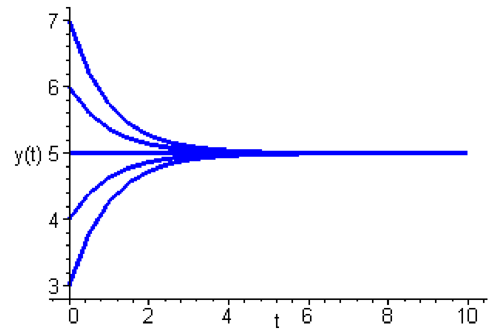
Section 1.2

$1(a)$ The differential equation can be rewritten as

$$\frac{dy}{5-y} = dt\,.$$

Integrating both sides of this equation results in $-ln|5-y| = t + c_1$, or equivalently, $5 - y = c\,e^{-t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - y_0$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{-t}$.

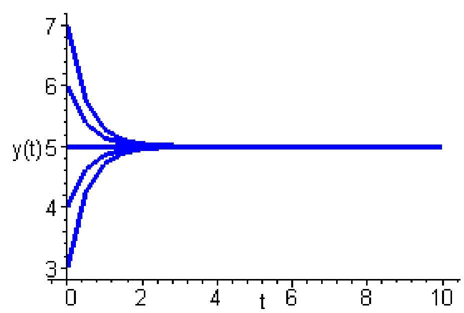


All solutions appear to converge to the equilibrium solution $y(t) = 5$.

$1(c)$. Rewrite the differential equation as

$$\frac{dy}{10-2y} = dt\,.$$

Integrating both sides of this equation results in $-\frac{1}{2}ln|10-2y| = t + c_1$, or equivalently,
$5 - y = c\,e^{-2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - y_0$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{-2t}$.

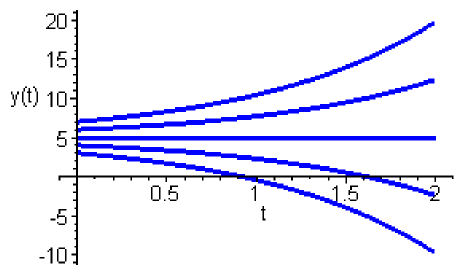


All solutions appear to converge to the equilibrium solution $y(t) = 5$, but at a *faster* rate than in Problem 1a .

$2(a)$. The differential equation can be rewritten as

$$\frac{dy}{y-5} = dt\,.$$

Integrating both sides of this equation results in $ln|y-5| = t + c_1$, or equivalently, $y - 5 = c\,e^t$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^t$.

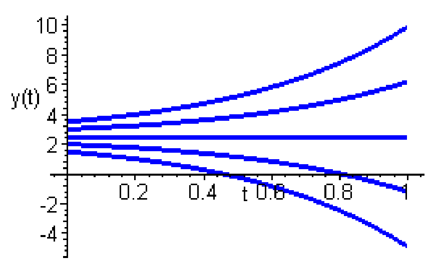


All solutions appear to diverge from the equilibrium solution $y(t) = 5$.

$2(b)$. Rewrite the differential equation as

$$\frac{dy}{2y-5} = dt\,.$$

Integrating both sides of this equation results in $\frac{1}{2}ln|2y-5| = t + c_1$, or equivalently, $2y - 5 = c\,e^{2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 2y_0 - 5$. Hence the solution is $y(t) = 2.5 + (y_0 - 2.5)e^{2t}$.

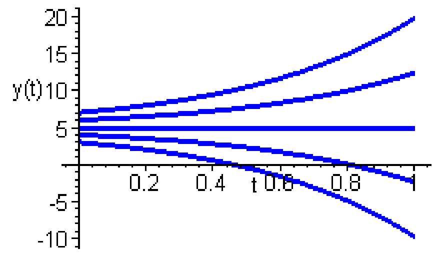


All solutions appear to diverge from the equilibrium solution $y(t) = 2.5$.

$2(c)$. The differential equation can be rewritten as

$$\frac{dy}{2y-10} = dt\,.$$

Integrating both sides of this equation results in $\frac{1}{2}ln|2y-10| = t + c_1$, or equivalently, $y - 5 = c\,e^{2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{2t}$.



