**Section 11.3**

4. The eigensystem of the associated homogeneous problem is given in Prob. 11 of Section 11.2. The normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{2}\,cos\,\sqrt{\lambda_n}\,x}{\sqrt{1 + sin^2\sqrt{\lambda_n}}}\,,$$

in which the eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\,sin\sqrt{\lambda_n} = 0$. Rewrite the given differential equation as $-y'' = 2y + x$. Since $\mu = 2 \neq \lambda_n$, the formal solution of the nonhomogeneous problem is

$$y(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - 2}\phi_n(x),$$

in which

$$\begin{aligned} c_n &= \int_0^1 f(x)\phi_n(x)dx \\ &= \frac{\sqrt{2}}{\sqrt{1 + sin^2\sqrt{\lambda_n}}}\int_0^1 x\,cos\,\sqrt{\lambda_n}\,x\,dx \\ &= \frac{\sqrt{2}\left(2\,cos\,\sqrt{\lambda_n} - 1\right)}{\lambda_n\sqrt{1 + sin^2\sqrt{\lambda_n}}}\,. \end{aligned}$$

Therefore we obtain the formal expansion

$$y(x) = 2\sum_{n=1}^{\infty} \frac{\sqrt{2}\left(2\,cos\,\sqrt{\lambda_n} - 1\right)\,cos\,\sqrt{\lambda_n}\,x}{\lambda_n(\lambda_n - 2)\left(1 + sin^2\sqrt{\lambda_n}\right)}\,.$$

5. The solution follows that in Prob. 1, except that the coefficients are given by

$$\begin{aligned} c_n &= \int_0^1 f(x)\phi_n(x)dx \\ &= \sqrt{2}\int_0^{1/2} 2x\,sin\,n\pi x\,dx + \sqrt{2}\int_{1/2}^1 (2 - 2x)\,sin\,n\pi x\,dx \\ &= 4\frac{\sqrt{2}\,sin(n\pi/2)}{n^2\pi^2}\,. \end{aligned}$$

Therefore the formal solution is

$$y(x) = 8\sum_{n=1}^{\infty} \frac{sin(n\pi/2)\;sin\,n\pi x}{n^2\pi^2(n^2\pi^2 - 2)}\,.$$

6. The differential equation can be written as $-y'' = \mu y + f(x)$. Note that $q(x) = 0$ and $r(x) = 1$. As shown in Prob. 1 in Section 11.2, the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\, sin\frac{(2n-1)x}{2},$$

with associated eigenvalues $\lambda_n = (2n-1)^2\pi^2/4$. Based on Theorem 11.3.1, the formal solution is given by

$$y(x) = \sqrt{2}\sum_{n=1}^{\infty}\frac{c_n}{(\lambda_n - \mu)}\, sin\frac{(2n-1)x}{2},$$

as long as $\mu \neq \lambda_n$. The coefficients in the series expansion are computed as

$$c_n = \sqrt{2}\int_0^1 f(x) sin\frac{(2n-1)x}{2}\, dx.$$

7. As shown in Prob. 1 in Section 11.2, the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\, cos\frac{(2n-1)x}{2},$$

with associated eigenvalues $\lambda_n = (2n-1)^2\pi^2/4$. Based on Theorem 11.3.1, the formal solution is given by

$$y(x) = \sqrt{2}\sum_{n=1}^{\infty}\frac{c_n}{(\lambda_n - \mu)}\, cos\frac{(2n-1)x}{2},$$

as long as $\mu \neq \lambda_n$. The coefficients in the series expansion are computed as

$$c_n = \sqrt{2}\int_0^1 f(x) cos\frac{(2n-1)x}{2}\, dx.$$

9. The normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{2}\, cos\sqrt{\lambda_n}\, x}{\sqrt{1 + sin^2\sqrt{\lambda_n}}}.$$

The eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. Based on Theorem 11.3.1, the formal solution is given by

$$y(x) = \sqrt{2}\sum_{n=1}^{\infty}\frac{c_n\, cos\sqrt{\lambda_n}\, x}{(\lambda_n - \mu)\sqrt{1 + sin^2\sqrt{\lambda_n}}},$$

as long as $\mu \neq \lambda_n$. The coefficients in the series expansion are computed as

$$c_n = \frac{\sqrt{2}}{\sqrt{1 + sin^2\sqrt{\lambda_n}}} \int_0^1 f(x) cos\,\sqrt{\lambda_n}\, x\, dx\,.$$

