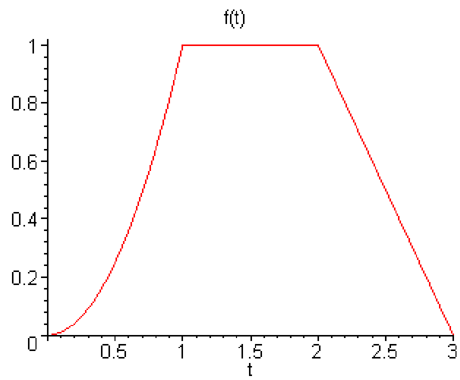
Chapter Six

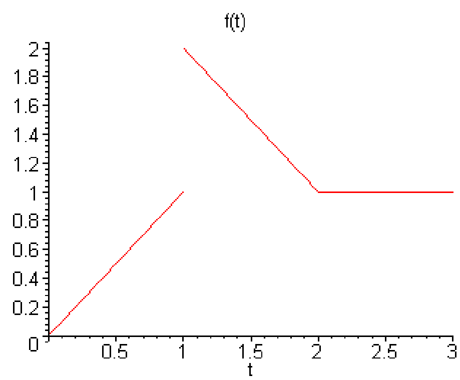
Section 6.1

3.



The function $f(t)$ is *continuous*.

4.



The function $f(t)$ has a *jump discontinuity* at $t = 1$.

7. Integration is a linear operation. It follows that

$$\begin{aligned}\int_0^A \cosh bt \cdot e^{-st}dt &= \frac{1}{2}\int_0^A e^{bt} \cdot e^{-st}dt + \frac{1}{2}\int_0^A e^{-bt} \cdot e^{-st}dt \\ &= \frac{1}{2}\int_0^A e^{(b-s)t}dt + \frac{1}{2}\int_0^A e^{-(b+s)t}dt\,.\end{aligned}$$

Hence

$$\int_0^A \cosh bt \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] + \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a *limit*, as $A \to \infty$,

$$\begin{aligned}\int_0^\infty cosh\, bt \cdot e^{-st} dt &= \frac{1}{2}\left[\frac{1}{s-b}\right] + \frac{1}{2}\left[\frac{1}{s+b}\right] \\ &= \frac{s}{s^2-b^2}\,.\end{aligned}$$

Note that the above is valid for $s > |b|$.

8. Proceeding as in Prob. 7,

$$\int_0^A sinh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] - \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a *limit*, as $A \to \infty$,

$$\begin{aligned}\int_0^\infty sinh\, bt \cdot e^{-st} dt &= \frac{1}{2}\left[\frac{1}{s-b}\right] - \frac{1}{2}\left[\frac{1}{s+b}\right] \\ &= \frac{b}{s^2-b^2}\,.\end{aligned}$$

The limit exists as long as $s > |b|$.

10. Observe that $e^{at} sinh\, bt = \left(e^{(a+b)t} - e^{(a-b)t}\right)/2$. It follows that

$$\int_0^A e^{at}\, sinh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1-e^{(a+b-s)A}}{s-a+b}\right] - \frac{1}{2}\left[\frac{1-e^{-(b-a+s)A}}{s+b-a}\right].$$

Taking a *limit*, as $A \to \infty$,

$$\begin{aligned}\int_0^\infty e^{at}\, sinh\, bt \cdot e^{-st} dt &= \frac{1}{2}\left[\frac{1}{s-a+b}\right] - \frac{1}{2}\left[\frac{1}{s+b-a}\right] \\ &= \frac{b}{(s-a)^2-b^2}\,.\end{aligned}$$

The limit exists as long as $s - a > |b|$.

11. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[\, sin\, bt] = \frac{1}{2i}\mathcal{L}\left[e^{ibt}\right] - \frac{1}{2i}\mathcal{L}\left[e^{-ibt}\right].$$

Since

$$\int_0^\infty e^{(a+ib)t} e^{-st} dt = \frac{1}{s-a-ib}\,,$$

we have

$$\int_0^\infty e^{\pm\, ibt}\, e^{-st} dt = \frac{1}{s \mp ib}\,.$$

Therefore

$$\begin{aligned}\mathcal{L}[sin\, bt] \; &= \frac{1}{2i}\left[\frac{1}{s\, -\, ib} - \frac{1}{s\, +\, ib}\right] \\ &= \frac{b}{s^2 + b^2}\,.\end{aligned}$$

12. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[\, cos\, bt] = \frac{1}{2}\mathcal{L}\big[e^{ibt}\big] + \frac{1}{2}\mathcal{L}\big[e^{-ibt}\big].$$

From Prob. 11 , we have

$$\int_0^\infty e^{\pm\, ibt}\, e^{-st} dt = \frac{1}{s \mp ib}\,.$$

Therefore

$$\begin{aligned}\mathcal{L}[cos\, bt] \; &= \frac{1}{2}\left[\frac{1}{s\, -\, ib} + \frac{1}{s\, +\, ib}\right] \\ &= \frac{s}{s^2 + b^2}\,.\end{aligned}$$

14. Using the *linearity* of the Laplace transform,

$$\mathcal{L}\big[\, e^{at} cos\, bt\big] = \frac{1}{2}\mathcal{L}\big[e^{(a+ib)t}\big] + \frac{1}{2}\mathcal{L}\big[e^{(a-ib)t}\big].$$

Based on the integration in Prob. 11 ,

$$\int_0^\infty e^{(a\,\pm\, ib)t} e^{-st} dt = \frac{1}{s - a \mp ib}\,.$$

Therefore

$$\begin{aligned}\mathcal{L}\big[e^{at} cos\, bt\big] \; &= \frac{1}{2}\left[\frac{1}{s\, - a -\, ib} + \frac{1}{s\, - a +\, ib}\right] \\ &= \frac{s - a}{(s-a)^2 + b^2}\,.\end{aligned}$$

The above is valid for $s > a$.

15. Integrating *by parts*,

$$\int_0^A te^{at}\cdot e^{-st}dt \;=\; -\left.\frac{t\,e^{(a-s)t}}{s-a}\right|_0^A + \int_0^A \frac{1}{s-a}e^{(a-s)t}dt$$
$$= \frac{1-e^{A(a-s)}+A(a-s)e^{A(a-s)}}{(s-a)^2}\,.$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty te^{at}\cdot e^{-st}dt = \frac{1}{(s-a)^2}\,.$$

Note that the limit exists as long as $s > a$.

17. Observe that $t\,cosh\,at = (t\,e^{at} + t\,e^{-at})/2$. For any value of c,

$$\int_0^A t\,e^{ct}\cdot e^{-st}dt \;=\; -\left.\frac{t\,e^{(c-s)t}}{s-c}\right|_0^A + \int_0^A \frac{1}{s-c}e^{(c-s)t}dt$$
$$= \frac{1-e^{A(c-s)}+A(c-s)e^{A(c-s)}}{(s-c)^2}\,.$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty te^{ct}\cdot e^{-st}dt = \frac{1}{(s-c)^2}\,.$$

Note that the limit exists as long as $s > |c|$. Therefore,

$$\int_0^\infty t\,cosh\,at\cdot e^{-st}dt \;=\; \frac{1}{2}\left[\frac{1}{(s-a)^2}+\frac{1}{(s+a)^2}\right]$$
$$= \frac{s^2+a^2}{(s-a)^2(s+a)^2}\,.$$

18. Integrating *by parts*,

$$\int_0^A t^n e^{at}\cdot e^{-st}dt \;=\; -\left.\frac{t^n\,e^{(a-s)t}}{s-a}\right|_0^A + \int_0^A \frac{n}{s-a}\,t^{n-1}e^{(a-s)t}dt$$
$$= -\frac{A^n e^{-(s-a)A}}{s-a} + \int_0^A \frac{n}{s-a}\,t^{n-1}e^{(a-s)t}dt\,.$$

