Section 10.2

1. The period of the function $sin\,\alpha x$ is $T = 2\pi/\alpha$. Therefore the function $sin\,5x$ has period $T = 2\pi/5$.

2. The period of the function $cos\,\alpha x$ is also $T = 2\pi/\alpha$. Therefore the function cos $2\pi x$
has period $T = 2\pi/2\pi = 1$.

4. Based on Prob. 1, the period of the function $sin\,\pi x/L$ is $T = 2\pi/(\pi/L) = 2L$.

6. Let $T > 0$ and consider the equation $(x+T)^2 = x^2$. It follows that $2Tx + T^2 = 0$ and $2x + T = 0$. Since the latter equation is *not* an identity, the function x^2 cannot be periodic with finite period.

8. The function is defined on intervals of length $(2n+1) - (2n-1) = 2$. On any two *consecutive* intervals, $f(x)$ is identically equal to 1 on one of the intervals and alternates between 1 and -1 on the other. It follows that the period is $T = 4$.

9. On the interval $L < x < 2L$, a simple *shift to the right* results in

$$f(x) = -(x - 2L) = 2L - x\,.$$

On the interval $-3L < x < -2L$, a simple *shift to the left* results in

$$f(x) = -(x + 2L) = -2L - x\,.$$

11. The next fundamental period *to the left* is on the interval $-2L < x < 0$. Hence the interval $-L < x < 0$ is the second half of a fundamental period. A simple *shift to the left* results in

$$f(x) = L - (x + 2L) = -L - x\,.$$

12. First note that

$$cos\frac{m\pi x}{L}cos\frac{n\pi x}{L} = \frac{1}{2}\left[cos\frac{(m-n)\pi x}{L} + cos\frac{(m+n)\pi x}{L}\right]$$

and

$$cos\frac{m\pi x}{L}sin\frac{n\pi x}{L} = \frac{1}{2}\left[sin\frac{(n-m)\pi x}{L} + sin\frac{(m+n)\pi x}{L}\right].$$

It follows that

$$\begin{aligned}\int_{-L}^{L} cos\frac{m\pi x}{L} cos\frac{n\pi x}{L} dx &= \frac{1}{2}\int_{-L}^{L}\left[cos\frac{(m-n)\pi x}{L} + cos\frac{(m+n)\pi x}{L}\right]dx \\ &= \frac{1}{2}\frac{L}{\pi}\left\{\frac{sin[(m-n)\pi x/L]}{m-n} + \frac{sin[(m+n)\pi x/L]}{m+n}\right\}\Bigg|_{-L}^{L} \\ &= 0\,,\end{aligned}$$

as long as $m+n$ and $m-n$ are not zero. For the case $m=n$,

$$\begin{aligned}\int_{-L}^{L}\Big(cos\frac{n\pi x}{L}\Big)^2 dx &= \frac{1}{2}\int_{-L}^{L}\left[1 + cos\frac{2n\pi x}{L}\right]dx \\ &= \frac{1}{2}\left\{x + \frac{sin(2n\pi x/L)}{2n\pi/L}\right\}\Bigg|_{-L}^{L} \\ &= L\,.\end{aligned}$$

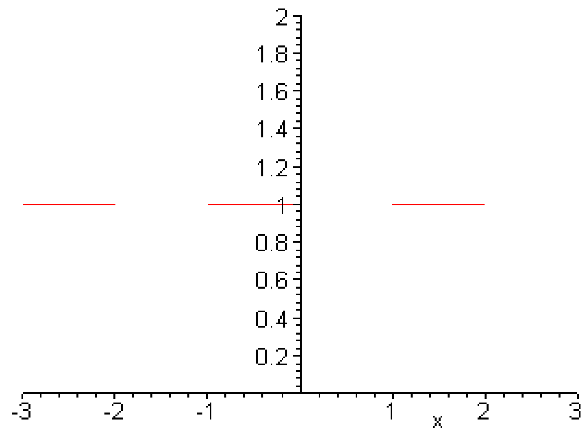
Likewise,

$$\begin{aligned}\int_{-L}^{L} cos\frac{m\pi x}{L} sin\frac{n\pi x}{L} dx &= \frac{1}{2}\int_{-L}^{L}\left[sin\frac{(n-m)\pi x}{L} + sin\frac{(m+n)\pi x}{L}\right]dx \\ &= \frac{1}{2}\frac{L}{\pi}\left\{\frac{cos[(n-m)\pi x/L]}{m-n} - \frac{cos[(m+n)\pi x/L]}{m+n}\right\}\Bigg|_{-L}^{L} \\ &= 0\,,\end{aligned}$$

