Section 10.3

$1(a)$. The given function is assumed to be periodic with $2L = 2$. The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \int_{-1}^{0}(-1)dx + \int_{0}^{1}(1)dx \\ &= 0\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= -\int_{-1}^{0} cos\, n\pi x\, dx + \int_{0}^{1} cos\; n\pi x\, dx \\ &= 0\,. \end{aligned}$$

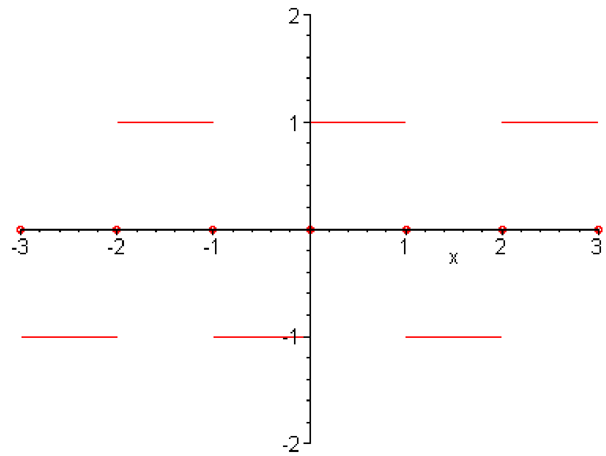
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= -\int_{-1}^{0} sin\; n\pi x\, dx + \int_{0}^{1} sin\; n\pi x\, dx \\ &= 2\frac{1 - \,cos\, n\pi}{n\pi}\,. \end{aligned}$$

Therefore the Fourier series for the specified function is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\, sin\,(2n-1)\pi x\,.$$

(b).



The function is piecewise continuous on each finite interval. The points of discontinuity are at *integer* values of x. At these points, the series converges to

$$|f(x-)+f(x+)|=0\,.$$

$3(a)$. The given function is assumed to be periodic with $T=2L$. The Fourier *cosine* coefficients are given by

$$\begin{aligned}a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx\\ &= \frac{1}{L}\int_{-L}^{0}(L+x)dx+\frac{1}{L}\int_{0}^{L}(L-x)dx\\ &= L\,,\end{aligned}$$

and for $n>0$,

$$\begin{aligned}a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx\\ &= \frac{1}{L}\int_{-L}^{0}(L+x)cos\frac{n\pi x}{L}\,dx+\frac{1}{L}\int_{0}^{L}(L-x)cos\frac{n\pi x}{L}\,dx\\ &= 2L\,\frac{1-\,cos\,n\pi}{n^2\pi^2}\,.\end{aligned}$$

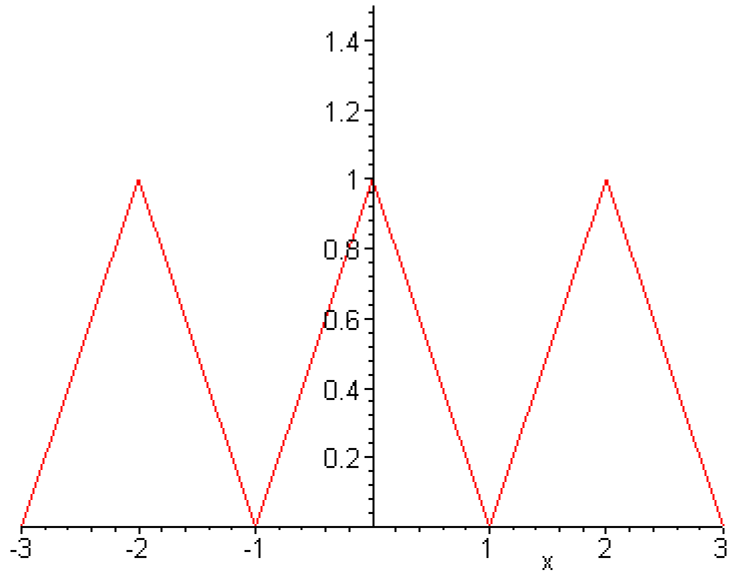
The Fourier *sine* coefficients are given by

$$\begin{aligned}b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx\\ &= \frac{1}{L}\int_{-L}^{0}(L+x)sin\frac{n\pi x}{L}\,dx+\frac{1}{L}\int_{0}^{L}(L-x)sin\frac{n\pi x}{L}\,dx\\ &= 0\,.\end{aligned}$$