Continuing to integrate by parts, it follows that

$$\int_0^A t^n e^{at} \cdot e^{-st} dt = -\frac{A^n e^{(a-s)A}}{s-a} - \frac{nA^{n-1}e^{(a-s)A}}{(s-a)^2} - \\ - \frac{n!Ae^{(a-s)A}}{(n-2)!(s-a)^3} - \cdots - \frac{n!\left(e^{(a-s)A}-1\right)}{(s-a)^{n+1}} .$$

That is,

$$\int_0^A t^n e^{at} \cdot e^{-st} dt = p_n(A) \cdot e^{(a-s)A} + \frac{n!}{(s-a)^{n+1}},$$

in which $p_n(\xi)$ is a *polynomial* of degree n. For any *given* polynomial,

$$\lim_{A\to\infty} p_n(A) \cdot e^{-(s-a)A} = 0 ,$$

as long as $s > a$. Therefore,

$$\int_0^\infty t^n e^{at} \cdot e^{-st} dt = \frac{n!}{(s-a)^{n+1}} .$$

20. Observe that $t^2 sinh\, at = (t^2 e^{at} - t^2 e^{-at})/2$. Using the result in Prob. 18,

$$\int_0^\infty t^2 sinh\, at \cdot e^{-st} dt = \frac{1}{2}\left[\frac{2!}{(s-a)^3} - \frac{2!}{(s+a)^3}\right] \\ = \frac{2a(3s^2+a^2)}{(s^2-a^2)^3} .$$

The above is valid for $s > |a|$.

22. Integrating *by parts*,

$$\int_0^A t\, e^{-t} dt = -t\, e^{-t}\Big|_0^A + \int_0^A e^{-t} dt \\ = 1 - e^{-A} - Ae^{-A}.$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty t\, e^{-t} dt = 1 - e^{-A} .$$

Hence the integral *converges*.

23. Based on a series expansion, note that for $t > 0$,

$$e^t > 1 + t + t^2/2 > t^2/2 .$$

It follows that for $t > 0$,

$$t^{-2}e^t > \frac{1}{2}.$$

Hence for any finite $A > 1$,

$$\int_1^A t^{-2}e^t dt > \frac{A-1}{2}.$$

It is evident that the limit as $A \to \infty$ does not exist.

24. Using the fact that $|cos\, t| \leq 1$, and the fact that

$$\int_0^\infty e^{-t}dt = 1,$$

it follows that the given integral *converges*.

25(a). Let $p > 0$. Integrating *by parts*,

$$\begin{aligned}\int_0^A e^{-x}x^p dx &= -e^{-x}x^p\Big|_0^A + p\int_0^A e^{-x}x^{p-1}dx \\ &= -A^p e^{-A} + p\int_0^A e^{-x}x^{p-1}dx.\end{aligned}$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty e^{-x}x^p dx = p\int_0^\infty e^{-x}x^{p-1}dx.$$

That is, $\Gamma(p+1) = p\,\Gamma(p)$.

(b). Setting $p = 0$,

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1.$$

(c). Let $p = n$. Using the result in Part (b),

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &\;\;\vdots \\ &= n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1).\end{aligned}$$

Since $\Gamma(1) = 1$, $\Gamma(n+1) = n!$.

(d). Using the result in Part (b),

$$\begin{aligned}\Gamma(p+n) &= (p+n-1)\,\Gamma(p+n-1)\\ &= (p+n-1)(p+n-2)\Gamma(p+n-2)\\ &\ \vdots\\ &= (p+n-1)(p+n-2)\cdots(p+1)p\,\Gamma(p)\,.\end{aligned}$$

Hence

$$\frac{\Gamma(p+n)}{\Gamma(p)} = p(p+1)(p+1)\cdots(p+n-1)\,.$$

Given that $\Gamma(1/2) = \sqrt{\pi}$, it follows that

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{945\sqrt{\pi}}{32}\,.$$