as long as $m+n$ and $m-n$ are not zero. For the case $m=n$,

$$\begin{aligned}\int_{-L}^{L} cos\frac{m\pi x}{L} sin\frac{n\pi x}{L} dx &= \frac{1}{2}\int_{-L}^{L} sin\frac{2n\pi x}{L} dx \\ &= -\frac{1}{2}\left\{\frac{cos(2n\pi x/L)}{2n\pi/L}\right\}\Bigg|_{-L}^{L} \\ &= 0\,.\end{aligned}$$

$14(a)$. For $L = 1$,



(b). The Fourier coefficients are calculated using the *Euler-Fourier* formulas:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{L}\int_{-L}^{0} dx \\ &= 1\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0} cos\frac{n\pi x}{L}dx \\ &= 0\,. \end{aligned}$$

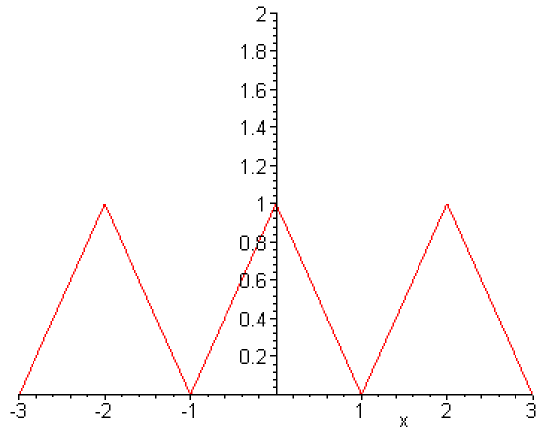
Likewise,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0} sin\frac{n\pi x}{L}dx \\ &= \frac{-1+(-1)^n}{n\pi}\,. \end{aligned}$$

It follows that $b_{2k} = 0$ and $b_{2k-1} = -2/[(2k-1)\pi]$, $k = 1, 2, 3, \cdots$. Therefore the Fourier series for the given function is

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1} sin\frac{(2k-1)\pi x}{L}\,.$$

$16(a)$.



(b). The Fourier coefficients are calculated using the *Euler-Fourier* formulas:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \int_{-1}^{0}(x+1)dx + \int_{0}^{1}(1-x)dx \\ &= 1\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \int_{-1}^{0}(x+1)cos\, n\pi x\, dx + \int_{0}^{1}(1-x)cos\, n\pi x\, dx \\ &= -2\frac{-1+(-1)^n}{n^2\pi^2}\,. \end{aligned}$$

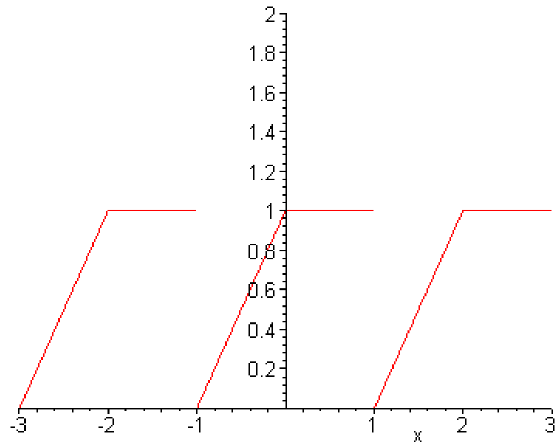
It follows that $a_{2k} = 0$ and $a_{2k-1} = 4/\left[(2k-1)^2\pi^2\right]$, $k = 1, 2, 3, \cdots$. Likewise,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \int_{-1}^{0}(x+1)sin\, n\pi x\, dx + \int_{0}^{1}(1-x)sin\, n\pi x\, dx \\ &= 0\,. \end{aligned}$$

Therefore the Fourier series for the given function is

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\,cos(2k-1)\pi x\,.$$

$17(a)$. For $L = 1$,



(b). The Fourier coefficients are calculated using the *Euler-Fourier* formulas:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{L}\int_{-L}^{0}(x+L)dx + \frac{1}{L}\int_{0}^{L} L\,dx \\ &= 3L/2\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0}(x+L)cos\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} L\,cos\frac{n\pi x}{L}dx \\ &= \frac{L(1-cos\,n\pi)}{n^2\pi^2}\,. \end{aligned}$$

