Chapter Nine

Section 9.1

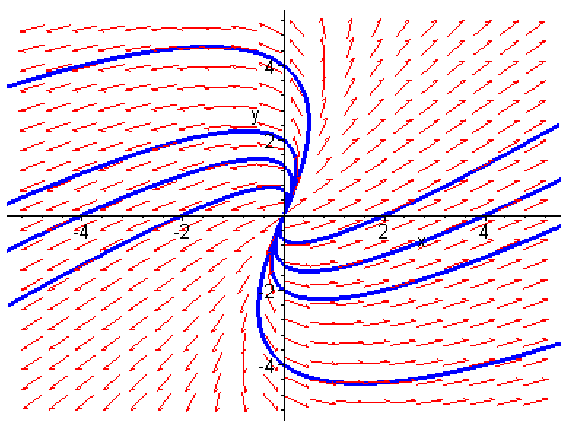
$2(a)$. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

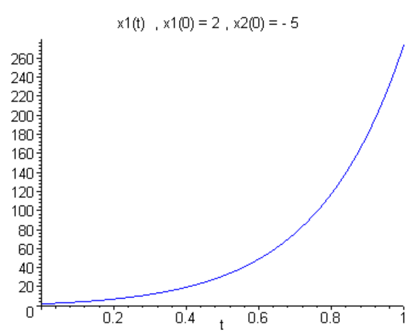
$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

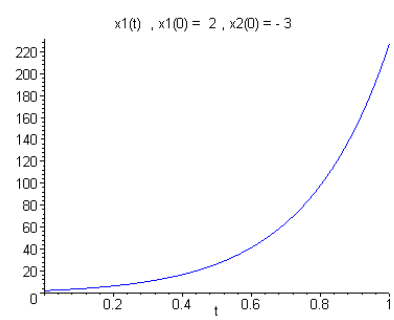
For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 6\,r + 8 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = 4$. For $r = 2$, the system of equations reduces to $3\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,3)^T$. Substitution of $r = 4$ results in the single equation $\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,1)^T$.

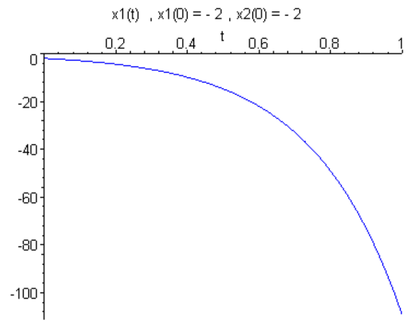
(b). The eigenvalues are *real* and *positive*, hence the critical point is an *unstable node*.

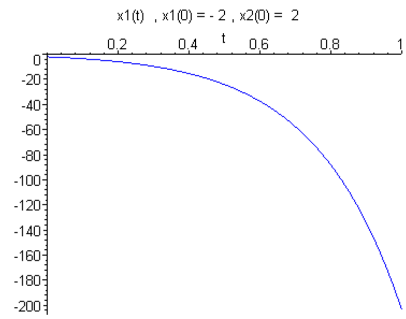
(c, d).











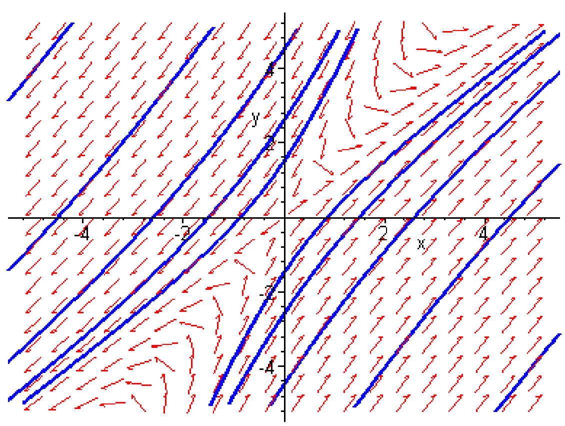
$3(a)$. Solution of the ODE requires analysis of the algebraic equations

