Section 7.2

$2(a)$.

$$\mathbf{A}-2\,\mathbf{B}=\begin{pmatrix} 1+i-2i & -1+2i-6 \\ 3+2i-4 & 2-i+4i \end{pmatrix}=\begin{pmatrix} 1-i & -7+2i \\ -1+2i & 2+3i \end{pmatrix}.$$

(b).

$$3\,\mathbf{A}+\,\mathbf{B}=\begin{pmatrix} 3+3i+i & -3+6i+3 \\ 9+6i+2 & 6-3i-2i \end{pmatrix}=\begin{pmatrix} 3+4i & 6i \\ 11+6i & 6-5i \end{pmatrix}.$$

(c).

$$\begin{aligned}\mathbf{AB}\ &=\begin{pmatrix} (1+i)i+2(-1+2i) & 3(1+i)+(-1+2i)(-2i) \\ (3+2i)i+2(2-i) & 3(3+2i)+(2-i)(-2i) \end{pmatrix}\\ &=\begin{pmatrix} -3+5i & 7+5i \\ 2+i & 7+2i \end{pmatrix}.\end{aligned}$$

(d).

$$\begin{aligned}\mathbf{BA}\ &=\begin{pmatrix} (1+i)i+3(3+2i) & (-1+2i)i+3(2-i) \\ 2(1+i)+(-2i)(3+2i) & 2(-1+2i)+(-2i)(2-i) \end{pmatrix}\\ &=\begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -4 \end{pmatrix}.\end{aligned}$$

3.

$$\begin{aligned}\mathbf{A}^T+\mathbf{B}^T\ &=\begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 1 \end{pmatrix}+\begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 1 \\ 3 & -1 & 0 \end{pmatrix}\\ &=\begin{pmatrix} -1 & 4 & 0 \\ 3 & -1 & 0 \\ 5 & -4 & 1 \end{pmatrix}\\ &=(\mathbf{A}+\mathbf{B})^T.\end{aligned}$$

$4(b)$.

$$\overline{\mathbf{A}}=\begin{pmatrix} 3+2i & 1-i \\ 2+i & -2-3i \end{pmatrix}.$$

(c). By definition, $\mathbf{A}^*=(\overline{\mathbf{A}^T})=(\overline{\mathbf{A}})^T$.

5.

$$2(\mathbf{A}+\mathbf{B}) = 2\begin{pmatrix} 5 & 3 & -2 \\ 0 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 6 & -4 \\ 0 & 4 & 10 \\ 2 & 4 & 6 \end{pmatrix}.$$

7. Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. The given operations in $(a) - (d)$ are performed elementwise. That is,

(a). $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.
(b). $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$.
(c). $\alpha(a_{ij} + b_{ij}) = \alpha\, a_{ij} + \alpha\, b_{ij}$.
(d). $(\alpha + \beta)\, a_{ij} = \alpha\, a_{ij} + \alpha\, a_{ij}$.

In the following, let $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$and $\mathbf{C} = (c_{ij})$.

(e). Calculating the generic element,

$$(\mathbf{BC})_{ij} = \sum_{k=1}^{n} b_{ik}\, c_{kj}\,.$$

Therefore

$$\begin{aligned} [\mathbf{A}(\mathbf{BC})]_{ij} &= \sum_{r=1}^{n} a_{ir} \left(\sum_{k=1}^{n} b_{rk}\, c_{kj} \right) \\ &= \sum_{r=1}^{n} \sum_{k=1}^{n} a_{ir}\, b_{rk}\, c_{kj} \\ &= \sum_{k=1}^{n} \left[\left(\sum_{r=1}^{n} a_{ir}\, b_{rk} \right) c_{kj} \right]. \end{aligned}$$

The last summation is recognized as

$$\sum_{r=1}^{n} a_{ir}\, b_{rk} = (\mathbf{AB})_{ik}\,,$$

which is the ik-*th* element of the matrix $\mathbf{AB}$.

