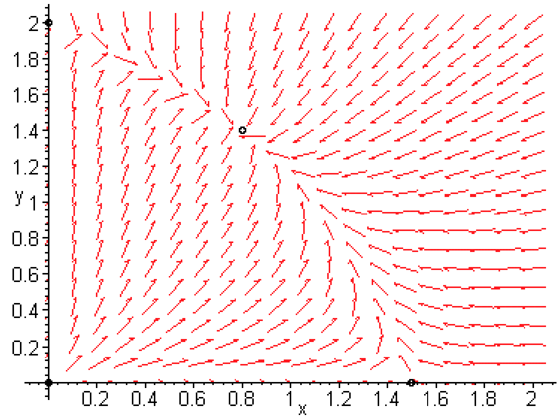
Section 9.4

$1(a)$.



(b). The critical points are solutions of the system of equations

$$x(1.5 - x - 0.5\,y) = 0$$
$$y(2 - y - 0.75\,x) = 0\,.$$

The four critical points are $(0\,,0)$, $(0\,,2)$, $(1.5\,,0)$ and $(0.8\,,1.4)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -\,x/2 \\ -\,3y/4 & 2 - 3x/4 - 2y \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ \ r_2 = 2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.

At the critical point $(0\,,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,2) = \begin{pmatrix} 1/2 & 0 \\ -\,3/2 & -\,2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -0.6 \end{pmatrix};\ \ r_2 = -2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

At the critical point $(1.5\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1.5\,,0) = \begin{pmatrix} -1.5 & -0.75 \\ 0 & 0.875 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1.5\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \ r_2 = 0.875\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -0.31579 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is also a *saddle*, which is *unstable*.

At the critical point $(0.8\,,1.4)$, the coefficient matrix of the linearized system is

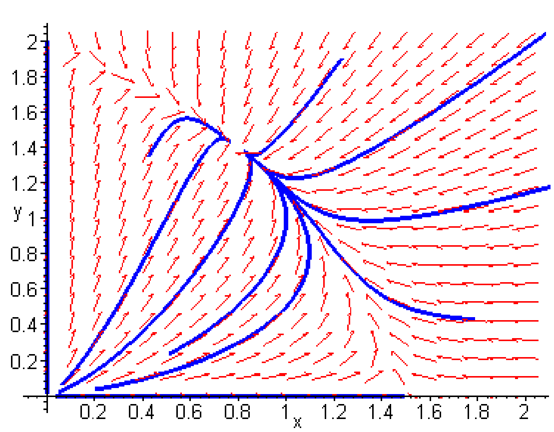
$$\mathbf{J}(0.8\,,1.4) = \begin{pmatrix} -0.8 & -0.4 \\ -1.05 & -1.4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{11}{10} + \frac{\sqrt{51}}{10}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \frac{3-\sqrt{51}}{4} \end{pmatrix};\ \ r_2 = -\frac{11}{10} - \frac{\sqrt{51}}{10}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \frac{3+\sqrt{51}}{4} \end{pmatrix}.$$

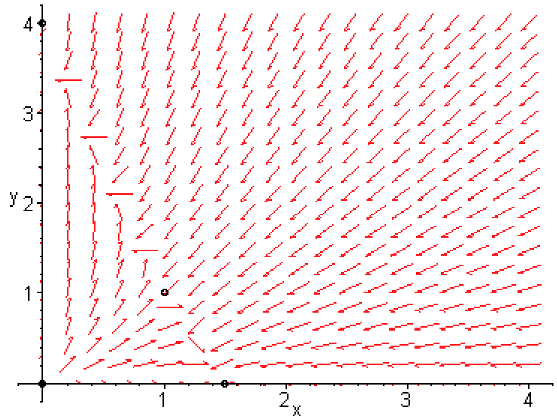
The eigenvalues are both negative. Hence the critical point is a *stable node*, which is *asymptotically stable*.

(d, e).