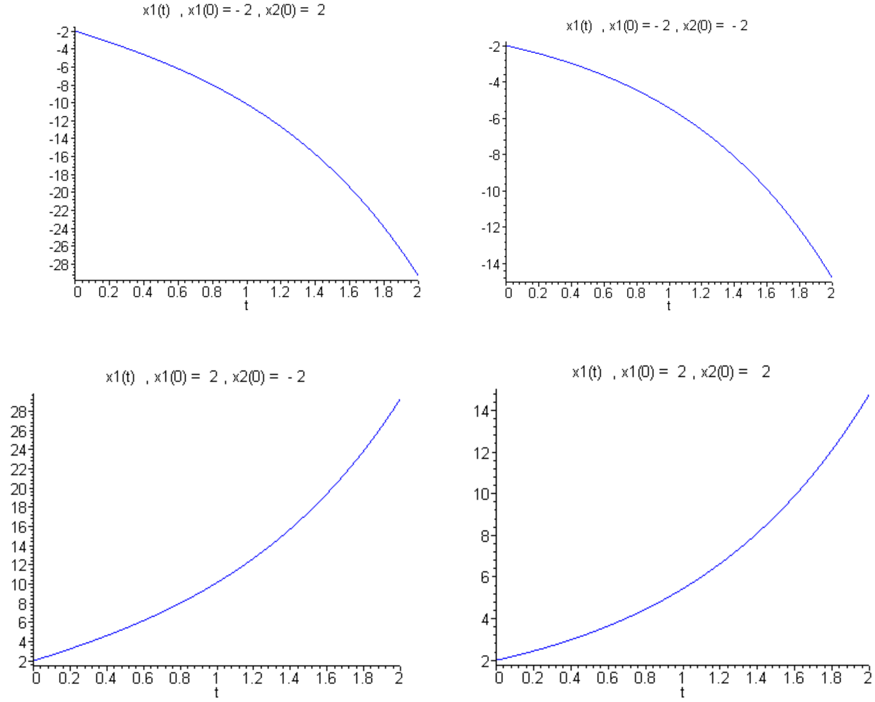
$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -1$ results in the single equation $3\,\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$.

(b). The eigenvalues are *real*, with $r_1\,r_2 < 0$. Hence the critical point is a *saddle*.

(c, d).





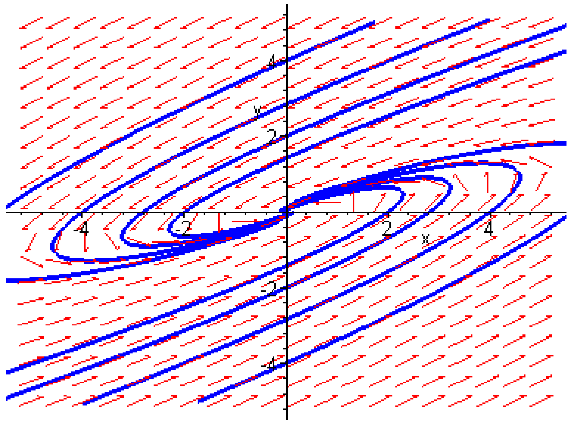
$5(a)$. The characteristic equation is given by

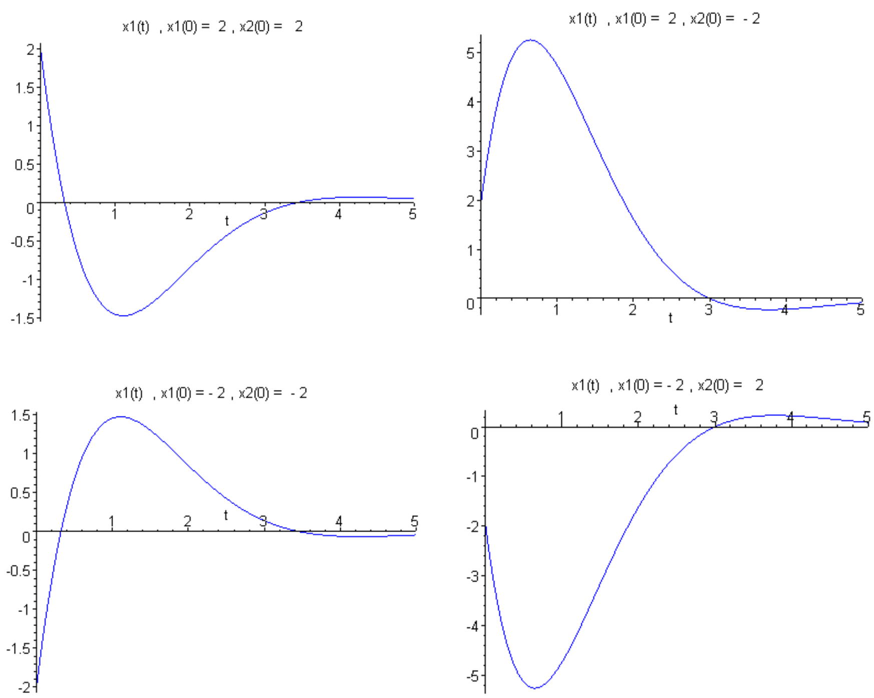
$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = r^2 + 2r + 2 = 0\,.$$