13. The differential equation can be written as $-y'' = \pi^2 y + cos\,\pi x - a$. Note that $\mu = \pi^2$ and $f(x) = cos\,\pi x - a$. Furthermore, $\mu = \pi^2$ is an eigenvalue corresponding to the eigenfunction $\phi_1(x) = \sqrt{2}\,sin\,\pi x$. A solution exists only if $f(x)$ and $\phi_1(x)$ are *orthogonal*. Since

$$\int_0^1 (cos\,\pi x - a) sin\,\pi x\, dx = -2a/\pi\,,$$

there exists a solution as long as $a = 0$. In that case, the ODE is

$$y'' + \pi^2 y = -cos\,\pi x\,.$$

The complementary solution is $y_c(x) = c_1 cos\,\pi x + c_2 sin\,\pi x$. A particular solution is $Y(x) = Ax\,cos\,\pi x + Bx\,sin\,\pi x$. Using the *method of undetermined coefficients*, we find that $A = 0$ and $B = -1/2\pi$. Therefore the general solution is

$$y(x) = c_1 cos\,\pi x + c_2 sin\,\pi x - \frac{x}{2\pi} sin\,\pi x\,.$$

The boundary conditions require that $c_1 = 0$. Hence the solution of the boundary value problem is

$$y(x) = c_2 sin\,\pi x - \frac{x}{2\pi} sin\,\pi x\,.$$

15. Let $y(x) = \phi_1(x) + \phi_2(x)$. It follows that $L[y] = L[\phi_1] + L[\phi_2] = f(x)$. Also,

$$\begin{aligned} a_1 y(0) + a_2 y'(0) &= a_1\phi_1(0) + a_1\phi_2(0) + a_2\phi_1'(0) + a_2\phi_2'(0) \\ &= a_1\phi_1(0) + a_2\phi_1'(0) + a_1\phi_2(0) + a_2\phi_2'(0) \\ &= \alpha\,. \end{aligned}$$

Similarly, the boundary condition at $x = 1$ is satisfied as well.

16. The complementary solution is $y_c(x) = c_1 cos\,\pi x + c_2 sin\,\pi x$. A particular solution is $Y(x) = A + Bx$. Using the *method of undetermined coefficients*, we find that $A = 0$ and $B = 1$. Therefore the general solution is

$$y(x) = c_1 cos\,\pi x + c_2 sin\,\pi x + x\,.$$

Imposing the boundary conditions, we find that $c_1 = 1$. Therefore the solution of the BVP is

$$y(x) = cos\,\pi x + c_2 sin\,\pi x + x\,.$$

Now attempt to solve the problem as shown in Prob. 15. Let BVP-1 be given by

$$\begin{aligned} &u'' + \pi^2 u = \pi^2 x\,, \\ &u(0) = 0\,, \quad u(1) = 0\,. \end{aligned}$$

The general solution of the ODE is

$$u(x) = c_1 cos\,\pi x + c_2 sin\,\pi x + x\,.$$

The boundary conditions require that $c_1 = 0$ and $-c_1 + 1 = 0$. We find that BVP-1 has no solution. Let BVP-2 be given by

$$\begin{aligned} &v'' + \pi^2 v = 0\,, \\ &v(0) = 1\,, \quad v(1) = 0\,. \end{aligned}$$

The general solution of the ODE is $v(x) = c_1 cos\,\pi x + c_2 sin\,\pi x$. Imposing the boundary conditions, we obtain $c_1 = 1$ and $-c_1 = 0$. Thus BVP-2 has no solution.