Therefore the Fourier series of the specified function is

$$f(x) = \frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\, cos\frac{(2n-1)\pi x}{L}\,.$$

(b). For $L = 1$,



Note that $f(x)$ is *continuous*. Based on Theorem 10.3.1 , the series converges to the continuous function $f(x)$.

$5(a)$. The given function is assumed to be periodic with $2L = 2\pi$. The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (1)dx \\ &= 1\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos\frac{n\pi x}{L}dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (1) cos\; nx\, dx \\ &= \frac{2}{n\pi}\, sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin\frac{n\pi x}{L}dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1) sin\ nx\, dx \\ &= 0\,. \end{aligned}$$

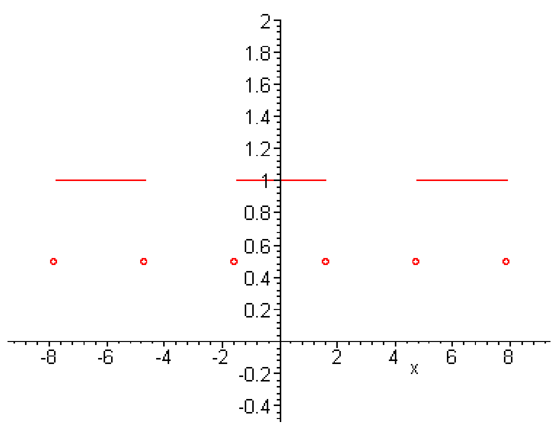
Observe that

$$sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0\,, & n = 2k \\ (-1)^{k+1}\,, & n = 2k-1 \end{cases} \quad , k = 1, 2, \cdots.$$

Therefore the Fourier series of the specified function is

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} cos\,(2n-1)x\,.$$

(b).



The given function is piecewise continuous, with discontinuities at *odd* multiples of $\pi/2$. At $x_d = (2k-1)\pi/2$, $k = 0, 1, 2, \cdots$, the series converges to

$$|f(x_d -) + f(x_d +)| = 1/2\,.$$

$6(a)$. The given function is assumed to be periodic with $2L = 2$. The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \int_0^1 x^2 dx \\ &= 1/3\,, \end{aligned}$$

and for $n > 0$,

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos \frac{n\pi x}{L} dx \\
&= \int_0^1 x^2 cos\ n\pi x\, dx \\
&= \frac{2\, cos\, n\pi}{n^2\pi^2}\,.
\end{aligned}
$$

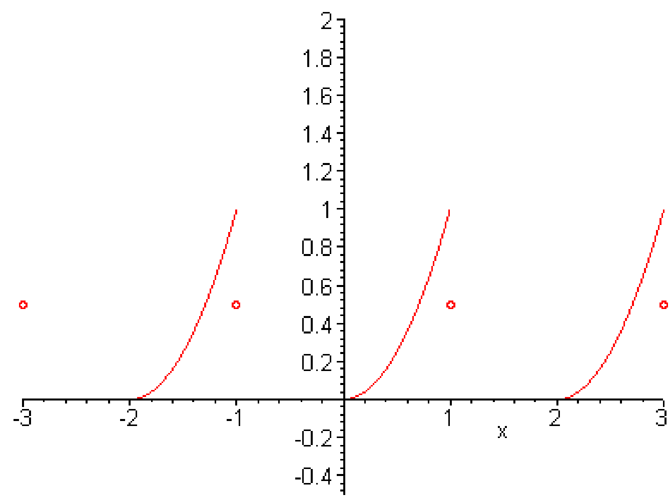
The Fourier *sine* coefficients are given by

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin \frac{n\pi x}{L} dx \\
&= \int_0^1 x^2 sin\ n\pi x\, dx \\
&= -\ \frac{2 - 2\, cos\, n\pi + n^2\pi^2\, cos\, n\pi}{n^3\pi^3}\,.
\end{aligned}
$$

Therefore the Fourier series for the specified function is

$$
\begin{aligned}
f(x) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\, cos\, n\pi x\, - \\
- \sum_{n=1}^{\infty}\left[\frac{2[1-(-1)^n]}{n^3\pi^3} + \frac{(-1)^n}{n\pi}\right] sin\, n\pi x\,.
\end{aligned}
$$

(b).