The equation has *complex* roots $r_1 = -1 + i$ and $r_2 = -1 - i$. For $r = -1 + i$, the components of the solution vector must satisfy $\xi_1 - (2+i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$. Substitution of $r = -1 - i$ results in the single equation $\xi_1 - (2-i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$.

(b). The eigenvalues are *complex conjugates*, with negative real part. Hence the origin is a *stable spiral*.

(c, d).





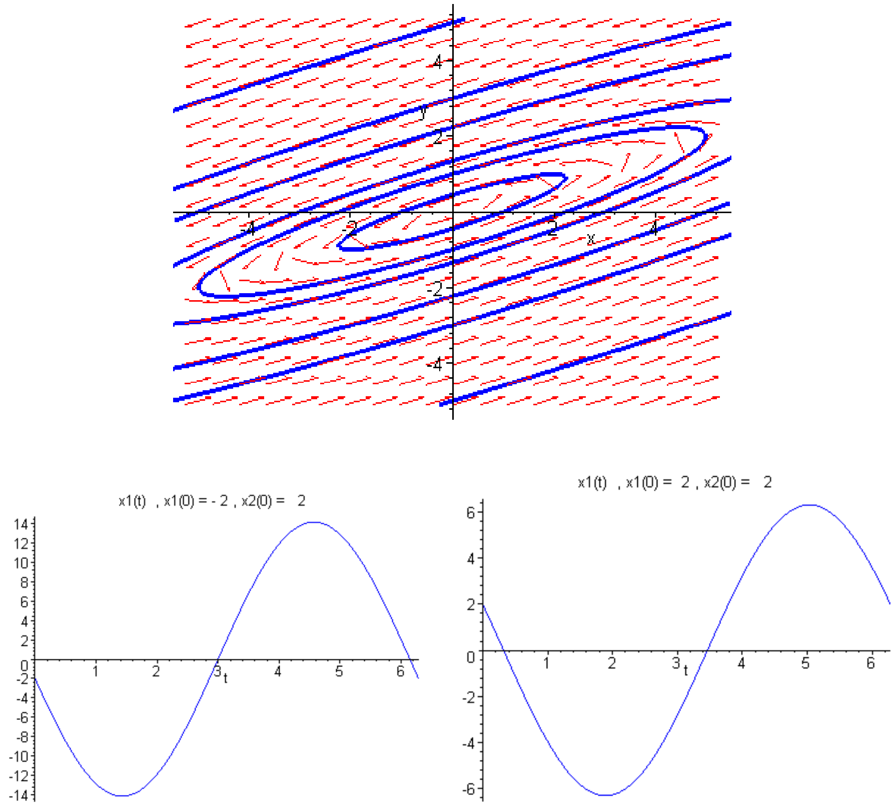
$6(a)$. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$.

(b). The eigenvalues are *purely imaginary*. Hence the critical point is a *center*.

(c, d).