17. Setting $y(x) = u(x) + v(x)$, substitution results in

$$\begin{aligned} u'' + v'' + p(x)[u' + v'] + q(x)[u + v] = \; & u'' + p(x)u' + q(x)u + \\ & + v'' + p(x)v' + q(x)v\,. \end{aligned}$$

Since the left hand side of the equation is *zero*,

$$u'' + p(x)u' + q(x)u = -\,[v'' + p(x)v' + q(x)v].$$

Furthermore, $u(0) = y(0) - v(0) = 0$ and $u(1) = y(1) - v(1) = 0$. The simplest function having the assumed properties is $v(x) = (b - a)x + a$. In this case,

$$g(x) = (a - b)p(x) + (a - b)x\,q(x) - a\,q(x)\,.$$

20. The associated homogeneous PDE is $u_t = u_{xx}$, $0 < x < 1$, with

$$u_x(0\,,t) = 0,\;\; u_x(1, t) + u(1, t) = 0 \text{ and } u(x\,,0) = 1 - x.$$

Applying the method of *separation of variables*, we obtain the eigenvalue problem $X'' + \lambda X = 0$, with boundary conditions $X'(0) = 0$ and $X'(1) + X(1) = 0$. It was shown in Prob. $4$, in Section $11.2$, that the normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{2}\,cos\,\sqrt{\lambda_n}\,x}{\sqrt{1 + sin^2\sqrt{\lambda_n}}}\,,$$

where $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\,sin\sqrt{\lambda_n} = 0.$

We assume a solution of the form

$$u(x\,,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)\,.$$

Substitution into the given PDE results in

$$\begin{aligned}\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) &= \sum_{n=1}^{\infty} b_n(t)\phi_n''(x) + e^{-t} \\ &= -\sum_{n=1}^{\infty} \lambda_n b_n(t)\phi_n(x) + e^{-t},\end{aligned}$$

that is,

$$\sum_{n=1}^{\infty} [b_n'(t) + \lambda_n b_n(t)]\phi_n(x) = e^{-t}.$$

We now note that

$$1 = \sum_{n=1}^{\infty} \frac{\sqrt{2}\,sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}\sqrt{1 + sin^2\sqrt{\lambda_n}}}\,\phi_n(x)\,.$$

Therefore

$$e^{-t} = \sum_{n=1}^{\infty} \beta_n\, e^{-t}\phi_n(x)\,,$$

in which $\beta_n = \sqrt{2}\,sin\sqrt{\lambda_n}/\left[\sqrt{\lambda_n}\sqrt{1 + sin^2\sqrt{\lambda_n}}\right]$. Combining these results,

$$\sum_{n=1}^{\infty} \left[b_n'(t) + \lambda_n b_n(t) - \beta_n e^{-t}\right]\phi_n(x) = 0\,.$$

Since the resulting equation is valid for $0 < x < 1$, it follows that

$$b_n'(t) + \lambda_n b_n(t) = \beta_n e^{-t},\ n = 1, 2, \cdots.$$

Prior to solving the sequence of ODEs, we establish the initial conditions. These are obtained from the expansion

$$u(x\,,0) = 1 - x = \sum_{n=1}^{\infty} \alpha_n \phi_n(x)\,,$$

in which $\alpha_n = \sqrt{2}\left(1 - cos\sqrt{\lambda_n}\right)/\left[\lambda_n\sqrt{1 + sin^2\sqrt{\lambda_n}}\right]$. That is, $b_n(0) = \alpha_n$.

Therefore the solutions of the first order ODEs are

$$b_n(t) = \frac{\beta_n\left(e^{-t} - e^{-\lambda_n t}\right)}{(\lambda_n - 1)} + \alpha_n e^{-\lambda_n t},\ n = 1, 2, \cdots.$$

Hence the solution of the boundary value problem is

$$u(x\,,t)=\sum_{n=1}^{\infty}\left[\frac{\beta_n\left(e^{-t}-e^{-\lambda_n t}\right)}{(\lambda_n-1)}+\alpha_n e^{-\lambda_n t}\right]\phi_n(x)\,.$$

21. Based on the boundary conditions, the normalized eigenfunctions are given by

$$\phi_n(x)=\sqrt{2}\,sin\,n\pi x\,,$$

with associated eigenvalues $\lambda_n=n^2\pi^2$. We now assume a solution of the form

$$u(x\,,t)=\sum_{n=1}^{\infty}b_n(t)\phi_n(x)\,.$$

Substitution into the given PDE results in

$$\begin{aligned}\sum_{n=1}^{\infty}b_n'(t)\phi_n(x) &= \sum_{n=1}^{\infty}b_n(t)\phi_n''(x)+1-|1-2x| \\ &= -\sum_{n=1}^{\infty}\lambda_n b_n(t)\phi_n(x)+1-|1-2x|,\end{aligned}$$

that is,

$$\sum_{n=1}^{\infty}[b_n'(t)+\lambda_n b_n(t)]\phi_n(x)=1-|1-2x|.$$

It was shown in Prob. 5 that

$$1-|1-2x|=\sum_{n=1}^{\infty}4\frac{\sqrt{2}\,sin(n\pi/2)}{n^2\pi^2}\phi_n(x)\,.$$