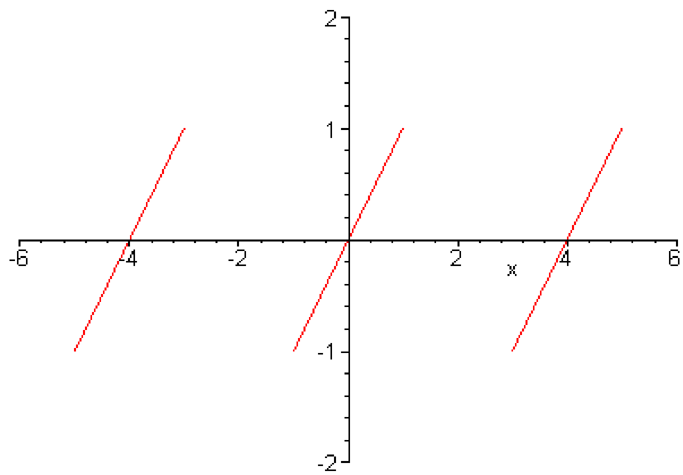
Likewise,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0}(x+L)sin\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} L\,sin\frac{n\pi x}{L}dx \\ &= -\frac{L\,cos\,n\pi}{n\pi}\,. \end{aligned}$$

Note that $cos\,n\pi = (-1)^n$. It follows that the Fourier series for the given function is

$$f(x) = \frac{3L}{4} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)^2}cos\frac{(2n-1)\pi x}{L} - \frac{(-1)^n\pi}{n}sin\frac{n\pi x}{L}\right].$$

$18(a)$.



(b). The Fourier coefficients are calculated using the *Euler-Fourier* formulas:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2}\int_{-1}^{1} x\,dx \\ &= 0\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-1}^{1} x\,cos\frac{n\pi x}{L}dx \\ &= 0\,. \end{aligned}$$

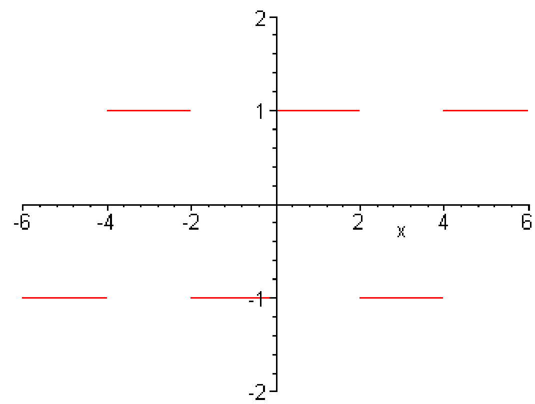
Likewise,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-1}^{1} x\,sin\frac{n\pi x}{L}dx \\ &= \frac{2}{n^2\pi^2}\left(2\,sin\frac{n\pi}{2} - n\pi\,cos\frac{n\pi}{2}\right). \end{aligned}$$

Therefore the Fourier series for the given function is

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{4}{n^2\pi^2} \, sin \frac{n\pi}{2} - \frac{2}{n\pi} \, cos \frac{n\pi}{2} \right] sin \frac{n\pi x}{2} \, .$$

$19(a)$.



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos \frac{n\pi x}{L} dx \\ &= \frac{1}{2}\int_{-2}^{0} - cos \frac{n\pi x}{2} dx + \frac{1}{2}\int_{0}^{2} cos \frac{n\pi x}{2} dx \\ &= 0 \, . \end{aligned}$$

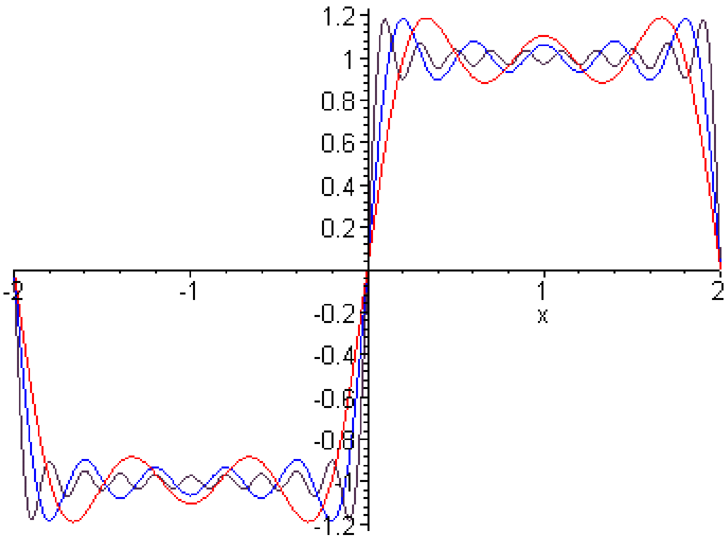
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin \frac{n\pi x}{L} dx \\ &= \frac{1}{2}\int_{-2}^{0} - sin \frac{n\pi x}{2} dx + \frac{1}{2}\int_{0}^{2} sin \frac{n\pi x}{2} dx \\ &= 2\frac{1 - cos\, n\pi}{n\pi} \, . \end{aligned}$$