The given function is piecewise continuous, with discontinuities at the *odd* integers.

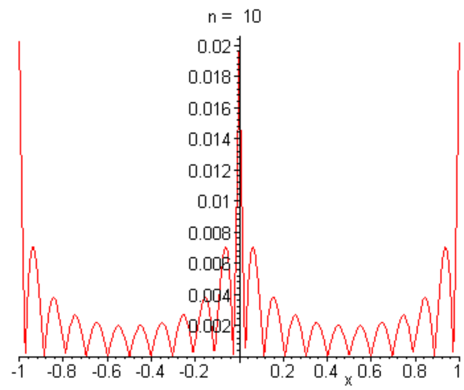
At $x_d = 2k - 1$, $k = 0, 1, 2, \cdots$, the series converges to

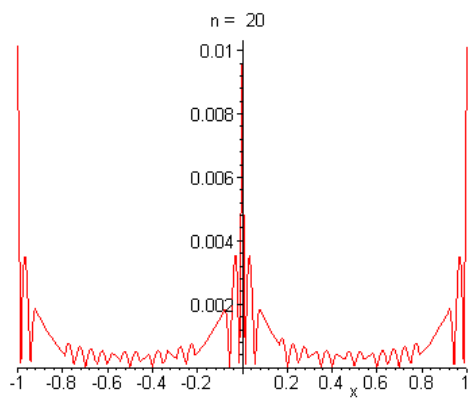
$$|f(x_d-) + f(x_d+)| = 1/2 .$$

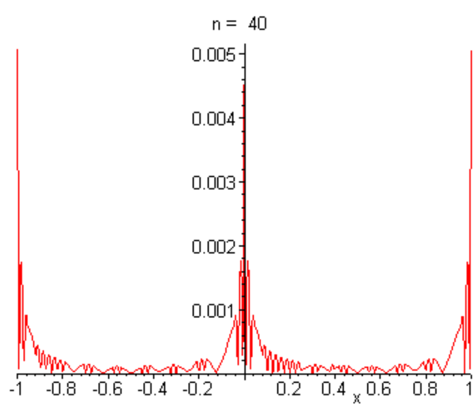
$8(a)$. As shown in Problem 16 of Section 10.2,

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \, cos(2n-1)\pi x .$$

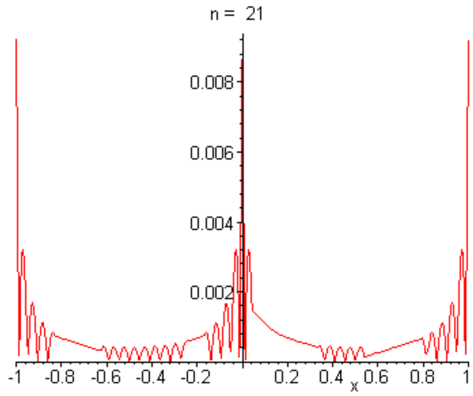
(b).







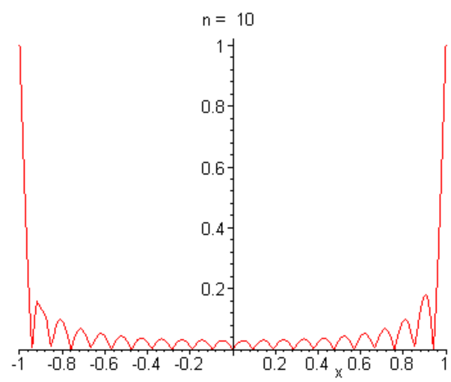
(c).