(f). Likewise,

$$
\begin{aligned}
[\mathbf{A}(\mathbf{B}+\mathbf{C})]_{ij} &= \sum_{k=1}^{n} a_{ik}(\, b_{kj} + \, c_{kj}) \\
&= \sum_{k=1}^{n} a_{ik}\, b_{kj} + \sum_{k=1}^{n} a_{ik}\, c_{kj} \\
&= (\mathbf{AB})_{ij} + (\mathbf{AC})_{ij}\,.
\end{aligned}
$$

$8(a)$. $\mathbf{x}^T\mathbf{y} = 2(\,-1+i) + 2(3i) + (1-i)(3-i) = 4i\,.$
(b). $\mathbf{y}^T\mathbf{y} = (\,-1+i)^2 + 2^2 + (3-i)^2 = 12 - 8i\,.$
(c). $(\mathbf{x}\,,\mathbf{y}) = 2(\,-1-i) + 2(3i) + (1-i)(3+i) = 2 + 2i\,.$
(d). $(\mathbf{y}\,,\mathbf{y}) = (\,-1+i)(\,-1-i) + 2^2 + (3-i)(3+i) = 16\,.$

9. Indeed,

$$
\mathbf{x}^T\mathbf{y} = \sum_{j=1}^{n} x_j\, y_j = \mathbf{y}^T\mathbf{x}\,,
$$

and

$$
(\mathbf{x}\,,\mathbf{y}) = \sum_{j=1}^{n} x_j\, \overline{y}_j = \sum_{j=1}^{n} \overline{y}_j\, x_j = \overline{\sum_{j=1}^{n} y_j\, \overline{x}_j} = \overline{(\mathbf{y}\,,\mathbf{x})}\,.
$$

11. First *augment* the given matrix by the identity matrix:

$$
[\mathbf{A}\,|\,\mathbf{I}\,] = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.
$$

Divide the *first row* by 3 , to obtain

$$
\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.
$$

Adding -6 times the *first row* to the *second row* results in

$$
\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}.
$$

Divide the *second row* by 4 , to obtain

$$
\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}.
$$

Finally, adding $1/3$ times the *second row* to the *first row* results in

$$\begin{pmatrix} 1 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}^{-1} = \frac{1}{12}\begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}.$$

13. The augmented matrix is

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

Combining the elements of the *first row* with the elements of the *second* and *third* rows results in

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{pmatrix}.$$

Divide the elements of the *second row* by -3, and the elements of the *third row* by 3. Now subtracting the new *second row* from the *first row* yields

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

Finally, combine the *third row* with the *second row* to obtain

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

15. Elementary row operations yield

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Finally, combining the $first$ and $third$ rows results in

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

16. Elementary row operations yield

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{10}{3} & -\frac{7}{3} & -\frac{1}{3} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ 0 & 1 & 0 & -\frac{3}{15} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{10}{3} & -\frac{7}{3} & -\frac{1}{3} & 1 \end{pmatrix}.$$

Finally, normalizing the *last row* results in

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ 0 & 1 & 0 & -\frac{3}{15} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \end{pmatrix}.$$

17. Elementary row operations on the augmented matrix yield the row-reduced form of the augmented matrix

$$\begin{pmatrix} 1 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & \frac{3}{7} & 0 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 1 & -2 & -1 \end{pmatrix}.$$

The *left submatrix* cannot be converted to the identity matrix. Hence the given matrix is singular.

18. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

19. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 10 & 4 & -4 & 1 \end{pmatrix}.$$

Normalizing the *last row* and combining it with the others results in

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & 1 & 0 & 0 & 5 & \frac{11}{5} & -\frac{6}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{pmatrix}.$$

20. Suppose that $\mathbf{A}$ is *nonsingular*, and that there exist matrices $\mathbf{B}$ and $\mathbf{C}$, such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$. Based on the properties of matrices, it follows that

$$\mathbf{A}(\mathbf{B} - \mathbf{C}) = \mathbf{AY} = \mathbf{0}_{n\times n}\,.$$

Write the *difference* of the two matrices, $\mathbf{Y}$, in terms of its *columns* as

$$\mathbf{Y} = \left[\mathbf{y^1}\,|\,\mathbf{y^2}|\cdots|\,\mathbf{y^n}\right].$$

The *j-th* column of the product matrix, $\mathbf{AY}$, can be expressed as $\mathbf{A\,y^j}$. Now since *all* columns of the product matrix consist only of *zeros*, we end up with n homogeneous systems of linear equations

$$\mathbf{A\,y^j} = \mathbf{0}_{n\times 1}\,,\ j = 1, 2, \cdots, n\,.$$

Since $\mathbf{A}$ is *nonsingular*, each system must have a *trivial solution*. That is, $\mathbf{y^j} = \mathbf{0}_{n\times 1}$, for $j = 1, 2, \cdots, n$. Hence $\mathbf{Y} = \mathbf{0}_{n\times n}$ and $\mathbf{B} = \mathbf{C}$.