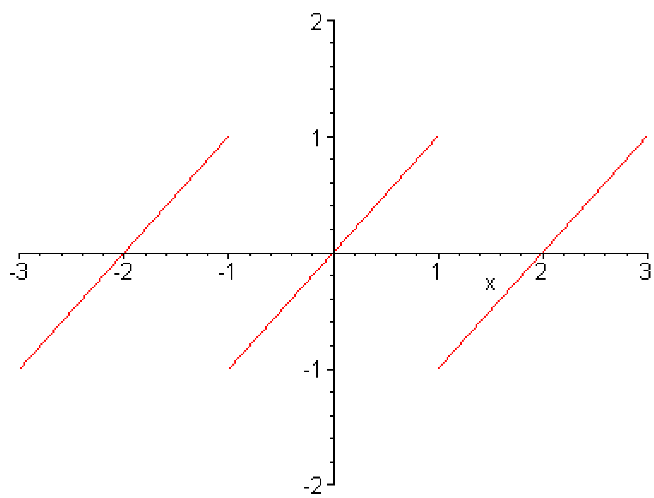
Therefore the Fourier series for the given function is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} sin \frac{(2n-1)\pi x}{2} \, .$$

(c).



$20(a)$.



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos\frac{n\pi x}{L} dx \\ &= \int_{-1}^{1} x\, cos\, n\pi x dx \\ &= 0\,. \end{aligned}$$

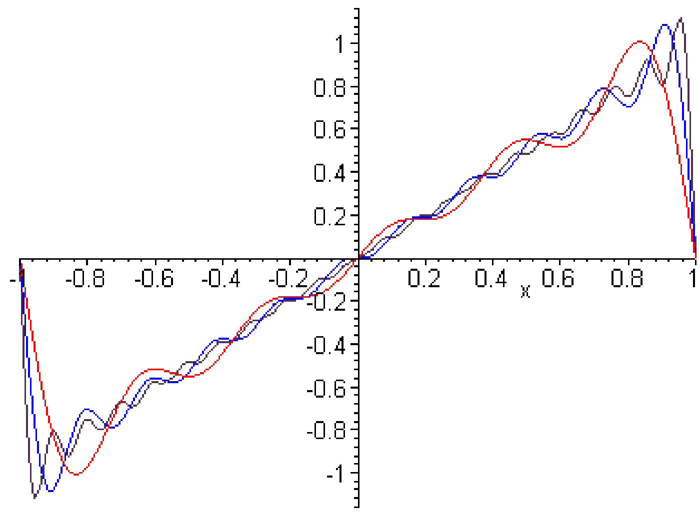
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin \frac{n\pi x}{L} dx \\ &= \int_{-1}^{1} x\, sin\, n\pi x dx \\ &= -2\frac{cos\, n\pi}{n\pi}\,. \end{aligned}$$

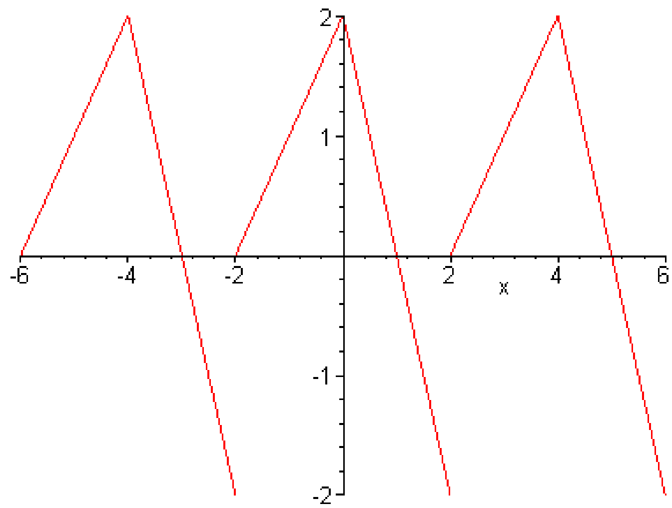
Therefore the Fourier series for the given function is

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} sin\, n\pi x\,.$$

(c).