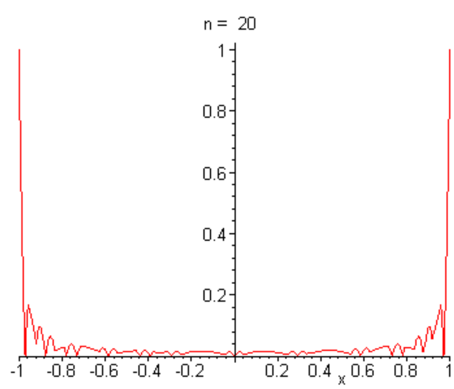


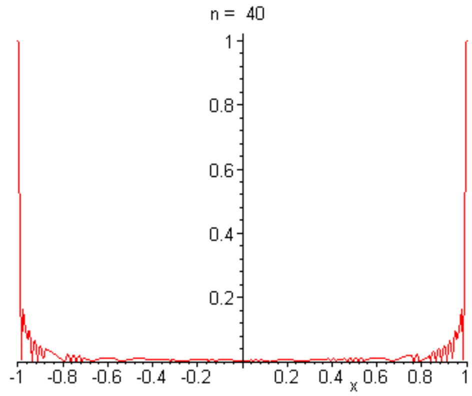
$9(a)$. As shown in Problem 20 of Section 10.2,

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} sin\, n\pi x\,.$$

(b).





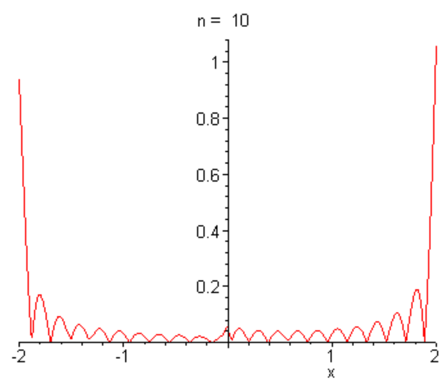


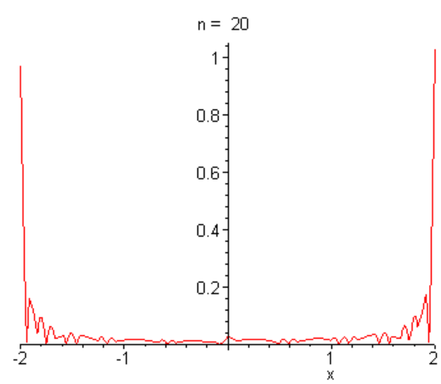
(c). The given function is discontinuous at $x = \pm 1$. At these points, the series will converge to a value of *zero*. The error can never be made arbitrarily small.

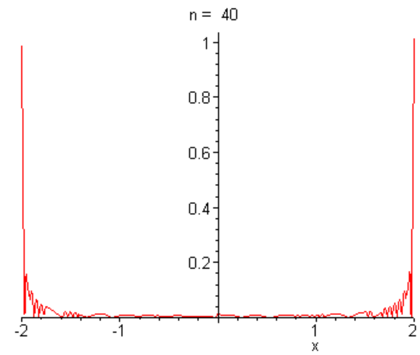
$10(a)$. As shown in Problem 22 of Section 10.2,

$$f(x) = \frac{1}{2} + \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} cos \frac{(2n-1)\pi x}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} sin \frac{n\pi x}{2} .$$

(b).





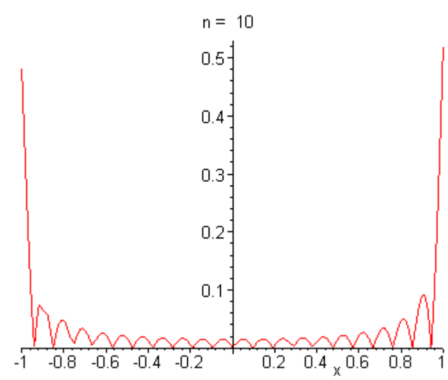


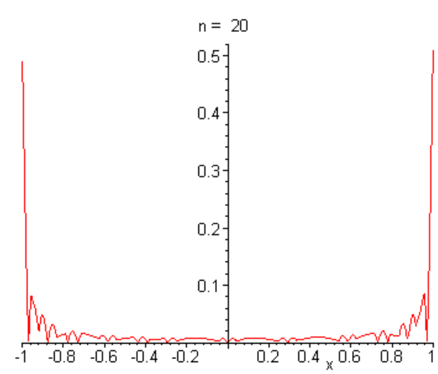
(c). The given function is discontinuous at $x = \pm 2$. At these points, the series will converge to a value of -1. The error can never be made arbitrarily small.