21(a).

$$\mathbf{A}+3\mathbf{B} = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} + \begin{pmatrix} 6e^t & 3e^{-t} & 9e^{2t} \\ -3e^t & 6e^{-t} & 3e^{2t} \\ 9e^t & -3e^{-t} & -3e^{2t} \end{pmatrix}$$
$$= \begin{pmatrix} 7e^t & 5e^{-t} & 10e^{2t} \\ -e^t & 7e^{-t} & 2e^{2t} \\ 8e^t & 0 & -e^{2t} \end{pmatrix}.$$

(b). Based on the standard definition of *matrix multiplication*,

$$\mathbf{AB} = \begin{pmatrix} 2e^{2t}-2+3e^{3t} & 1+4e^{-2t}-e^t & 3e^{3t}+2e^t-e^{4t} \\ 4e^{2t}-1-3e^{3t} & 2+2e^{-2t}+e^t & 6e^{3t}+e^t+e^{4t} \\ -2e^{2t}-3+6e^{3t} & -1+6e^{-2t}-2e^t & -3e^{3t}+3e^t-2e^{4t} \end{pmatrix}.$$

(c).

$$\frac{d\mathbf{A}}{dt} = \begin{pmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{pmatrix}.$$

(d). Note that

$$\int \mathbf{A}(t)dt = \begin{pmatrix} e^t & -2e^{-t} & e^{2t}/2 \\ 2e^t & -e^{-t} & -e^{2t}/2 \\ -e^t & -3e^{-t} & e^{2t} \end{pmatrix} + \mathbf{C}\,.$$

Therefore

$$\int_0^1 \mathbf{A}(t)dt = \begin{pmatrix} e & -2e^{-1} & e^2/2 \\ 2e & -e^{-1} & -e^2/2 \\ -e & -3e^{-1} & e^2 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 1/2 \\ 2 & -1 & -1/2 \\ -1 & -3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} e-1 & 2-2e^{-1} & e^2/2-1/2 \\ 2e-2 & 1-e^{-1} & 1/2-e^2/2 \\ 1-e & 3-3e^{-1} & e^2-1 \end{pmatrix}.$$

The result can also be written as

$$(e-1)\begin{pmatrix} 1 & \frac{2}{e} & \frac{1}{2}(e+1) \\ 2 & \frac{1}{e} & -\frac{1}{2}(e+1) \\ -1 & \frac{3}{e} & e+1 \end{pmatrix}.$$

23. First note that

$$\mathbf{x}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(e^t + t\,e^t\right) = \begin{pmatrix} 3e^t + 2t\,e^t \\ 2e^t + 2t\,e^t \end{pmatrix}.$$

We also have

$$\begin{aligned}
\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} &= \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}\left(t\,e^t\right) \\
&= \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix}\left(t\,e^t\right) \\
&= \begin{pmatrix} 2e^t + 2t\,e^t \\ 3e^t + 2t\,e^t \end{pmatrix}.
\end{aligned}$$

It follows that

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t = \begin{pmatrix} 3e^t + 2t\,e^t \\ 2e^t + 2t\,e^t \end{pmatrix}.$$

24. It is easy to see that

$$\mathbf{x}' = \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t} = \begin{pmatrix} -6e^{-t} \\ 8e^{-t} + 4e^{2t} \\ 4e^{-t} - 4e^{-2t} \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}
\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\mathbf{x} &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} e^{2t} \\
&= \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t}.
\end{aligned}$$

26. Differentiation, elementwise, results in

$$\mathbf{\Psi}' = \begin{pmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{pmatrix}.$$

On the other hand,

$$
\begin{aligned}
\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{\Psi} &= \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^{t} & e^{-2t} & e^{3t} \\ -4e^{t} & -e^{-2t} & 2e^{3t} \\ -e^{t} & -e^{-2t} & e^{3t} \end{pmatrix} \\
&= \begin{pmatrix} e^{t} & -2e^{-2t} & 3e^{3t} \\ -4e^{t} & 2e^{-2t} & 6e^{3t} \\ -e^{t} & 2e^{-2t} & 3e^{3t} \end{pmatrix}.
\end{aligned}
$$