(f). Except for initial conditions lying on the coordinate axes, almost all trajectories converge to the stable node at $(0.8\,,1.4)$.

$2(a)$.



(b). The critical points are the solution set of the system of equations

$$x(1.5 - x - 0.5\,y) = 0$$
$$y(2 - 0.5\,y - 1.5\,x) = 0\,.$$

The four critical points are $(0\,,0)$, $(0\,,4)$, $(1.5\,,0)$ and $(1\,,1)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -\,x/2 \\ -\,3y/2 & 2 - 3x/2 - y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ \ r_2 = 2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.

At the critical point $(0\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,4) = \begin{pmatrix} -\,1/2 & 0 \\ -\,6 & -\,2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix};\ r_2 = -2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0\,,4)$ is a *stable node*, which is *asymptotically stable*.

At the critical point $(3/2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3/2\,,0) = \begin{pmatrix} -3/2 & -3/4 \\ 0 & -1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ r_2 = -1/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point is a *stable node*, which is *asymptotically stable*.

At the critical point $(1, 1)$, the coefficient matrix of the linearized system is

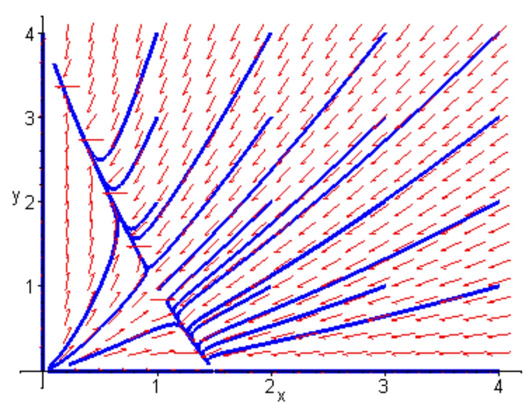
$$\mathbf{J}(1\,,1) = \begin{pmatrix} -1 & -1/2 \\ -3/2 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3+\sqrt{13}}{4}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{13}}{2} \end{pmatrix};\ r_2 = -\frac{3+\sqrt{13}}{4}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ \frac{-1+\sqrt{13}}{2} \end{pmatrix}.$$

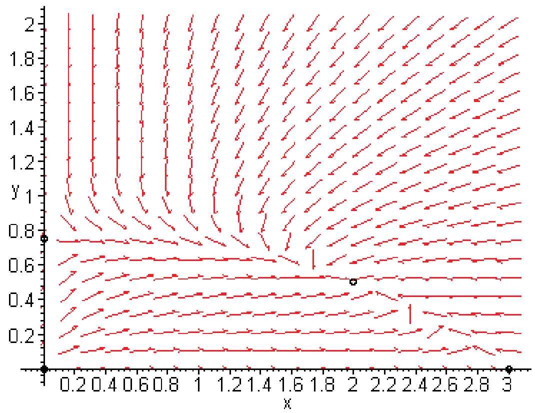
The eigenvalues are of opposite sign, hence $(1, 1)$ is a *saddle*, which is *unstable*.

(d, e).



(f). Trajectories *approaching* the critical point $(1\,,1)$ form a *separatrix*. Solutions on either side of the separatrix approach either $(0\,,4)$ or $(1.5\,,0)$.

$4(a)$.



(b). The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,x - y) = 0$$
$$y(0.75 - y - 0.125\,x) = 0\,.$$

The four critical points are $(0\,,0)$, $(0\,,3/4)$, $(3\,,0)$ and $(2\,,1/2)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - x - y & -\,x \\ -\,y/8 & 3/4 - x/8 - 2y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \ r_2 = 3/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.

At the critical point $(0\,,3/4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,3/4) = \begin{pmatrix} 3/4 & 0 \\ -\,3/32 & -\,3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/4\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -\,16 \\ 1 \end{pmatrix};\ \ r_2 = -\,3/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0\,,3/4)$ is a *saddle*, which is *unstable*.