$22(a)$.



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2}\int_{-2}^{0}(x+2)dx + \frac{1}{2}\int_{0}^{2}(2-2x)dx \\ &= 1\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-2}^{0}(x+2)cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2}(2-2x)cos\,\frac{n\pi x}{2}dx \\ &= 6\frac{(1-cos\,n\pi)}{n^2\pi^2}\,. \end{aligned}$$

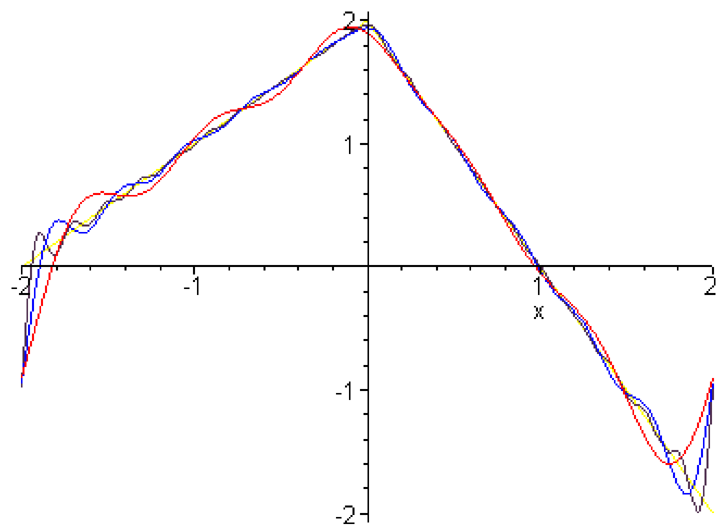
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-2}^{0}(x+2)sin\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2}(2-2x)sin\,\frac{n\pi x}{2}dx \\ &= 2\frac{cos\,n\pi}{n\pi}\,. \end{aligned}$$

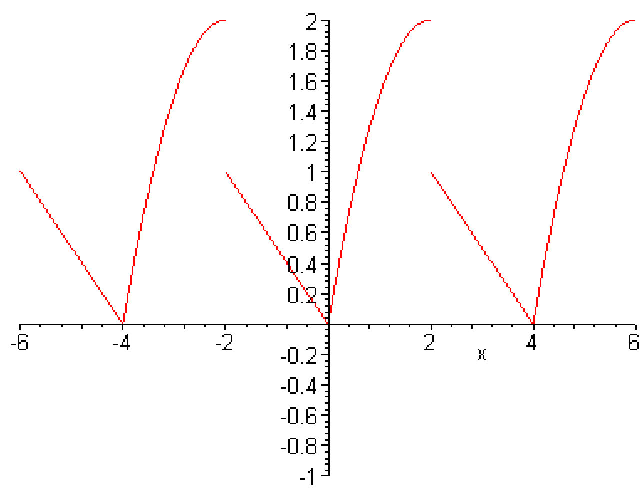
Therefore the Fourier series for the given function is

$$f(x) = \frac{1}{2} + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,cos\frac{(2n-1)\pi x}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}sin\frac{n\pi x}{2}\,.$$

(c).



$23(a)$.



(b). The Fourier *cosine* coefficients are given by

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2}\int_{-2}^{0}\Big(-\frac{x}{2}\Big)dx + \frac{1}{2}\int_{0}^{2}\Big(2x-\frac{1}{2}x^2\Big)dx \\
&= 11/6\,,
\end{aligned}
$$

and for $n > 0$,

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\
&= \frac{1}{2}\int_{-2}^{0}\Big(-\frac{x}{2}\Big)cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2}\Big(2x-\frac{1}{2}x^2\Big)cos\,\frac{n\pi x}{2}dx \\
&= -\,\frac{(5-cos\,n\pi)}{n^2\pi^2}\,.
\end{aligned}
$$

