(x)dx - \frac{1}{L}\int_0^{L} f(u)du \\ &= 0\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{2L}\int_0^{2L} \tilde{f}\,(x) cos\frac{n\pi x}{2L}\,dx \\ &= \frac{1}{L}\int_0^{L} f(x) cos\frac{n\pi x}{2L}dx - \frac{1}{L}\int_L^{2L} f(2L-x) cos\frac{n\pi x}{2L}dx\,. \end{aligned}$$

For the second integral, let $u = 2L - x$. Then

$$cos\frac{n\pi x}{2L} = cos\frac{n\pi(2L+u)}{2L} = (-1)^n cos\frac{n\pi u}{2L}$$

and therefore

$$\int_L^{2L} f(2L-x)\cos\frac{n\pi x}{2L}dx = (-1)^n \int_0^L f(u)\cos\frac{n\pi u}{2L}du\,.$$

Hence

$$a_n = \frac{1-(-1)^n}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}dx\,.$$

It immediately follows that $a_n = 0$ for $n = 2k$, $k = 0, 1, 2, \cdots$, and

$$a_{2k-1} = \frac{2}{L}\int_0^L f(x)\cos\frac{(2k-1)\pi x}{2L}dx\,,\ \text{for}\ k = 1, 2, \cdots.$$

The associated Fourier series representation

$$f(x) = \sum_{n=0}^{\infty} a_{2n-1}\cos\frac{(2n-1)\pi x}{2L}$$

converges almost everywhere on $(-2L\,, 2L)$ and hence on $(0\,, L)$.

For example, if $f(x) = x$ for $0 \le x \le L = 1$, the graph of the extended function is: