Section 4.4

2. The characteristic equation is $r(r^2-1)=0$. Hence the homogeneous solution is $y_c(t)=c_1+c_2e^t+c_3e^{-t}$. The Wronskian is evaluated as $W(1,e^t,e^{-t})=2$. Now compute the three determinants

$$W_1(t)=\begin{vmatrix}0 & e^t & e^{-t}\\ 0 & e^t & -e^{-t}\\ 1 & e^t & e^{-t}\end{vmatrix}=-2$$

$$W_2(t)=\begin{vmatrix}1 & 0 & e^{-t}\\ 0 & 0 & -e^{-t}\\ 0 & 1 & e^{-t}\end{vmatrix}=e^{-t}$$

$$W_3(t)=\begin{vmatrix}1 & e^t & 0\\ 0 & e^t & 0\\ 0 & e^t & 1\end{vmatrix}=e^t$$

The solution of the system of equations (10) is

$$u_1'(t)=\frac{t\,W_1(t)}{W(t)}=-t$$

$$u_2'(t)=\frac{t\,W_2(t)}{W(t)}=te^{-t}/2$$

$$u_3'(t)=\frac{t\,W_3(t)}{W(t)}=te^{t}/2$$

Hence $u_1(t)=-t^2/2$, $u_2(t)=-e^{-t}(t+1)/2$, $u_3(t)=e^t(t-1)/2$. The particular solution becomes $Y(t)=-t^2/2-(t+1)/2+(t-1)/2=-t^2/2-1$. The constant is a solution of the homogeneous equation, therefore the general solution is

$$y(t)=c_1+c_2e^t+c_3e^{-t}-t^2/2\,.$$

3. From Prob. 13 in Section 4.2, $y_c(t)=c_1e^{-t}+c_2e^t+c_3e^{2t}$. The Wronskian is evaluated as $W(e^{-t},e^t,e^{2t})=6\,e^{2t}$. Now compute the three determinants

$$W_1(t)=\begin{vmatrix}0 & e^t & e^{2t}\\ 0 & e^t & 2e^{2t}\\ 1 & e^t & 4e^{2t}\end{vmatrix}=e^{3t}$$

$$W_2(t) = \begin{vmatrix} e^{-t} & 0 & e^{2t} \\ -e^{-t} & 0 & 2e^{2t} \\ e^{-t} & 1 & 4e^{2t} \end{vmatrix} = -3e^t$$

$$W_3(t) = \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2$$

Hence $u_1'(t) = e^{5t}/6$, $u_2'(t) = -e^{3t}/2$, $u_3'(t) = e^{2t}/3$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= e^{-t}\left[e^{5t}/30\right] - e^t\left[e^{3t}/6\right] + e^{2t}\left[e^{2t}/6\right] \\ &= e^{4t}/30\,. \end{aligned}$$

6. From Prob. 22 in Section 4.2, $y_c(t) = c_1 cos\,t + c_2 sin\,t + t[c_3 cos\,t + c_4 sin\,t]$. The Wronskian is evaluated as $W(cos\,t, sin\,t, t\,cos\,t, t\,sin\,t) = 4$. Now compute the four auxiliary determinants

$$W_1(t) = \begin{vmatrix} 0 & sin\,t & t\,cos\,t & t\,sin\,t \\ 0 & cos\,t & cos\,t - t\,sin\,t & sin\,t + t\,cos\,t \\ 0 & -sin\,t & -2sin\,t - t\,cos\,t & 2cos\,t - t\,sin\,t \\ 1 & -cos\,t & -3cos\,t + t\,sin\,t & -3sin\,t - t\,cos\,t \end{vmatrix} = -2sin\,t + 2t\,cos\,t$$

$$W_2(t) = \begin{vmatrix} cos\,t & 0 & t\,cos\,t & t\,sin\,t \\ -sin\,t & 0 & cos\,t - t\,sin\,t & sin\,t + t\,cos\,t \\ -cos\,t & 0 & -2sin\,t - t\,cos\,t & 2cos\,t - t\,sin\,t \\ sin\,t & 1 & -3cos\,t + t\,sin\,t & -3sin\,t - t\,cos\,t \end{vmatrix} = 2t\,sin\,t + 2cos\,t$$

$$W_3(t) = \begin{vmatrix} cos\,t & sin\,t & 0 & t\,sin\,t \\ -sin\,t & cos\,t & 0 & sin\,t + t\,cos\,t \\ -cos\,t & -sin\,t & 0 & 2cos\,t - t\,sin\,t \\ sin\,t & -cos\,t & 1 & -3sin\,t - t\,cos\,t \end{vmatrix} = -2cos\,t$$

$$W_4(t) = \begin{vmatrix} cos\,t & sin\,t & t\,cos\,t & 0 \\ -sin\,t & cos\,t & cos\,t - t\,sin\,t & 0 \\ -cos\,t & -sin\,t & -2sin\,t - t\,cos\,t & 0 \\ sin\,t & -cos\,t & -3cos\,t + t\,sin\,t & 1 \end{vmatrix} = -2sin\,t$$

It follows that $u_1'(t) = [-sin^2 t + t\,sin\,t\,cos\,t]/2$, $u_2'(t) = [t sin^2 t + sin\,t\,cos\,t]/2$, $u_3'(t) = -sin\,t\,cos\,t/2$, $u_4'(t) = -sin^2 t/2$. Hence

$$u_1(t) = \left[3sin\,t\,cos\,t - 2t\,cos^2 t - t\right]/8$$

$$u_2(t) = \left[sin^2 t - 2cos^2 t - 2t\, sin\, t\, cos\, t + t^2\right]/8$$

$$u_3(t) = -\,sin^2 t/4$$

$$u_4(t) = [cos\, t\, sin\, t - t]/4$$

Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= cos\, t\,[u_1(t)] + sin\, t\,[u_2(t)] + t\, cos\, t\,[u_3(t)] + t\, sin\, t\,[u_4(t)] \\ &= \left[sin\, t - 3t\, cos\, t - t^2 sin\, t\right]/8\,. \end{aligned}$$

Note that only the *last term* is not a solution of the homogeneous equation. Hence the general solution is

$$y(t) = c_1 cos\, t + c_2 sin\, t\, + t[c_3 cos\, t + c_4 sin\, t\,] - t^2 sin\, t\,/8\,.$$

8. Based on the results in Prob. 2, $y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}$. It was also shown that $W(1, e^t, e^{-t}) = 2$, with $W_1(t) = -2$, $W_2(t) = e^{-t}$, $W_3(t) = e^t$. Therefore we have $u_1'(t) = -\,csc\, t$, $u_2'(t) = e^{-t} csc\, t\,/2$, $u_3'(t) = e^t csc\, t\,/2$. The particular solution can be expressed as $Y(t) = [u_1(t)] + e^{-t}[u_2(t)] + e^t\,[u_3(t)]$. More specifically,

$$\begin{aligned} Y(t) &= ln|csc(t) + cot(t)| + \frac{e^t}{2}\int_{t_0}^{t} e^{-s} csc(s)ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s} csc(s)ds \\ &= ln|csc(t) + cot(t)| + \int_{t_0}^{t} cosh(t-s)csc(s)ds\,. \end{aligned}$$

9. Based on Prob. 4, $u_1'(t) = sec\, t$, $u_2'(t) = -1$, $u_3'(t) = -\,tan\, t$. The particular solution can be expressed as $Y(t) = [u_1(t)] + cos\, t\,[u_2(t)] + sin\, t\,[u_3(t)]$. That is,