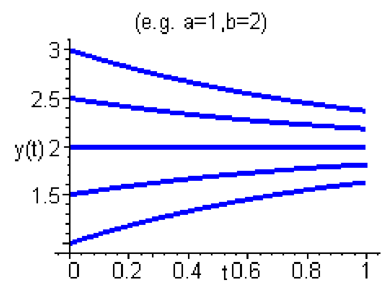
All solutions appear to diverge from the equilibrium solution $y(t) = 5$.

$3(a)$. Rewrite the differential equation as

$$\frac{dy}{b - ay} = dt\,,$$

which is valid for $y \neq b\,/a$. Integrating both sides results in $\frac{-1}{a} ln|b - ay| = t + c_1$, or equivalently, $b - ay = c\,e^{-at}$. Hence the general solution is $y(t) = (b - c\,e^{-at})/a$. Note that if $y = b/a$, then $dy/dt = 0$, and $y(t) = b/a$ is an equilibrium solution.

(b)



(i) As a increases, the equilibrium solution gets closer to $y(t) = 0$, from above. Furthermore, the *convergence rate* of all solutions, that is, a , also increases.
(ii) As b increases, then the equilibrium solution $y(t) = b/a$ also becomes larger. In this case, the convergence rate remains the same.
(iii) If a and b both increase (*but* $b/a =$ *constant*), then the equilibrium solution $y(t) = b/a$ remains the same, but the *convergence rate* of all solutions increases.

$5(a)$. Consider the simpler equation $dy_1/dt = \, - ay_1$. As in the previous solutions, rewrite the equation as

$$\frac{dy_1}{y_1} = \, - a\,dt\,.$$

Integrating both sides results in $y_1(t) = c\,e^{-at}$.
(b). Now set $y(t) = y_1(t) + k$, and substitute into the original differential equation. We find that

$$-ay_1 + 0 = -a(y_1 + k) + b\,.$$

That is, $-ak + b = 0$, and hence $k = b/a$.
(c). The general solution of the differential equation is $y(t) = c\,e^{-at} + b/a$. This is exactly the form given by Eq. (17) in the text. Invoking an initial condition $y(0) = y_0$, the solution may also be expressed as $y(t) = b/a + (y_0 - b/a)e^{-at}$.

$6(a)$. The general solution is $p(t) = 900 + c\,e^{t/2}$, that is, $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the specific solution becomes $p(t) = 900 - 50e^{t/2}$. This solution is a *decreasing* exponential, and hence the time of extinction is equal to the number of months
it takes, say t_f, for the population to reach *zero*. Solving $900 - 50e^{t_f/2} = 0$, we find that $t_f = 2\,ln(900/50) = 5.78$ *months*.
(b) The solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, is a *decreasing* exponential as long as $p_0 < 900$. Hence $900 + (p_0 - 900)e^{t_f/2} = 0$ has only *one* root, given by

$$t_f = 2\,ln\left(\frac{900}{900 - p_0}\right).$$

(c). The answer in part (b) is a general equation relating time of extinction to the value of
the initial population. Setting $t_f = 12$ *months*, the equation may be written as

$$\frac{900}{900 - p_0} = e^6,$$

which has solution $p_0 = 897.7691$. Since p_0 is the initial population, the appropriate answer is $p_0 = 898$ *mice*.

$7(a)$. The general solution is $p(t) = p_0\,e^{rt}$. Based on the discussion in the text, time t is measured in *months*. Assuming 1 *month* $= 30$ *days*, the hypothesis can be expressed as $p_0\,e^{r\cdot 1} = 2p_0$. Solving for the rate constant, $r = ln(2)$, with units of *per month*.

(b). N *days* $= N/30$ *months*. The hypothesis is stated mathematically as $p_0 e^{rN/30} = 2p_0$.
It follows that $rN/30 = ln(2)$, and hence the rate constant is given by $r = 30\,ln(2)/N$. The units are understood to be *per month*.