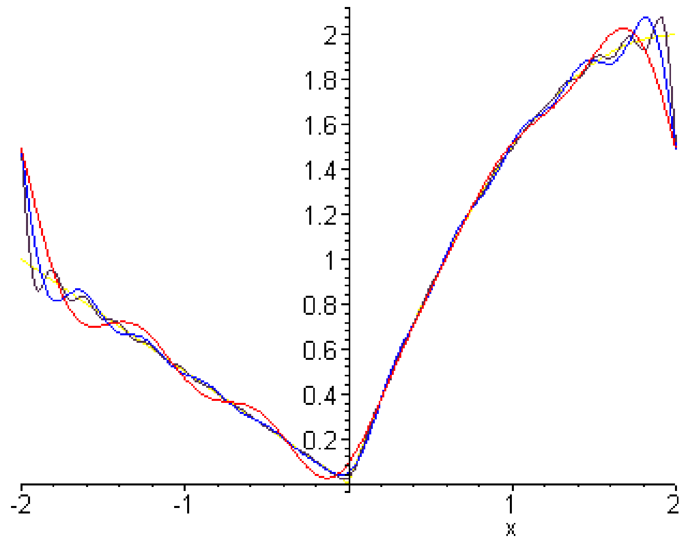
The Fourier *sine* coefficients are given by

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\
&= \frac{1}{2}\int_{-2}^{0}\Big(-\frac{x}{2}\Big)sin\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2}\Big(2x-\frac{1}{2}x^2\Big)sin\,\frac{n\pi x}{2}dx \\
&= \frac{4-(4+n^2\pi^2)cos\,n\pi}{n^3\pi^3}\,.
\end{aligned}
$$

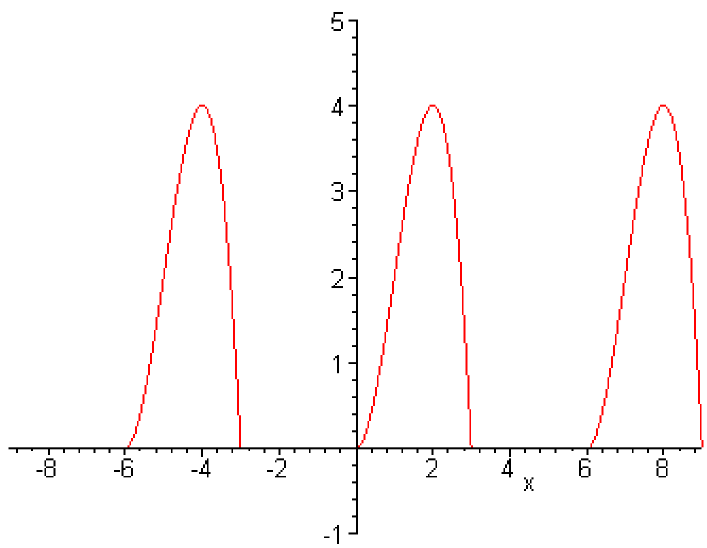
Therefore the Fourier series for the given function is

$$
\begin{aligned}
f(x) &= \frac{11}{12} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n-5]}{n^2}\,cos\frac{n\pi x}{2}\; + \\
&\quad + \frac{1}{\pi^3}\sum_{n=1}^{\infty}\frac{[4-(4+n^2\pi^2)(-1)^n]}{n^3}sin\frac{n\pi x}{2}\,.
\end{aligned}
$$

(c).



$24(a)$.



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{3}\int_0^3 x^2(3-x)dx \\ &= 9/4\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos \frac{n\pi x}{L} dx \\ &= \frac{1}{3}\int_0^3 x^2(3-x) cos\, \frac{n\pi x}{3} dx \\ &= -27\frac{(6-6\,cos\,n\pi + n^2\pi^2 cos\,n\pi)}{n^4\pi^4}\,. \end{aligned}$$

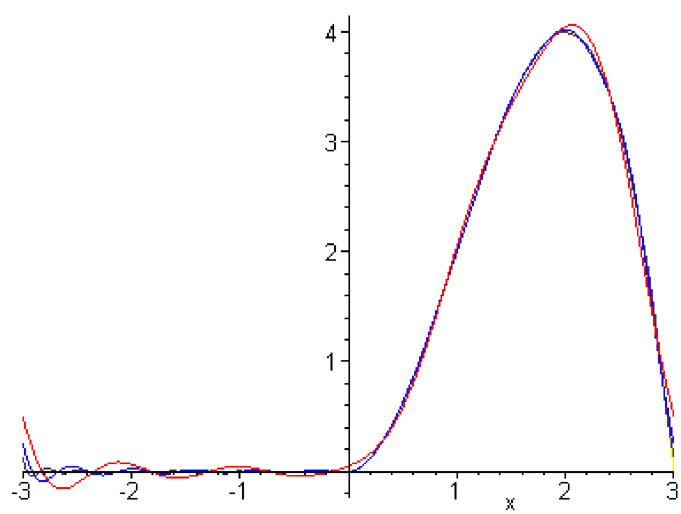
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin \frac{n\pi x}{L} dx \\ &= \frac{1}{3}\int_0^3 x^2(3-x) sin\, \frac{n\pi x}{3} dx \\ &= -54\frac{1+2\,cos\,n\pi}{n^3\pi^3}\,. \end{aligned}$$