At the critical point $(3\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3\,,0) = \begin{pmatrix} -\,3/2 & -\,3 \\ 0 & 3/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \,-\,3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,;\ \ r_2 = 3/8\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -\,8 \\ 5 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0\,,3/4)$ is a *saddle*, which is *unstable*.

At the critical point $(2\,,1/2)$, the coefficient matrix of the linearized system is

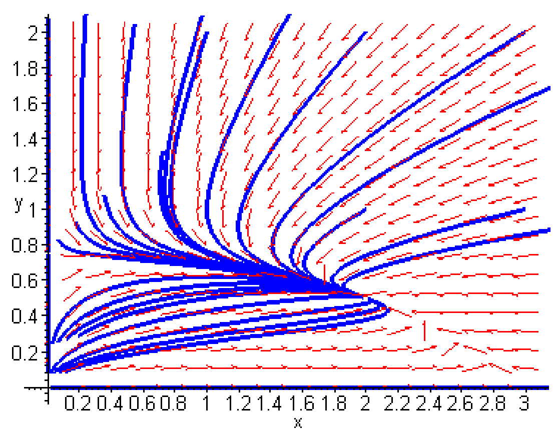
$$\mathbf{J}(2\,,1/2) = \begin{pmatrix} -\,1 & -\,2 \\ -\,1/16 & -\,1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-\,3+\sqrt{3}}{4}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\,\frac{1+\sqrt{3}}{8} \end{pmatrix}\,;\ r_2 = -\,\frac{3+\sqrt{3}}{4}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ \frac{-1+\sqrt{3}}{8} \end{pmatrix}.$$

The eigenvalues are negative, hence the critical point $(2\,,1/2)$ is a *stable node*, which is *asymptotically stable*.

(d, e).



(f). Except for initial conditions along the coordinate axes, almost all solutions converge
to the stable node $(2\,,1/2)$.

7. It follows immediately that

$$\begin{aligned}(\sigma_1 X+\sigma_2 Y)^2-4\sigma_1\sigma_2 XY &= \sigma_1^2X^2+2\sigma_1\sigma_2XY+\sigma_2^2Y^2-4\sigma_1\sigma_2XY\\ &=(\sigma_1X-\sigma_2Y)^2.\end{aligned}$$

Since all parameters and variables are *positive*, it follows that

$$(\sigma_1 X+\sigma_2 Y)^2-4(\sigma_1\sigma_2-\alpha_1\alpha_2)XY\geq 0\,.$$

Hence the radicand in Eq.(39) is *nonnegative*.

$10(a)$. The critical points consist of the solution set of the equations

$$\begin{aligned}x(\epsilon_1-\sigma_1x-\alpha_1y)&=0\\ y(\epsilon_2-\sigma_2y-\alpha_2x)&=0\,.\end{aligned}$$

If $x=0$, then either $y=0$ or $y=\epsilon_2/\sigma_2$. If $\epsilon_1-\sigma_1x-\alpha_1y=0$, then solving for x results in $x=(\epsilon_1-\alpha_1y)/\sigma_1$. Substitution into the *second* equation yields

$$(\sigma_1\sigma_2-\alpha_1\alpha_2)y^2-(\sigma_1\epsilon_2-\epsilon_1\alpha_2)y=0\,.$$

Based on the hypothesis, it follows that $(\sigma_1\epsilon_2-\epsilon_1\alpha_2)y=0$. So if $\sigma_1\epsilon_2-\epsilon_1\alpha_2\neq 0$, then $y=0$, and the critical points are located at $(0\,,0)$, $(0\,,\epsilon_2/\sigma_2)$ and $(\epsilon_1/\sigma_1,0)$.

For the case $\sigma_1\epsilon_2-\epsilon_1\alpha_2=0$, y can be arbitrary. From the relation $x=(\epsilon_1-\alpha_1y)/\sigma_1$, we conclude that all points on the line $\sigma_1x+\alpha_1y=\epsilon_1$ are critical points, in addition to the point $(0\,,0)$.