$9(a)$. Assuming *no air resistance*, with the positive direction taken as *downward*, Newton's
Second Law can be expressed as

$$m\frac{dv}{dt} = mg$$

in which g is the *gravitational constant* measured in appropriate units. The equation can be

written as $dv/dt = g$, with solution $v(t) = gt + v_0$. The object is released with an initial
velocity v_0.

(b). Suppose that the object is released from a height of h *units* above the ground. Using the
fact that $v = dx/dt$, in which x is the *downward displacement* of the object, we obtain the
differential equation for the displacement as $dx/dt = gt + v_0$. With the origin placed at
the point of release, direct integration results in $x(t) = gt^2/2 + v_0\,t$. Based on the chosen
coordinate system, the object reaches the ground when $x(t) = h$. Let $t = T$ be the time
that it takes the object to reach the ground. Then $gT^2/2 + v_0T = h$. Using the quadratic
formula to solve for T,

$$T = \frac{-\,v_0 \pm \sqrt{v_0 + 2gh}}{g}\,.$$

The *positive* answer corresponds to the time it takes for the object to fall to the ground. The
negative answer represents a previous instant at which the object could have been launched
upward (*with the same impact speed*), only to ultimately fall downward with speed v_0,
from a height of h *units* above the ground.

(c). The impact speed is calculated by substituting $t = T$ into $v(t)$ in part (a). That is,
$v(T) = \sqrt{v_0 + 2gh}$.

10(a,b). The general solution of the differential equation is $Q(t) = c\,e^{-rt}$. Given that
$Q(0) = 100$ *mg*, the value of the constant is given by $c = 100$. Hence the amount of
thorium-234 present at any time is given by $Q(t) = 100\,e^{-rt}$. Furthermore, based on the
hypothesis, setting $t = 1$ results in $82.04 = 100\,e^{-r}$. Solving for the rate constant, we
find that $r = -\,ln(82.04/100) = .19796$/*week* or $r = .02828$/*day*.

(c). Let T be the time that it takes the isotope to decay to *one-half* of its original amount.
From part (a), it follows that $50 = 100\,e^{-rT}$, in which $r = .19796$/*week*. Taking the
natural logarithm of both sides, we find that $T = 3.5014$ *weeks* or $T = 24.51$ *days*.

11. The general solution of the differential equation $dQ/dt = -\,r\,Q$ is $Q(t) = Q_0e^{-rt}$,
in which $Q_0 = Q(0)$ is the initial amount of the substance. Let τ be the time that it takes
the substance to decay to *one-half* of its original amount, Q_0. Setting $t = \tau$ in the solution,
we have $0.5\,Q_0 = Q_0e^{-r\tau}$. Taking the natural logarithm of both sides, it follows that
$-\,r\tau = ln(0.5)$ or $r\tau = ln\,2$.

12. The differential equation governing the amount of radium-226 is $dQ/dt = -r\,Q$, with solution $Q(t) = Q(0)e^{-rt}$. Using the result in Problem 11, and the fact that the half-life $\tau = 1620$ *years*, the decay rate is given by $r = ln(2)/1620$ *per year*. The amount of radium-226, after t years, is therefore $Q(t) = Q(0)e^{-0.00042786t}$. Let T be the time that it takes the isotope to decay to $3/4$ of its original amount. Then setting $t = T$,
and $Q(T) = \frac{3}{4}Q(0)$, we obtain $\frac{3}{4}Q(0) = Q(0)e^{-0.00042786T}$. Solving for the decay time, it follows that $-0.00042786\,T = ln(3/4)$ or $T = 672.36$ *years*.

13. The solution of the differential equation, with $Q(0) = 0$, is
$Q(t) = CV(1 - e^{-t/CR})$.
As $t \to \infty$, the exponential term vanishes, and hence the limiting value is $Q_L = CV$.

14(a). The *accumulation* rate of the chemical is $(0.01)(300)$ *grams per hour*. At any given time t, the *concentration* of the chemical in the pond is $Q(t)/10^6$ *grams per gallon*.
Consequently, the chemical *leaves* the pond at a rate of $(3 \times 10^{-4})Q(t)$ *grams per hour*. Hence, the rate of change of the chemical is given by

$$\frac{dQ}{dt} = 3 - 0.0003\,Q(t) \ \ \textit{gm/hr}\,.$$

Since the pond is initially free of the chemical, $Q(0) = 0$.