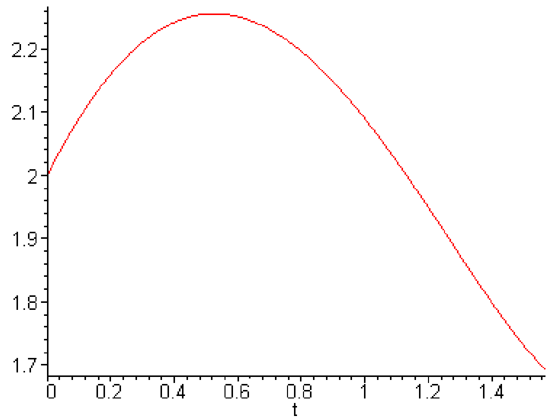
$$Y(t) = ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|.$$

Hence the general solution of the initial value problem is

$$y(t) = c_1 + c_2 cos\, t + c_3 sin\, t\, + ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|.$$

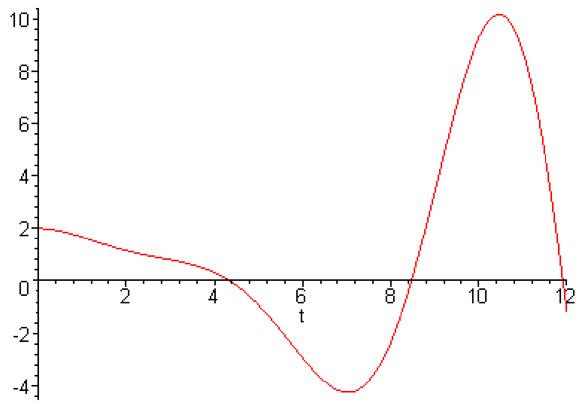
Invoking the initial conditions, we require that $c_1 + c_2 = 2$, $c_3 = 1$, $-\,c_2 = -2$. Therefore

$$y(t) = 2cos\, t + sin\, t + ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|$$



10. From Prob. 6, $y(t) = c_1 cos\,t + c_2 sin\,t + c_3 t\,cos\,t + c_4 t\,sin\,t - t^2 sin\,t/8$. In order to satisfy the initial conditions, we require that $c_1 = 2$, $c_2 + c_3 = 0$, $-c_1 + 2c_4 = -1$, $-3/4 - c_2 - 3c_3 = 1$. Therefore

$$y(t) = 2cos\,t + \left[7sin\,t - 7t\,cos\,t + 4t\,sin\,t - t^2 sin\,t\right]/8\,.$$



12. From Prob. 8, the general solution of the initial value problem is

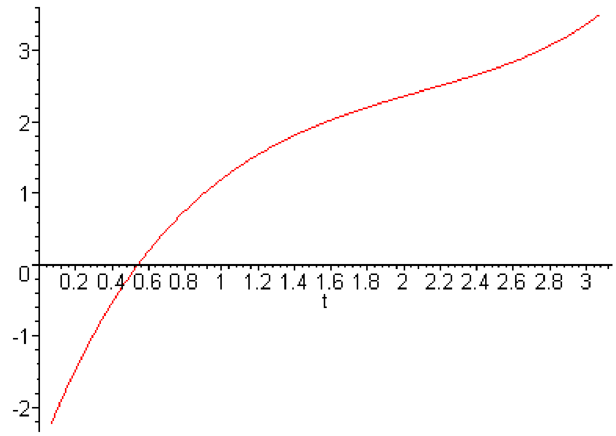
$$y(t) = c_1 + c_2 e^t + c_3 e^{-t} + ln|csc(t) + cot(t)| + \frac{e^t}{2}\int_{t_0}^{t} e^{-s} csc(s)ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s} csc(s)ds.$$

In this case, $t_0 = \pi/2$. Observe that $y(\pi/2) = y_c(\pi/2)$, $y'(\pi/2) = y_c'(\pi/2)$, and $y''(\pi/2) = y_c''(\pi/2)$. Therefore we obtain the system of equations

$$\begin{aligned} c_1 + c_2 e^{\pi/2} + c_3 e^{-\pi/2} &= 2 \\ c_2 e^{\pi/2} - c_3 e^{-\pi/2} &= 1 \\ c_2 e^{\pi/2} + c_3 e^{-\pi/2} &= -1 \end{aligned}$$