Substituting on the right hand side and collecting terms, we obtain

$$\sum_{n=1}^{\infty}\left[b_n'(t)+\lambda_n b_n(t)-4\frac{\sqrt{2}\,sin(n\pi/2)}{n^2\pi^2}\right]\phi_n(x)=0.$$

Since the resulting equation is valid for $0<x<1$, it follows that

$$b_n'(t)+n^2\pi^2 b_n(t)=4\frac{\sqrt{2}\,sin(n\pi/2)}{n^2\pi^2},\ n=1,2,\cdots.$$

Based on the given initial condition, we also have $b_n(0)=0$, for $n=1,2,\cdots$. The solutions of the first order ODEs are

$$b_n(t)=4\frac{\sqrt{2}\,sin(n\pi/2)}{n^4\pi^4}\left(1-e^{-n^2\pi^2 t}\right),\ n=1,2,\cdots.$$

Hence the solution of the boundary value problem is

$$u(x\,,t) = \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{sin(n\pi/2)}{n^4} \left(1 - e^{-n^2\pi^2 t}\right) sin\, n\pi x\,.$$

$23(a)$. Let $u(x\,,t)$ be a solution of the boundary value problem and $v(x)$ be a solution of the related BVP. Substituting for $u(x\,,t) = w(x\,,t) + v(x)$, we have

$$r(x)u_t = r(x)w_t$$

and

$$\begin{aligned} [p(x)u_x]_x - q(x)u + F(x) &= [p(x)w_x]_x - q(x)w + [p(x)v']' - q(x)v + F(x) \\ &= [p(x)w_x]_x - q(x)w - F(x) + F(x) \\ &= [p(x)w_x]_x - q(x)w\,. \end{aligned}$$

Hence $w(x\,,t)$ is a solution of the *homogeneous* PDE

$$r(x)w_t = [p(x)w_x]_x - q(x)w\,.$$

The required *boundary conditions* are

$$\begin{aligned} w(0\,,t) &= u(0\,,t) - v(0) = 0\,, \\ w(1\,,t) &= u(1\,,t) - v(1) = 0\,. \end{aligned}$$

The associated *initial condition* is $w(x\,,0) = u(x\,,0) - v(x) = f(x) - v(x)$.

$(b)$. Let $v(x)$ be a solution of the ODE

$$[p(x)v']' - q(x)v = \, -F(x)\,,$$

and satisfying the boundary conditions $v'(0) - h_1 v(0) = T_1$ , $v'(1) + h_2 v(1) = T_2$. If $w(x\,,t) = u(x\,,t) - v(x)$, then it is easy to show the $w$ satisfies the PDE and initial condition given in Part $(a)$. Furthermore,

$$\begin{aligned} w_x(0\,,t) - h_1 w(0\,,t) &= u_x(0\,,t) - v'(0) - h_1 u(0\,,t) + h_1 v(0) \\ &= u_x(0\,,t) - h_1 u(0\,,t) - v'(0) + h_1 v(0) \\ &= 0\,. \end{aligned}$$

Similarly, the other boundary condition is also homogeneous.

25. In this problem, $F(x) = -\pi^2 cos\,\pi x$ . First find a solution of the boundary value problem

$$v'' = \pi^2 cos\,\pi x \ , \ v'(0) = 0 \ , \ v(1) = 1 \, .$$

The general solution is $v(x) = Ax + B - cos\,\pi x$ . Imposing the initial conditions, the solution of the related BVP is $v(x) = -cos\,\pi x$ . Now let $w(x\,,t) = u(x\,,t) + cos\,\pi x$ . It follows that $w(x\,,t)$ satisfies the *homogeneous* boundary value problem, and the initial condition $w(x\,,0) = cos(3\pi x/2) - cos\,\pi x - (-cos\,\pi x) = cos(3\pi x/2)$ .