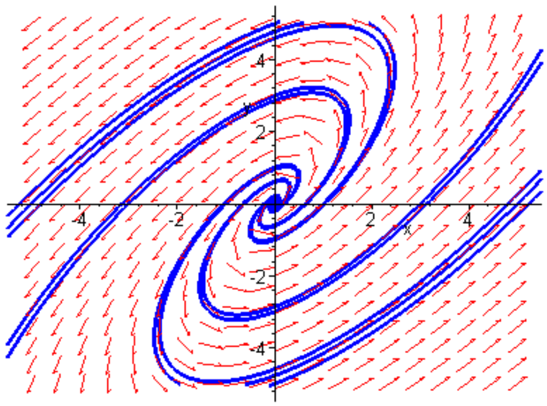
$7(a)$. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

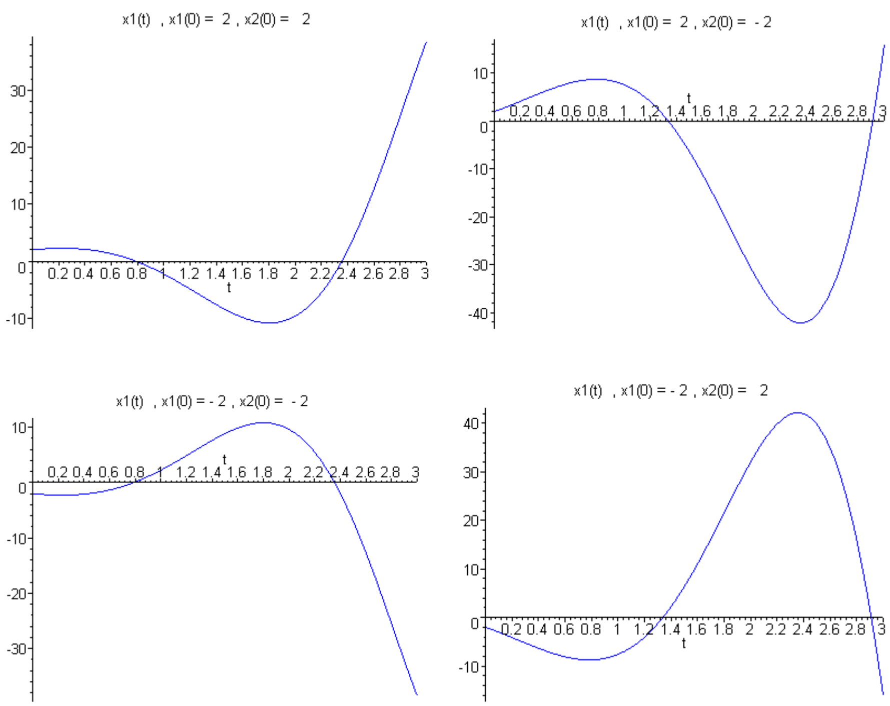
$$\begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 2r + 5 = 0$. The roots of the characteristic equation are $r = 1 \pm 2i$. Substituting $r = 1 - 2i$, the two equations reduce to $(1+i)\xi_1 - \xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (1\,, 1+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1\,, 1-i)^T$.

(b). The eigenvalues are *complex conjugates*, with positive real part. Hence the origin is an *unstable spiral*.

(c, d).





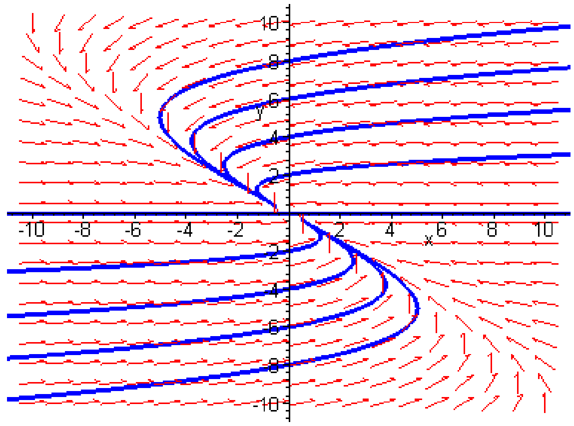
$8(a)$. The characteristic equation is given by

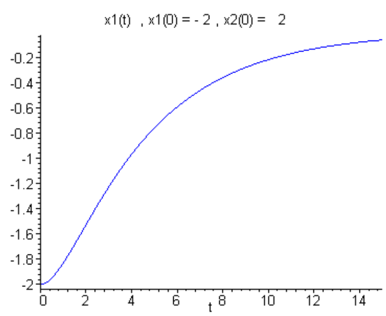
$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = (r+1)(r+0.25) = 0\,,$$