(b). The Jacobian matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix}\epsilon_1-2\sigma_1x-\alpha_1y & -\alpha_1x\\ -\alpha_2y & \epsilon_2-2\sigma_2y-\alpha_2x\end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix}\epsilon_1 & 0\\ 0 & \epsilon_2\end{pmatrix},$$

with eigenvalues $r_1=\epsilon_1$ and $r_2=\epsilon_2$. Since both eigenvalues are *positive*, the origin is an *unstable node*.

At the point $(0\,,\epsilon_2/\sigma_2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,\epsilon_2/\sigma_2)=\begin{pmatrix}(\epsilon_1\alpha_2-\sigma_1\epsilon_2)/\alpha_2 & 0\\ \epsilon_2\alpha_2/\sigma_2 & -\epsilon_2\end{pmatrix},$$

with eigenvalues $r_1=(\epsilon_1\alpha_2-\sigma_1\epsilon_2)/\alpha_2$ and $r_2=-\epsilon_2$. If $\sigma_1\epsilon_2-\epsilon_1\alpha_2>0$, then both eigenvalues are *negative*. Hence the point $(0\,,\epsilon_2/\sigma_2)$ is a *stable node*, which is *asymptotically stable*. If $\sigma_1\epsilon_2-\epsilon_1\alpha_2<0$, then the eigenvalues are of opposite sign. Hence the point $(0\,,\epsilon_2/\sigma_2)$ is a *saddle*, which is *unstable*.

At the point $(\epsilon_1/\sigma_1, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\epsilon_1/\sigma_1, 0) = \begin{pmatrix} -\epsilon_1 & -\epsilon_1\alpha_1/\sigma_1 \\ 0 & (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1 \end{pmatrix},$$

with eigenvalues $r_1 = (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1$ and $r_2 = -\epsilon_1$. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 > 0$, then the eigenvalues are of *opposite* sign. Hence the point $(\epsilon_1/\sigma_1, 0)$ is a *saddle*, which is *unstable.* If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 < 0$, then both eigenvalues are *negative*. In that case the point $(\epsilon_1/\sigma_1, 0)$ is a *stable node*, which is *asymptotically stable.*

(c). As shown in Part (a), when $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 = 0$, the set of critical points consists of $(0\,,0)$ and all of the points on the straight line $\sigma_1 x + \alpha_1 y = \epsilon_1$. Based on Part (b), the origin is still an *unstable node*. Setting $y = (\epsilon_1 - \sigma_1 x)/\alpha_1$, the Jacobian matrix of the vector field, *along the given straight line*, is

$$\mathbf{J} = \begin{pmatrix} -\sigma_1 x & -\alpha_1 x \\ -\alpha_2(\epsilon_1 - \sigma_1 x)/\alpha_1 & \alpha_2 x - \epsilon_1\alpha_2/\sigma_1 \end{pmatrix}.$$

The characteristic equation of the matrix is

$$r^2 + \left[\frac{\epsilon_1\alpha_2 - \alpha_2\sigma_1 x + \sigma_1^2 x}{\sigma_1}\right] r = 0\,.$$

Using the given hypothesis, $(\epsilon_1\alpha_2 - \alpha_2\sigma_1 x + \sigma_1^2 x)/\sigma_1 = \epsilon_2 - \alpha_2 x + \sigma_1 x$. Hence the characteristic equation can be written as

$$r^2 + [\epsilon_2 - \alpha_2 x + \sigma_1 x] r = 0\,.$$

First note that $0 \le x \le \epsilon_1/\sigma_1$. Since the coefficient in the quadratic equation is *linear*, and

$$\epsilon_2 - \alpha_2 x + \sigma_1 x = \begin{cases} \epsilon_2 & \text{at } x = 0 \\ \epsilon_1 & \text{at } x = \epsilon_1/\sigma_1 \,, \end{cases}$$

it follows that the coefficient is *positive* for $0 \le x \le \epsilon_1/\sigma_1$. Therefore, along the straight line $\sigma_1 x + \alpha_1 y = \epsilon_1$, one eigenvalue is *zero* and the other one is *negative*. Hence the continuum of critical points consists of *stable nodes*, which are *asymptotically stable*.