(b). The differential equation can be rewritten as

$$\frac{dQ}{10000 - Q} = 0.0003\,dt\,.$$

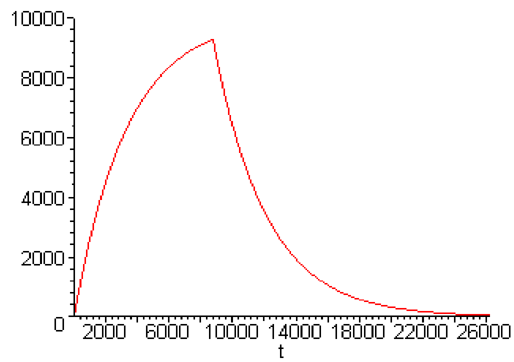
Integrating both sides of the equation results in $-ln|10000 - Q| = 0.0003t + C$. Taking
the natural logarithm of both sides gives $10000 - Q = c\,e^{-0.0003t}$. Since $Q(0) = 0$, the value of the constant is $c = 10000$. Hence the amount of chemical in the pond at any time
is $Q(t) = 10000(1 - e^{-0.0003t})$ *grams*. Note that 1 *year* $= 8760$ *hours*. Setting $t = 8760$, the amount of chemical present after *one year* is $Q(8760) = 9277.77$ *grams*, that is, 9.27777 *kilograms*.

(c). With the *accumulation* rate now equal to *zero*, the governing equation becomes $dQ/dt = -0.0003\,Q(t)$ *gm/hr*. Resetting the time variable, we now assign the new initial value as $Q(0) = 9277.77$ *grams*.

(d). The solution of the differential equation in Part (c) is $Q(t) = 9277.77\,e^{-0.0003t}$. Hence, one year *after* the source is removed, the amount of chemical in the pond is $Q(8760) = 670.1$ *grams*.

(e). Letting t be the amount of time after the source is removed, we obtain the equation $10 = 9277.77\, e^{-0.0003t}$. Taking the natural logarithm of both sides, $-\,0.0003\, t =$ $= ln(10/9277.77)$ or $t = 22,776$ *hours* $= 2.6$ *years* .

(f)



$15(a)$. It is assumed that dye is no longer entering the pool. In fact, the rate at which the dye *leaves* the pool is $200 \cdot [q(t)/60000]$ *kg/min* $= 200(60/1000)[q(t)/60]$ *gm per hour* .

Hence the equation that governs the amount of dye in the pool is

$$\frac{dq}{dt} = -\,0.2\, q \quad (gm/hr)\,.$$

The initial amount of dye in the pool is $q(0) = 5000$ *grams* .

(b). The solution of the governing differential equation, with the specified initial value, is $q(t) = 5000\, e^{-0.2\,t}$.

(c). The amount of dye in the pool after four hours is obtained by setting $t = 4$. That is, $q(4) = 5000\, e^{-0.8} = 2246.64$ *grams* . Since size of the pool is $60,000$ *gallons* , the *concentration* of the dye is 0.0374 *grams/gallon* .

(d). Let T be the time that it takes to reduce the concentration level of the dye to 0.02 *grams/gallon* . At that time, the amount of dye in the pool is $1,200$ *grams* . Using the answer in part (b), we have $5000\, e^{-0.2\,T} = 1200$. Taking the natural logarithm of both sides of the equation results in the required time $T = 7.14$ *hours* .

(e). Note that $0.2 = 200/1000$. Consider the differential equation

$$\frac{dq}{dt} = -\,\frac{r}{1000}\, q\,.$$

Here the parameter r corresponds to the *flow rate*, measured in *gallons per minute* . Using the same initial value, the solution is given by $q(t) = 5000\, e^{-r\,t/1000}$. In order to determine the appropriate flow rate, set $t = 4$ and $q = 1200$. (Recall that 1200 *gm* of dye has a concentration of 0.02 *gm/gal*). We obtain the equation $1200 = 5000\, e^{-r/250}$. Taking the natural logarithm of both sides of the equation results in the required flow rate $r = 357$ *gallons per minute*.