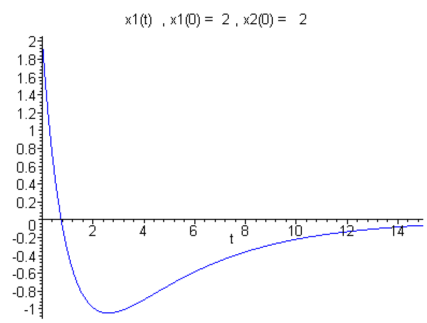
with roots $r_1 = -1$ and $r_2 = -0.25$. For $r = -1$, the components of the solution vector must satisfy $\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,0)^T$. Substitution of $r = -0.25$ results in the single equation $0.75\,\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (4\,,-3)^T$.

(b). The eigenvalues are *real* and both *negative*. Hence the critical point is a *stable* *node*.

(c, d).







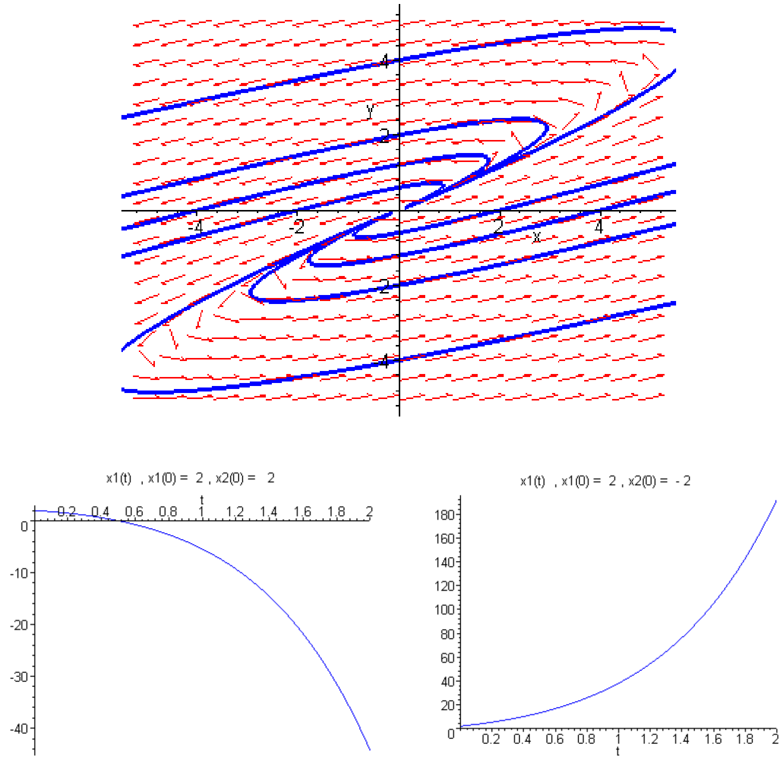
$9(a)$. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 2\,r + 1 = 0$. The single root of the characteristic equation is $r = 1$. Setting $r = 1$, the components of the solution vector must satisfy $\xi_1 - 2\,\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi} = (2\,,1)^T$.

(b). Since there is only one linearly independent eigenvector, the critical point is an $unstable$, *improper node*.

(c, d).



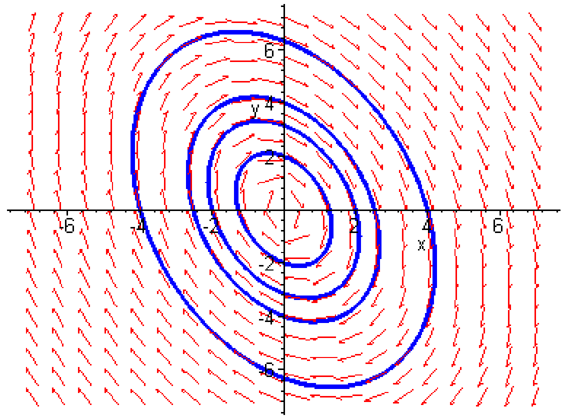
$10(a)$. The characteristic equation is given by