$11(a)$. The critical points are solutions of the system of equations

$$\begin{aligned} x(1 - x - y) + \delta a &= 0 \\ y(0.75 - y - 0.5\,x) + \delta b &= 0\,. \end{aligned}$$

Assume solutions of the form

$$\begin{aligned} x &= x_0 + x_1\delta + x_2\delta^2 + \cdots \\ y &= y_0 + y_1\delta + y_2\delta^2 + \cdots . \end{aligned}$$

Substitution of the series expansions results in

$$x_0(1 - x_0 - y_0) + (x_1 - 2x_1x_0 - x_0y_1 - x_1y_0 + a)\delta + \cdots = 0$$
$$y_0(0.75 - y_0 - 0.5\,x_0\,) + (0.75\,y_1 - 2y_0y_1 - x_1y_0/2 - x_0y_1/2 + b)\delta + \cdots = 0\,.$$

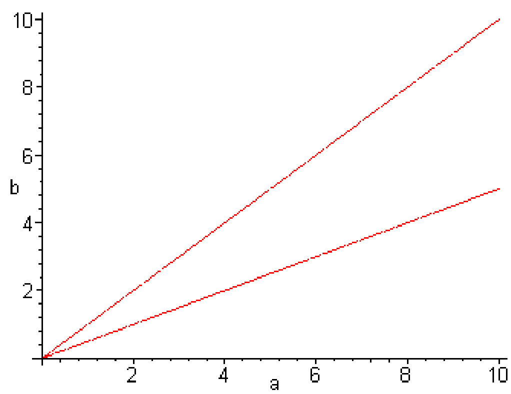
(b). Taking a limit as $\delta \to 0$, the equations reduce to the original system of equations. It follows that $x_0 = y_0 = 0.5$.

(c). Setting the coefficients of the *linear* terms equal to zero, we find that

$$-\,y_1/2 - x_1/2 + a = 0$$
$$-\,x_1/4 - y_1/2 + b = 0\,,$$

with solution $x_1 = 4a - 4b$ and $y_1 = -\,2a + 4b$.

(d). Consider the ab - parameter space. The collection of points for which $b < a$ represents an *increase* in the level of species 1. At points where $b > a$, $x_1\delta < 0$. Likewise, the collection of points for which $2b > a$ represents an *increase* in the level of species 2. At points where $2b < a$, $y_1\delta < 0$.



It follows that if $b < a < 2b$, the level of *both* species will *increase*. This condition is represented by the wedge-shaped region on the graph. Otherwise, the level of one species
will increase, whereas the level of the other species will simultaneously decrease. Only for $a = b = 0$ will both populations remain the same.

13(a). The critical points consist of the solution set of the equations

$$-\,y = 0$$
$$-\,\gamma\,y - x(x - 0.15)(x - 2) = 0\,.$$

Setting $y = 0$, the second equation becomes $x(x - 0.15)(x - 2) = 0$, with roots $x = 0$, 0.15 and 2. Hence the critical points are located at $(0\,,0)$, $(0.15\,,0)$ and $(2\,,0)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ -3x^2 + 4.3\,x - 0.3 & -\gamma \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & -1 \\ -0.3 & -\gamma \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{10}\sqrt{25\,\gamma^2 + 30}\,.$$

Regardless of the value of γ, the eigenvalues are real and of opposite sign. Hence $(0\,,0)$ is a *saddle*, which is *unstable*.