We now seek solutions of the homogeneous problem of the form

$$w(x\,,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)\,,$$

in which $\phi_n(x) = \sqrt{2}\,cos\,(2n-1)\pi x/2$ are the *normalized* eigenfunctions of the *homogeneous* problem and $\lambda_n = (2n-1)^2\pi^2/4$ , with $n = 1, 2, \cdots$ . Substitution into the PDE for $w$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) &= \sum_{n=1}^{\infty} b_n(t)\phi_n''(x)\\ &= -\sum_{n=1}^{\infty} \lambda_n b_n(t)\phi_n(x)\,.\end{aligned}$$

Since the latter equation is valid for $0 < x < 1$ , it follows that

$$b_n'(t) + \lambda_n b_n(t) = 0\,,\ n = 1, 2, \cdots,$$

with $b_n(t) = b_n(0)exp(-\lambda_n t)$. Hence

$$w(x\,,t) = \sum_{n=1}^{\infty} b_n(0)exp(-\lambda_n t)\phi_n(x)\,.$$

Imposing the initial condition, we require that

$$\sqrt{2}\sum_{n=1}^{\infty} b_n(0)cos\,\frac{(2n-1)\pi x}{2} = cos\frac{3\pi x}{2}\,.$$

It is evident that all of the coefficients are *zero*, except for $b_2(0) = 1/\sqrt{2}$ . Therefore

$$w(x\,,t) = exp\big(-9\pi^2 t/4\big)cos\frac{3\pi x}{2}\,,$$

and the solution of the original BVP is

$$u(x\,,t) = exp\big(-9\pi^2 t/4\big)cos\frac{3\pi x}{2} - cos\,\pi x\,.$$

26$(a)$. Let $u(x\,,t) = X(x)T(t)$. Substituting into the homogeneous form of $(i)$,

$$r(x)XT'' = [p(x)X']'T - q(x)XT\,.$$

Now divide both sides of the resulting equation by $XT$ to obtain

$$\frac{T''}{T} = \frac{[p(x)X']'}{r(x)X} - \frac{q(x)}{r(x)} = -\,\lambda\,.$$

It follows that

$$-\,[p(x)X']' + q(x)X = \lambda r(x)X\,.$$

Since the boundary conditions $(ii)$ are valid for all $t > 0$, we also have

$$X'(0) - h_1X(0) = 0\ ,\quad X'(1) + h_2X(1) = 0\,.$$

$(b)$. Let $\lambda_n$ and $\phi_n(x)$ denote the eigenvalues and eigenfunctions of the BVP in Part $(a)$. Assume a solution, of the PDE $(i)$, of the form

$$u(x\,,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)\,.$$

Substituting into $(i)$,

$$\begin{aligned} r(x)\sum_{n=1}^{\infty} b_n''(t)\phi_n &= \sum_{n=1}^{\infty} b_n(t)\big\{[p(x)\phi_n']' - q(x)\phi_n\big\} + F(x\,,t) \\ &= \sum_{n=1}^{\infty} b_n(t)[\,-\lambda_n r(x)\phi_n] + F(x\,,t)\,. \end{aligned}$$

Rearranging the terms,

$$r(x)\sum_{n=1}^{\infty}[b_n''(t) + \lambda_n b_n(t)]\phi_n = F(x\,,t)\,,$$

or

$$\sum_{n=1}^{\infty}[b_n''(t) + \lambda_n b_n(t)]\phi_n = \frac{F(x\,,t)}{r(x)}\,.$$

Now expand the right hand side in terms of the eigenfunctions. That is, write

$$\frac{F(x\,,t)}{r(x)} = \sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x)\,,$$

in which

$$\gamma_n(t) = \int_0^1 r(x)\frac{F(x\,,t)}{r(x)}\phi_n(x)dx$$
$$= \int_0^1 F(x\,,t)\phi_n(x)dx\,, \quad n = 1, 2, \cdots.$$