Hence the solution of the initial value problem is

$$y(t) = 3 - e^{-t+\pi/2} + ln|csc(t) + cot(t)| + \int_{t_0}^{t} cosh(t-s)csc(s)ds\,.$$



13. First write the equation as $y''' + x^{-1}y'' - 2x^{-2}y' + 2x^{-3}y = 2x$. The Wronskian is evaluated as $W(x, x^2, 1/x) = 6/x$. Now compute the three determinants

$$W_1(x) = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = -3$$

$$W_2(x) = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = 2/x$$

$$W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

Hence $u_1'(x) = -x^2$, $u_2'(x) = 2x/3$, $u_3'(x) = x^4/3$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(x) &= x\left[-x^3/3\right] + x^2\left[x^2/3\right] + \frac{1}{x}\left[x^5/15\right] \\ &= x^4/15\,. \end{aligned}$$

15. The homogeneous solution is $y_c(t) = c_1 cos\,t + c_2 sin\,t + c_3 cosh\,t + c_4 sinh\,t$. The Wronskian is evaluated as $W(cos\,t, sin\,t, cosh\,t, sinh\,t) = 4$. Now the four additional determinants are given by $W_1(t) = 2\,sin\,t$, $W_2(t) = -2\,cos\,t$, $W_3(t) = -2\,sinh\,t$, $W_4(t) = 2\,cosh\,t$. If follows that $u_1'(t) = g(t)\,sin(t)/2$, $u_2'(t) = -g(t)\,cos(t)/2$, $u_3'(t) = -g(t)\,sinh(t)/2$, $u_4'(t) = g(t)\,cosh(t)/2$. Therefore the particular solution

can be expressed as

$$\begin{aligned} Y(t) &= \frac{cos(t)}{2}\int_{t_0}^{t} g(s)\,sin(s)\,ds - \frac{sin(t)}{2}\int_{t_0}^{t} g(s)\,cos(s)\,ds - \\ &\quad - \frac{cosh(t)}{2}\int_{t_0}^{t} g(s)\,sinh(s)\,ds + \frac{sinh(t)}{2}\int_{t_0}^{t} g(s)\,cosh(s)\,ds\,. \end{aligned}$$

Using the appropriate identities, the integrals can be combined to obtain

$$Y(t) = \frac{1}{2}\int_{t_0}^{t} g(s)\,sinh(t-s)\,ds - \frac{1}{2}\int_{t_0}^{t} g(s)\,sin(t-s)\,ds\,.$$

17. First write the equation as $y''' - 3\,x^{-1}y'' + 6\,x^{-2}y' - 6\,x^{-3}y = g(x)\,/\,x^3$. It can be shown that $y_c(x) = c_1x + c_2\,x^2 + c_3\,x^3$ is a solution of the homogeneous equation. The Wronskian of this fundamental set of solutions is $W(x, x^2, x^3) = 2x^3$. The three additional determinants are given by $W_1(x) = x^4$, $W_2(x) = -2x^3$, $W_3(x) = x^2$. Hence $u_1'(x) = g(x)/2x^2$, $u_2'(x) = -g(x)\,/x^3$, $u_3'(x) = g(x)/2x^4$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(x) &= x\int_{x_0}^{x}\frac{g(t)}{2t^2}dt - x^2\int_{x_0}^{x}\frac{g(t)}{t^3}dt + x^3\int_{x_0}^{x}\frac{g(t)}{2t^4}dt \\ &= \frac{1}{2}\int_{x_0}^{x}\left[\frac{x}{t^2} - \frac{2\,x^2}{t^3} + \frac{x^3}{t^4}\right]g(t)dt\,. \end{aligned}$$