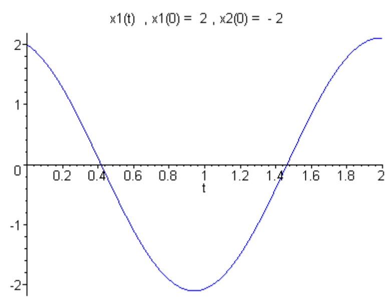
$$\begin{vmatrix} 1-r & 2 \\ -5 & -1-r \end{vmatrix} = r^2 + 9 = 0\,.$$

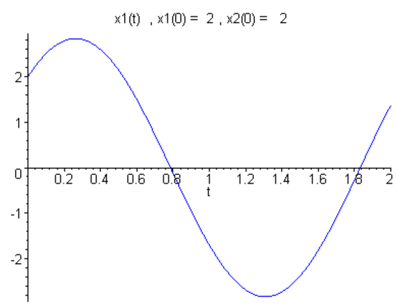
The equation has *complex* roots $r_{1,2} = \pm 3i$. For $r = -3i$, the components of the solution vector must satisfy $5\,\xi_1 + (1-3i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1-3i\,,\,-5)^T$. Substitution of $r = 3i$ results in $5\,\xi_1 + (1+3i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1+3i\,,\,-5)^T$.

(b). The eigenvalues are *purely imaginary*, hence the critical point is a *center*.

(c, d).



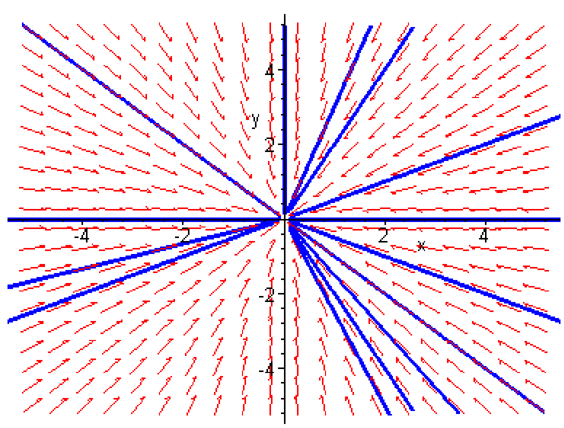


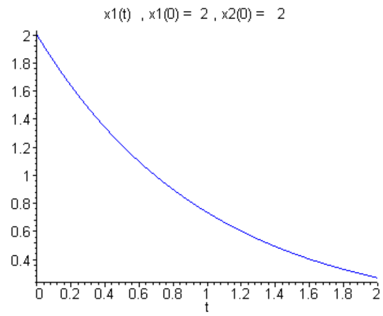


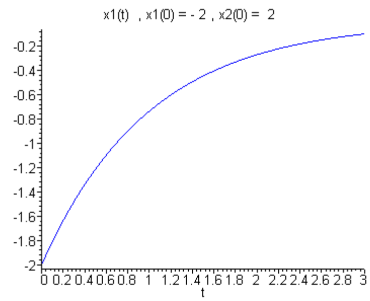
$11(a)$. The characteristic equation is $(r+1)^2 = 0$, with double root $r = -1$. It is easy to see that the two linearly independent eigenvectors are $\boldsymbol{\xi}^{(1)} = (1\,,0)^T$ and $\boldsymbol{\xi}^{(2)} = (0\,,1)^T$.

(b). Since there are two linearly independent eigenvectors, the critical point is a *stable proper node*.

(c, d).