Combining these results, we have

$$\sum_{n=1}^{\infty}[b_n''(t) + \lambda_n b_n(t) - \gamma_n(t)]\phi_n = 0\,.$$

It follows that

$$b_n''(t) + \lambda_n b_n(t) = \gamma_n(t) \ \ , \ n = 1, 2, \cdots.$$

In order to solve this sequence of ODEs, we require initial conditions $b_n(0)$ and $b_n'(0)$. Note that

$$u(x\,,0) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) \ \text{ and } \ u_t(x\,,0) = \sum_{n=1}^{\infty} b_n'(0)\phi_n(x)\,.$$

Based on the given initial conditions,

$$f(x) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) \ \text{ and } \ g(x) = \sum_{n=1}^{\infty} b_n'(0)\phi_n(x)\,.$$

Hence $b_n(0) = \alpha_n$ and $b_n'(0) = \beta_n$, the expansion coefficients for $f(x)$ and $g(x)$ in terms of the eigenfunctions, $\phi_n(x)$.

$27(a)$. Since the eigenvectors are *orthogonal*, they form a basis. Given any vector **b**,

$$\mathbf{b} = \sum_{i=1}^{n} b_i \boldsymbol{\xi}^{(i)}.$$

Taking the inner product, with $\boldsymbol{\xi}^{(j)}$, of both sides of the equation, we have

$$\left(\mathbf{b}\,, \boldsymbol{\xi}^{(j)}\right) = b_j\left(\boldsymbol{\xi}^{(j)}\,, \boldsymbol{\xi}^{(j)}\right).$$

$(b)$. Consider solutions of the form

$$\mathbf{x} = \sum_{i=1}^{n} a_i \boldsymbol{\xi}^{(i)}.$$

Substituting into Eq. $(i)$, and using the above form of **b**,

$$\sum_{i=1}^{n} a_i \mathbf{A}\boldsymbol{\xi}^{(i)} - \sum_{i=1}^{n} \mu\, a_i \boldsymbol{\xi}^{(i)} = \sum_{i=1}^{n} b_i \boldsymbol{\xi}^{(i)}.$$

It follows that

$$\sum_{i=1}^{n}[a_i\lambda_i - \mu\, a_i - b_i]\boldsymbol{\xi}^{(i)} = \mathbf{0}\,.$$

Since the eigenvectors are linearly independent,

$$a_i\lambda_i - \mu\, a_i - b_i = 0\,, \text{ for } i = 1, 2, \cdots, n\,.$$

That is,

$$a_i = b_i/(\lambda_i - \mu)\,, \quad i = 1, 2, \cdots, n\,.$$

Assuming that the eigenvectors are *normalized*, the solution is given by

$$\mathbf{x} = \sum_{i=1}^{n}\frac{(\mathbf{b}\,, \boldsymbol{\xi}^{(i)})}{\lambda_i - \mu}\boldsymbol{\xi}^{(i)},$$

as long as $\mu$ is *not* equal to one of the eigenvalues.

29. First write the ODE as $y'' + y = -f(x)$. A fundamental set of solutions of the homogeneous equation is given by

$$y_1 = cos\, x \text{ and } y_2 = sin\, x\,.$$

The Wronskian is equal to $W[cos\, x\,, sin\, x] = 1$. Applying the method of *variation of parameters*, a particular solution is

$$Y(x) = y_1(x)u_1(x) + y_2(x)u_2(x)\,,$$

in which

$$u_1(x) = \int_0^x sin(s)f(s)ds \text{ and } u_2(x) = -\int_0^x cos(s)f(s)ds\,.$$

Therefore the general solution is

$$y = \phi(x) = c_1 cos\, x + c_2 sin\, x + cos\, x\int_0^x sin(s)f(s)ds - sin\, x\int_0^x cos(s)f(s)ds.$$

Imposing the boundary conditions, we must have $c_1 = 0$ and

$$c_2\, sin\, 1 + cos\, 1\int_0^1 sin(s)f(s)ds - sin\, 1\int_0^1 cos(s)f(s)ds = 0\,.$$

It follows that

$$c_2 = \frac{1}{sin\, 1}\int_0^1 sin(1-s)f(s)ds\,,$$

and

$$\phi(x) = \frac{sin\,x}{sin\,1}\int_0^1 sin(1-s)f(s)ds - \int_0^x sin(x-s)f(s)ds.$$