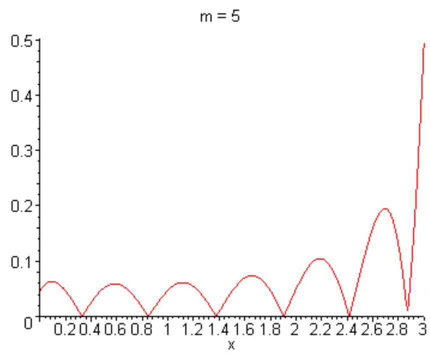
Therefore the Fourier series for the given function is

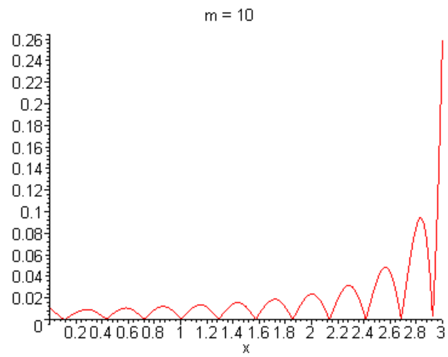
$$\begin{aligned} f(x) &= \frac{9}{8} - 27\sum_{n=1}^{\infty}\left[\frac{6[1-(-1)^n]}{n^4\pi^4} + \frac{(-1)^n}{n^2\pi^2}\right] cos \frac{n\pi x}{3} - \\ &- \frac{54}{\pi^3}\sum_{n=1}^{\infty}\frac{[1+2(-1)^n]}{n^3} sin \frac{n\pi x}{3}\,. \end{aligned}$$

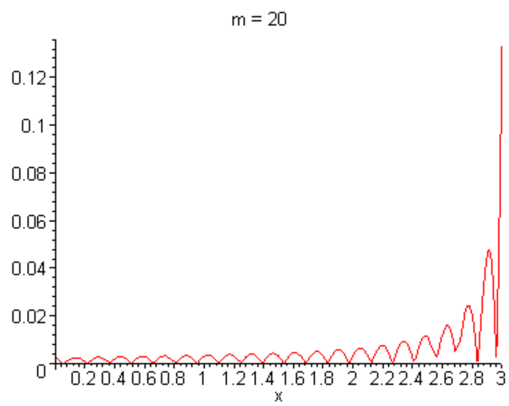
(c).



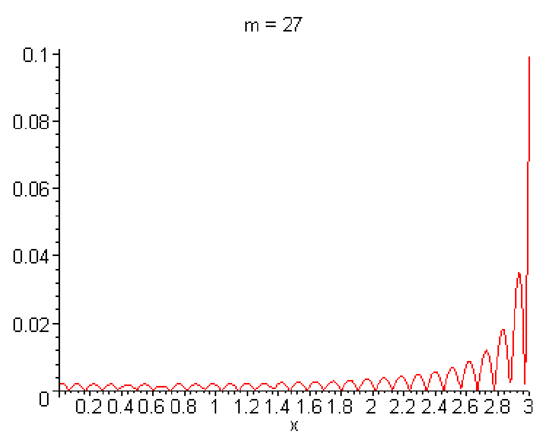
26.



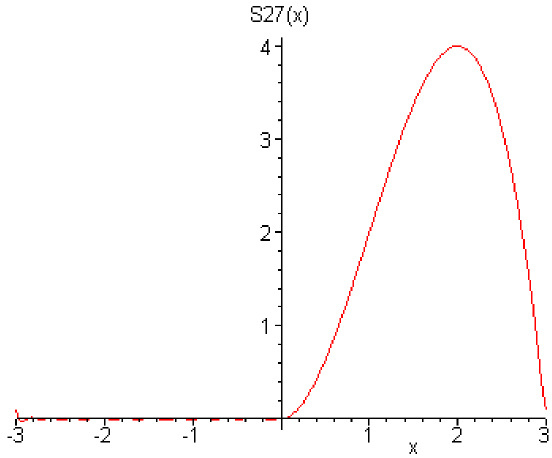




It is evident that $|e_m(x)|$ is greatest at $x = \pm 3$. Increasing the number of terms in the partials sums, we find that if $m \geq 27$, then $|e_m(x)| \leq 0.1$, for all $x \in [-3, 3]$.