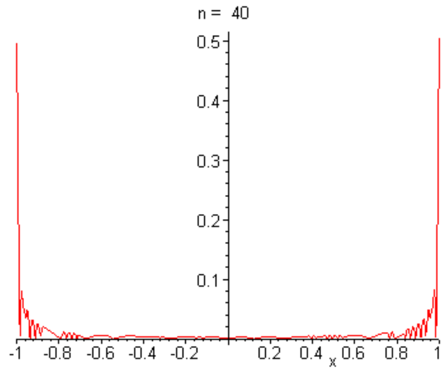
$11(a)$. As shown in Problem 6 , above ,

$$f(x) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,cos\,n\pi x -$$
$$-\sum_{n=1}^{\infty}\left[\frac{2[1-(-1)^n]}{n^3\pi^3} + \frac{(-1)^n}{n\pi}\right]sin\,n\pi x\,.$$

(b).







(c). The given function is piecewise continuous, with discontinuities at the *odd* integers. At $x_d = 2k-1$, $k = 0, 1, 2, \cdots$, the series converges to

$$|f(x_d-) + f(x_d+)| = 1/2\,.$$

At these points the error can never be made arbitrarily small.

13. The solution of the *homogenous* differential equation is

$$y_c(t) = c_1\, cos\, \omega t + c_2\, sin\, \omega t\,.$$

Given that $\omega^2 \neq n^2$, we can use the *method of undetermined coefficients* to find a particular solution

$$Y(t) = \frac{1}{\omega^2 - n^2}\, sin\, nt\,.$$

Hence the general solution of the ODE is

$$y(t) = c_1\, cos\, \omega t + c_2\, sin\, \omega t + \frac{1}{\omega^2 - n^2}\, sin\, nt\,.$$

Imposing the initial conditions, we obtain the equations

$$\begin{aligned} c_1 &= 0 \\ \omega\, c_2 + \frac{n}{\omega^2 - n^2} &= 0\,. \end{aligned}$$

It follows that $c_2 = -n/[\omega(\omega^2 - n^2)]$. The solution of the IVP is

$$y(t) = \frac{1}{\omega^2 - n^2}\, sin\, nt - \frac{n}{\omega(\omega^2 - n^2)}\, sin\, \omega t\,.$$

If $\omega^2 = n^2$, then the forcing function is also one of the fundamental solutions of the ODE.
The method of undetermined coefficients may still be used, with a more elaborate trial solution. Using the *method of variation of parameters*, we obtain

$$Y(t) = -\cos nt \int \frac{\sin^2 nt}{n} dt + \sin nt \int \frac{\cos nt \sin nt}{n} dt$$
$$= \frac{\sin nt - nt \cos nt}{2n^2}.$$

In this case, the general solution is

$$y(t) = c_1 \cos nt + c_2 \sin nt - \frac{t}{2n} \cos nt.$$

Invoking the initial conditions, we obtain $c_1 = 0$ and $c_2 = 1/2n^2$. Therefore the solution
of the IVP is

$$y(t) = \frac{1}{2n^2} \sin nt - \frac{t}{2n} \cos nt.$$

16. Note that the function $f(t)$ and the function given in Problem 8 have the same Fourier
series. Therefore

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\pi t.$$

The solution of the homogeneous problem is

$$y_c(t) = c_1 \cos \omega t + c_2 \sin \omega t.$$

Using the method of undetermined coefficients, we assume a particular solution of the form

$$Y(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi t.$$

Substitution into the ODE and equating like terms results in $A_0 = 1/2\omega^2$ and

$$A_n = \frac{a_n}{\omega^2 - n^2\pi^2}.$$

It follows that the general solution is

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{2\omega^2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi t}{(2n-1)^2 \left[\omega^2 - (2n-1)^2 \pi^2\right]}.$$