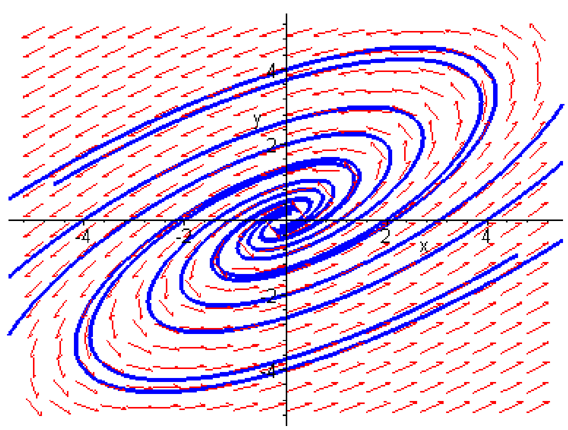
$12(a)$. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

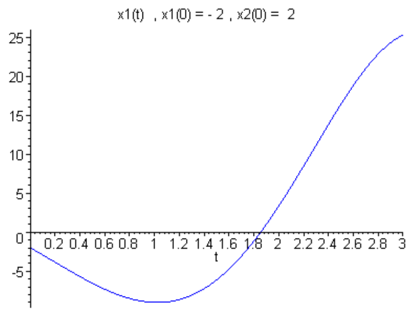
$$\begin{pmatrix} 2-r & -5/2 \\ 9/5 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

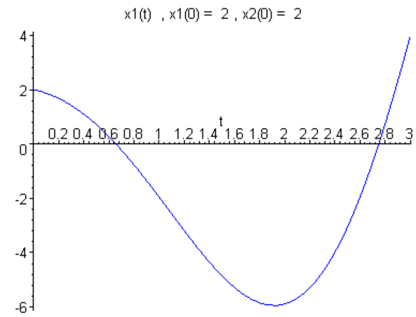
For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r + 5/2 = 0$. The roots of the characteristic equation are $r = 1/2 \pm 3i/2$. Substituting $r = 1/2 - 3i/2$, the equations reduce to $(3+3i)\xi_1 - 5\,\xi_2 = 0$. Therefore the two eigenvectors are $\boldsymbol{\xi}^{(1)} = (5\,, 3+3i)^T$ and $\boldsymbol{\xi}^{(2)} = (5\,, 3-3i)^T$.

(b). Since the eigenvalues are *complex*, with *positive* real part, the critical point is an *unstable spiral*.

(c, d).







14. Setting $\mathbf{x}' = \mathbf{0}$, that is,

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

we find that the critical point is $\mathbf{x}^0 = (-1, 0)^T$. With the change of dependent variable, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{u}\,.$$

The critical point for the transformed equation is the origin. Setting $\mathbf{u} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 4r + 3 = 0$. The roots of the characteristic equation are $r = -3\,,\ -1$. Hence the critical point is a *stable node*.

15. Setting $\mathbf{x}' = \mathbf{0}$, that is,

$$\begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 \\ -5 \end{pmatrix},$$

we find that the critical point is $\mathbf{x}^0 = (-2, 1)^T$. With the change of dependent variable, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{u}\,.$$

The characteristic equation is $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 2r + 3 = 0$, with complex conjugate roots $r = -1 \pm i\sqrt{2}$. Since the real parts of the eigenvalues are *negative*, the critical point is a *stable spiral*.

16. The critical point $\mathbf{x}^0$ satisfies the system of equations

$$\begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -\alpha \\ \gamma \end{pmatrix}.$$

It follows that $x^0 = \gamma/\delta$ and $y^0 = \alpha/\beta$. Using the transformation, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix}\mathbf{u}\,.$$

The characteristic equation is $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + \beta\,\delta = 0$. Since $\beta\,\delta > 0$, the roots are purely imaginary, with $r = \pm\,i\sqrt{\beta\delta}$. Hence the critical point is a *center*.

20. The system of ODEs can be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\mathbf{x}\,.$$

The characteristic equation is $r^2 - p\,r + q = 0$. The roots are given by

$$r_{1,2} = \frac{p \pm \sqrt{p^2 - 4q}}{2} = \frac{p \pm \sqrt{\Delta}}{2}\,.$$

The results can be verified using Table 9.1.1 .

21(a). If $q > 0$ and $p < 0$, then the roots are either complex conjugates with negative real parts, or both real and negative.

(b). If $q > 0$ and $p = 0$, then the roots are purely imaginary.

(c). If $q < 0$, then the roots are real, with $r_1 \cdot r_2 > 0$. If $p > 0$, then either the roots are real, with $r_1 \cdot r_2 \geq 0$ or the roots are complex conjugates with positive real parts.