At the critical point $(0.15\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0.15\,,0) = \begin{pmatrix} 0 & -1 \\ 0.2775 & -\gamma \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{20}\sqrt{100\,\gamma^2 - 111}\,.$$

If $100\,\gamma^2 - 111 \geq 0$, then the eigenvalues are real. Furthermore, since $r_1 r_2 = 0.2775$, both eigenvalues will have the same sign. Therefore the critical point is a *node*, with its stability dependent on the *sign* of γ. If $100\,\gamma^2 - 111 < 0$, the eigenvalues are complex conjugates. In that case the critical point $(0.15\,,0)$ is a *spiral*, with its stability dependent on the *sign* of γ.

At the critical point $(2\,,0)$, the coefficient matrix of the linearized system is

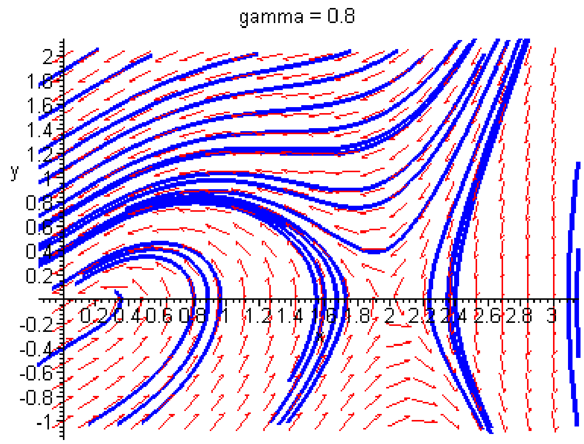
$$\mathbf{J}(2\,,0) = \begin{pmatrix} 0 & -1 \\ -3.7 & -\gamma \end{pmatrix},$$

with eigenvalues

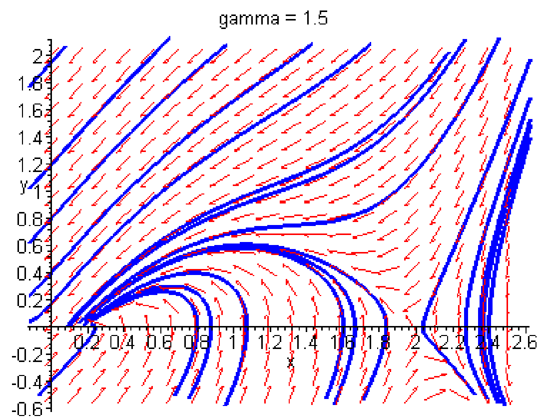
$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{10}\sqrt{25\,\gamma^2 + 370}\,.$$

Regardless of the value of γ, the eigenvalues are real and of opposite sign. Hence $(2\,,0)$ is a *saddle*, which is *unstable*.

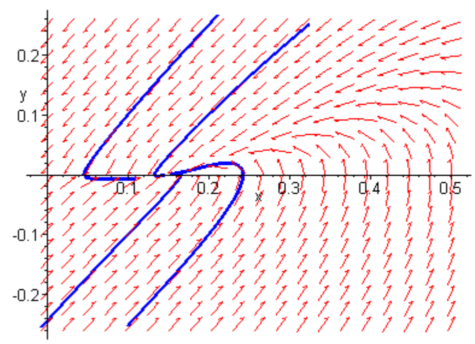
(b).



It is evident that for $\gamma = 0.8$, the critical point $(0.15\,, 0)$ is a *stable spiral*.



Closer examination shows that for $\gamma = 1.5$, the critical point $(0.15\,, 0)$ is a *stable node*.



(c). Based on the phase portraits in Part (b), it is apparent that the required value of γ satisfies $0.8 < \gamma < 1.5$. Using the initial condition $x(0) = 2$ and $y(0) = 0.01$, it is possible to solve the initial value problem for various values of γ. A reasonable first guess is $\gamma = \sqrt{1.11}$. This value marks the change in qualitative behavior of the critical point $(0.15\,,0)$. Numerical experiments show that the solution remains positive for $\gamma \approx 1.20\,.$