Graphing the partial sum $s_{27}(x)$, the convergence is as predicted:



28. Let $x = T + a$, for some $a \in [0, T]$. First note that for any value of h,

$$\begin{aligned} f(x+h) - f(x) &= f(T+a+h) - f(T+a) \\ &= f(a+h) - f(a) \,. \end{aligned}$$

Since f is *differentiable*,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= f'(a) \,. \end{aligned}$$

That is, $f'(a+T) = f'(a)$. By induction, it follows that $f'(a+T) = f'(a)$ for every value of a.

On the other hand, if $f(x) = 1 + cos\, x$, then the function

$$\begin{aligned} F(x) &= \int_0^x [1 + cos\, t] dt \\ &= x + sin\, x \end{aligned}$$

is *not* periodic, unless its definition is restricted to a specific interval.

29(a). Based on the hypothesis, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are a basis for $\mathbb{R}^3$. Given any vector $\mathbf{u} \in \mathbb{R}^3$, it can be expressed as a linear combination $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$. Taking the inner product of both sides of this equation with $\mathbf{v}_i$, we have

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v}_i &= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \cdot \mathbf{v}_i \\ &= a_i\, \mathbf{v}_i \cdot \mathbf{v}_i \,, \end{aligned}$$

since the basis vectors are mutually orthogonal. Hence

$$a_i = \frac{\mathbf{u} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\,,\ i = 1, 2, 3\,.$$

Recall that $\mathbf{u} \cdot \mathbf{v}_i = u\, v_i \cos\theta$, in which θ is the angle between $\mathbf{u}$ and $\mathbf{v}_i$. Therefore

$$a_i = \frac{u \cos\theta}{v_i}\,.$$

Here $u \cos\theta$ is interpreted as the magnitude of the projection of $\mathbf{u}$ in the direction of $\mathbf{v}_i$.

(b). Assuming that a Fourier series converges to a periodic function, $f(x)$,

$$f(x) = \frac{a_0}{2}\phi_0(x) + \sum_{m=1}^{\infty} a_m \phi_m(x) + \sum_{m=1}^{\infty} b_m \psi_m(x)\,.$$

Taking the inner product, defined by

$$(u\,, v) = \int_{-L}^{L} u(x) v(x) dx\,,$$

of both sides of the series expansion with the specified trigonometric functions, we have

$$(f\,, \phi_n) = \frac{a_0}{2}(\phi_0\,, \phi_n) + \sum_{m=1}^{\infty} a_m (\phi_m\,, \phi_n) + \sum_{m=1}^{\infty} b_m (\psi_m\,, \phi_n)$$

for $n = 0, 1, 2, \cdots$.

(c). It also follows that

$$(f\,, \psi_n) = \frac{a_0}{2}(\phi_0\,, \psi_n) + \sum_{m=1}^{\infty} a_m (\phi_m\,, \psi_n) + \sum_{m=1}^{\infty} b_m (\psi_m\,, \psi_n)$$

for $n = 1, 2, \cdots$. Based on the orthogonality conditions,

$$(\phi_m\,, \phi_n) = L\,\delta_{mn}\ ,\ (\psi_m\,, \psi_n) = L\,\delta_{mn}\ ,$$

and $(\psi_m\,, \phi_n) = L\,\delta_{mn}$. Note that $(\phi_0\,, \phi_0) = 2L$. Therefore

$$a_0 = \frac{2(f\,, \phi_0)}{(\phi_0\,, \phi_0)} = \frac{1}{L}\int_{-L}^{L} f(x)\phi_0(x) dx$$

and

$$a_n = \frac{(f\,, \phi_n)}{(\phi_n\,, \phi_n)} = \frac{1}{L}\int_{-L}^{L} f(x)\phi_n(x) dx\,, \quad n = 1, 2, \cdots.$$

Likewise,

$$b_n = \frac{(f\,,\psi_n)}{(\psi_n\,,\psi_n)} = \frac{1}{L}\int_{-L}^{L} f(x)\psi_n(x)dx\,, \quad n = 1, 2, \cdots .$$