Setting $y(0) = 1$, we find that

$$c_1 = 1 - \frac{1}{2\omega^2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi t}{(2n-1)^2 \left[\omega^2 - (2n-1)^2 \pi^2\right]}.$$

Invoking the initial condition $y'(0) = 0$, we obtain $c_2 = 0$. Hence the solution of the initial value problem is

$$y(t) = cos\,\omega t - \frac{1}{2\omega^2} cos\,\omega t + \frac{1}{2\omega^2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{cos(2n-1)\pi t - cos\,\omega t}{(2n-1)^2\left[\omega^2 - (2n-1)^2\pi^2\right]} \,.$$

17. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n\, cos\frac{n\pi x}{L} + b_n\, sin\frac{n\pi x}{L}\right].$$

Squaring both sides of the equation, we *formally* have

$$\begin{aligned} |f(x)|^2 &= \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\left[a_n^2\, cos^2\frac{n\pi x}{L} + b_n^2\, sin^2\frac{n\pi x}{L}\right] + a_0 \sum_{n=1}^{\infty}\left[a_n\, cos\frac{n\pi x}{L} + b_n\, sin\frac{n\pi x}{L}\right] + \\ &\quad + \sum_{m \neq n}\left[c_{mn}\, cos\frac{m\pi x}{L}\, sin\frac{n\pi x}{L}\right]. \end{aligned}$$

Integrating both sides of the last equation, and using the *orthogonality conditions*,

$$\begin{aligned} \int_{-L}^{L} |f(x)|^2 dx &= \int_{-L}^{L} \frac{a_0^2}{4} dx + \sum_{n=1}^{\infty}\left[\int_{-L}^{L} a_n^2\, cos^2\frac{n\pi x}{L} dx + \int_{-L}^{L} b_n^2\, sin^2\frac{n\pi x}{L} dx\right] \\ &= \frac{a_0^2}{2} L + \sum_{n=1}^{\infty}\left[a_n^2\, L + b_n^2 L\right]. \end{aligned}$$

Therefore,

$$\frac{1}{L}\int_{-L}^{L} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

19(a). As shown in the Example, the Fourier series of the function

$$f(x) = \begin{cases} 0, & -L < x < 0 \\ L, & 0 < x < L\,, \end{cases}$$

is given by

$$f(x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1} sin\frac{(2n-1)\pi x}{L}\,.$$

Setting $L = 1$,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1} sin(2n-1)\pi x\,.$$

It follows that

$$\sum_{n=1}^{\infty}\frac{1}{2n-1}sin(2n-1)\pi x = \frac{\pi}{2}\left[f(x)-\frac{1}{2}\right]. \qquad (ii)$$

(b). Given that

$$g(x) = \sum_{n=1}^{\infty}\frac{2n-1}{1+(2n-1)^2}\, sin(2n-1)\pi x\,, \qquad (i)$$

and subtracting Eq.(ii) from Eq.(i), we find that

$$\begin{aligned} g(x)-\frac{\pi}{2}\left[f(x)-\frac{1}{2}\right] \;&=\; \sum_{n=1}^{\infty}\frac{2n-1}{1+(2n-1)^2}\, sin(2n-1)\pi x - \\ &\quad - \sum_{n=1}^{\infty}\frac{1}{2n-1} sin(2n-1)\pi x\,. \end{aligned}$$

Based on the fact that

$$\frac{2n-1}{1+(2n-1)^2}-\frac{1}{2n-1} \;=\; -\,\frac{1}{(2n-1)\big[1+(2n-1)^2\big]}\,,$$

and the fact that we can combine the two series, it follows that

$$g(x) = \frac{\pi}{2}\left[f(x)-\frac{1}{2}\right]-\sum_{n=1}^{\infty}\frac{sin(2n-1)\pi x}{(2n-1)\big[1+(2n-1)^2\big]}\,.$$