Using standard identities,

$$sin\,x \cdot sin(1-s) - sin\,1 \cdot sin(x-s) = sin\,s \cdot sin(1-x)\,.$$

Therefore

$$\frac{sin\,x \cdot sin(1-s)}{sin\,1} - sin(x-s) = \frac{sin\,s \cdot sin(1-x)}{sin\,1}\,.$$

Splitting up the *first* integral, we obtain

$$\begin{aligned}\phi(x) &= \int_0^x \frac{sin\,s \cdot sin(1-x)}{sin\,1} f(s)ds + \int_x^1 \frac{sin\,x \cdot sin(1-s)}{sin\,1} f(s)ds \\ &= \int_0^1 G(x\,,s)f(s)ds\,,\end{aligned}$$

in which

$$G(x\,,s) = \begin{cases} \frac{sin\,s \cdot sin(1-x)}{sin\,1}\,, & 0 \le s \le x \\ \frac{sin\,x \cdot sin(1-s)}{sin\,1}\,, & x \le s \le 1\,. \end{cases}$$

31. The general solution of the homogeneous problem is

$$y = c_1 + c_2\,x\,.$$

By inspection, it is easy to see that $y_1(x) = 1$ satisfies the BC $y'(0) = 0$ and that the function $y_2(x) = 1 - x$ satisfies the BC $y(1) = 0$. The Wronskian of these solutions is $W[y_1\,,y_2] = -1$. Based on Prob. 30, with $p(x) = 1$, the Green's function is given by

$$G(x\,,s) = \begin{cases} (1-x)\,, & 0 \le s \le x \\ (1-s)\,, & x \le s \le 1\,. \end{cases}$$

Therefore the solution of the given BVP is

$$\phi(x) = \int_0^x (1-x)f(s)ds + \int_x^1 (1-s)f(s)ds\,.$$

32. The general solution of the homogeneous problem is

$$y = c_1 + c_2\,x\,.$$

We find that $y_1(x) = x$ satisfies the BC $y(0) = 0$. Imposing the boundary condition

$y(1)+y'(1)=0$, we must have $c_1+2c_2=0$. Hence choose $y_2(x)=-2+x$. The Wronskian of these solutions is $W[y_1\,,y_2]=2$. Based on Prob. 30, with $p(x)=1$, the Green's function is given by

$$G(x\,,s)=\begin{cases} s(x-2)/2\,, & 0\le s\le x \\ x(s-2)/2\,, & x\le s\le 1\,. \end{cases}$$

Therefore the solution of the given BVP is

$$\phi(x)=\frac{1}{2}\int_0^x s(x-2)f(s)ds+\frac{1}{2}\int_x^1 x(s-2)f(s)ds\,.$$

34. The general solution of the homogeneous problem is

$$y=c_1+c_2\,x\,.$$

By inspection, it is easy to see that $y_1(x)=x$ satisfies the BC $y(0)=0$ and that the function $y_2(x)=1$ satisfies the BC $y'(1)=0$. The Wronskian of these solutions is $W[y_1\,,y_2]=-1$. Based on Prob. 30, with $p(x)=1$, the Green's function is given by

$$G(x\,,s)=\begin{cases} s\,, & 0\le s\le x \\ x\,, & x\le s\le 1\,. \end{cases}$$

Therefore the solution of the given BVP is

$$\phi(x)=\int_0^x sf(s)ds+\int_x^1 xf(s)ds\,.$$

35$(a)$. We proceed to show that if the expression given by Eq. $(iv)$ is substituted into the
integral of Eq. $(iii)$, then the result is the solution of the nonhomogeneous problem. As long as we can interchange the summation and integration,

$$\begin{aligned} y=\phi(x)\ &=\int_0^1 G(x\,,s\,,\mu)f(s)ds \\ &=\sum_{n=1}^{\infty}\frac{\phi_i(x)}{\lambda_i-\mu}\int_0^1 f(s)\phi_i(s)ds\,. \end{aligned}$$

Note that

$$\int_0^1 f(s)\phi_i(s)ds=c_i\,.$$

Therefore

$$y=\phi(x)=\sum_{n=1}^{\infty}\frac{c_i\,\phi_i(x)}{\lambda_i-\mu}\,,$$

as given by Eq. (13) in the text. It is assumed that the eigenfunctions are *normalized* and $\lambda_i \neq \mu$ .

$(b)$. For any fixed value of $x$, $G(x\,,s\,,\mu)$ is a function of $s$ and the parameter $\mu$ . With appropriate assumptions on $G$ , we can write the eigenfunction expansion

$$G(x\,,s\,,\mu) = \sum_{i=1}^{\infty} a_i(x\,,\mu)\phi_i(s)\,.$$

Since the eigenfunctions are *orthonormal* with respect to $r(x)$,

$$\int_0^1 G(x\,,s\,,\mu)r(s)\phi_i(s)ds = a_i(x\,,\mu)\,.$$

Now let

$$y_i(x) = \int_0^1 G(x\,,s\,,\mu)r(s)\phi_i(s)ds\,.$$

Based on the association $f(x) = r(x)\phi_i(x)$ , it is evident that

$$L[y_i] = \mu\, r(x)y_i(x) + r(x)\phi_i(x)\,.$$

In order to evaluate the left hand side, we consider the eigenfunction expansion

$$y_i(x) = \sum_{k=1}^{\infty} b_{ik}\phi_k(x)\,.$$

It follows that

$$\begin{aligned} L[y_i] &= \sum_{k=1}^{\infty} b_{ik} L[\phi_k] \\ &= \sum_{k=1}^{\infty} b_{ik}\lambda_k\, r(x)\phi_k(x)\,. \end{aligned}$$

Therefore

$$r(x)\sum_{k=1}^{\infty} b_{ik}\lambda_k\phi_k(x) = \mu\, r(x)\sum_{k=1}^{\infty} b_{ik}\phi_k(x) + r(x)\phi_i(x)\,,$$

and since $r(x) \neq 0$ ,

$$\sum_{k=1}^{\infty} b_{ik}\lambda_k\phi_k(x) = \mu\sum_{k=1}^{\infty} b_{ik}\phi_k(x) + \phi_i(x)\,.$$

Rearranging the terms, we find that

$$\phi_i(x) = \sum_{k=1}^{\infty} b_{ik}(\lambda_k - \mu)\phi_k(x) \, .$$

Since the eigenfunctions are linearly independent, $b_{ik}(\lambda_k - \mu) = \delta_{ik}$ , and thus

$$y_i(x) = \sum_{k=1}^{\infty} \frac{\delta_{ik}}{\lambda_k - \mu}\phi_k(x) = \frac{1}{\lambda_i - \mu}\phi_i(x) \, .$$

We conclude that

$$a_i(x \, , \mu) = \frac{1}{\lambda_i - \mu}\phi_i(x) \, ,$$

which verifies that

$$G(x \, , s \, , \mu) = \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(s)}{\lambda_i - \mu} \, .$$

36. First note that $-\, d^2y/ds^2 = 0$ for $s \neq x$ . On the interval $0 < s < x$ , the solution of the ODE is $y_1(s) = c_1 + c_2 s$ . Given that $y(0) = 0$ , we have $y_1(s) = c_2 s$ . On the interval $x < s < 1$ , the solution is $y_2(s) = d_1 + d_2 s$ . Imposing the condition $y(1) = 0$ , we have $y_2(s) = d_1(1 - s)$. Assuming continuity of the solution, at $s = x$ ,

$$c_2 x = d_1(1 - x) \, ,$$

which gives $c_2 = d_1(1 - x)/x$ . Next, integrate both sides of the given ODE over an *infinitesimal* interval containing $s = x$ :

$$-\int_{x^-}^{x^+} \frac{d^2y}{ds^2} \, ds = \int_{x^-}^{x^+} \delta(s - x) ds = 1 \, .$$

It follows that

$$y'(x^-) - y'(x^+) = 1 \, ,$$

and hence $c_2 - (\, - d_1) = 1$ . Solving for the two coefficients, we obtain $c_2 = 1 - x$ and $d_1 = x$ . Therefore the solution of the BVP is given by

$$y(s) = \begin{cases} s(1 - x) \, , & 0 \leq s \leq x \\ x(1 - s) \, , & x \leq s \leq 1 \, , \end{cases}$$

which is identical to the Green's function in Prob. 28.