Section 7.8

2. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 4-r & -2 \\ 8 & -4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with the *single* root $r = 0$. Substituting $r = 0$ reduces the system of equations to $2\xi_1 - \xi_2 = 0$. Therefore the only eigenvector is $\boldsymbol{\xi} = (1\,,2)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which is a *constant* vector. In order to generate a second linearly independent solution, we must search for a *generalized eigenvector*. This leads to the system of equations

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This system also reduces to a single equation, $2\eta_1 - \eta_2 = 1/2$. Setting $\eta_1 = k$, some arbitrary constant, we obtain $\eta_2 = 2k - 1/2$. A second solution is

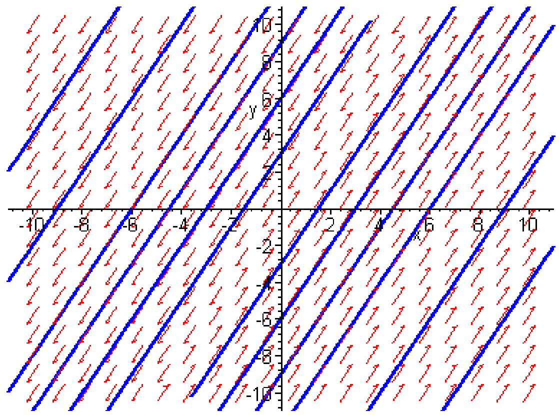
$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} k \\ 2k - 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + k\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \end{aligned}$$

Note that the *last* term is a multiple of $\mathbf{x}^{(1)}$ and may be dropped. Hence

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}\right].$$



All of the points on the line $x_2 = 2x_1$ are equilibrium points. Solutions starting at all other points become unbounded.

3. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{3}{2} - r & 1 \\ -\frac{1}{4} & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $\xi_1 - 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (2\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by finding a *generalized eigenvector*. We therefore analyze the system

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The equations reduce to the single equation $-\eta_1 + 2\eta_2 = 2$. Let $\eta_1 = 2k$. We obtain $\eta_2 = 1 + k$, and a second linearly independent solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 2k \\ 1+k \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + k \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}. \end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \right].$$

4. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -3-r & \frac{5}{2} \\ -\frac{5}{2} & 2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + r + \frac{1}{4} = 0$. The only root is $r = -1/2$, which is an eigenvalue of multiplicity *two*. Setting $r = -1/2$ is the coefficient matrix reduces the system to the single equation $-\xi_1 + \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}.$$

In order to obtain a second linearly independent solution, we find a solution of the system

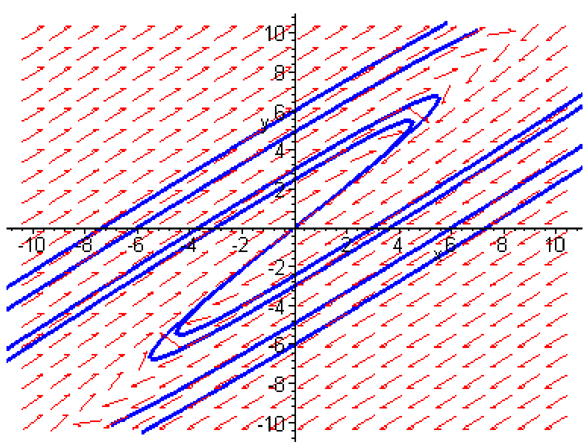
$$\begin{pmatrix} -\frac{5}{2} & \frac{5}{2} \\ -\frac{5}{2} & \frac{5}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

There equations reduce to $-5\eta_1 + 5\eta_2 = 2$. Set $\eta_1 = k$, some arbitrary constant. Then $\eta_2 = k + 2/5$. A second solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} k \\ k + 2/5 \end{pmatrix} e^{-t/2} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}. \end{aligned}$$

Dropping the *last* term, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2}\right].$$



6. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r - 2 = 0$, with roots $r_1 = 2$ and $r_{2,3} = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding eigenvector vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,1\,,1)^T$. Setting $r = -1$, the system of equations is reduced to the *single* equation

$$\xi_1 + \xi_2 + \xi_3 = 0\,.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,0\,,-1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r = -1$) is $\boldsymbol{\xi}^{(3)} = (0\,,1\,,-1)^T$. Therefore the general solution may be written as

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}e^{-t}.$$

7. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 6r + 9 = 0$. The only root is $r = -3$, which is an eigenvalue of multiplicity *two*. Substituting $r = 3$ into the coefficient matrix, the system reduces to the single equation $\xi_1 - \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}.$$

For a second linearly independent solution, we search for a *generalized eigenvector*. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, $4\eta_1 - 4\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. It follows that a second solution is given by

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} k + 1/4 \\ k \end{pmatrix} e^{-3t} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}. \end{aligned}$$

Dropping the last term, the general solution is

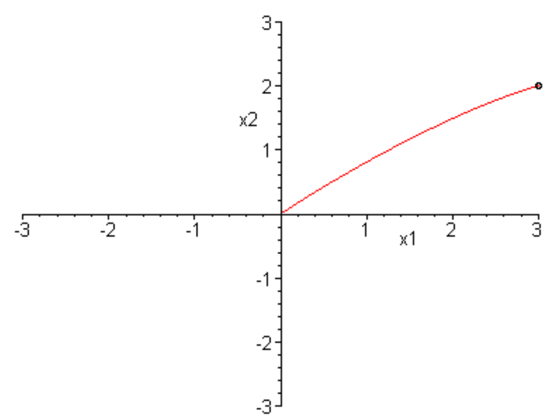
$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t}\right].$$

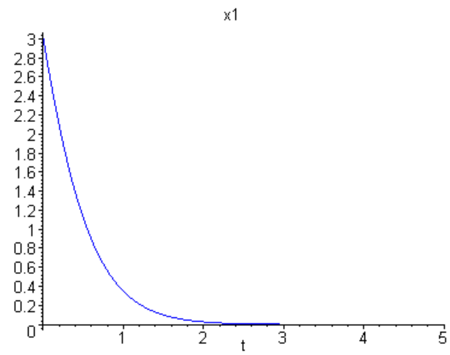
Imposing the initial conditions, we require that

$$\begin{aligned} c_1 + \frac{1}{4}c_2 &= 3 \\ c_1 &= 2, \end{aligned}$$

which results in $c_1 = 2$ and $c_2 = 4$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} te^{-3t}.$$





8. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{5}{2}-r & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2}-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $-\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by solving the system

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $-3\eta_1 + 3\eta_2 = 2$. Let $\eta_1 = k$. We obtain $\eta_2 = 2/3 + k$, and a second linearly independent solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} k \\ 2/3 + k \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \end{aligned}$$

Dropping the last term, the general solution is

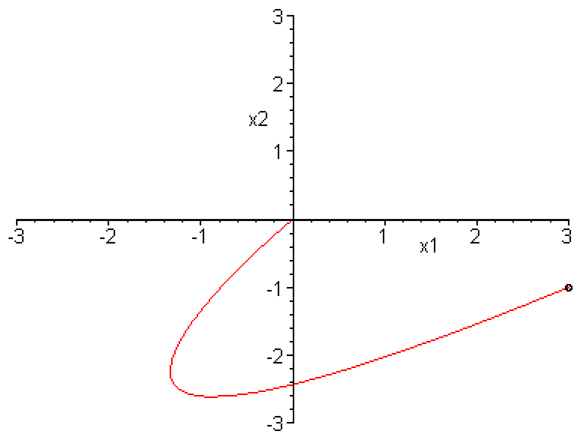
$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t}\right].$$

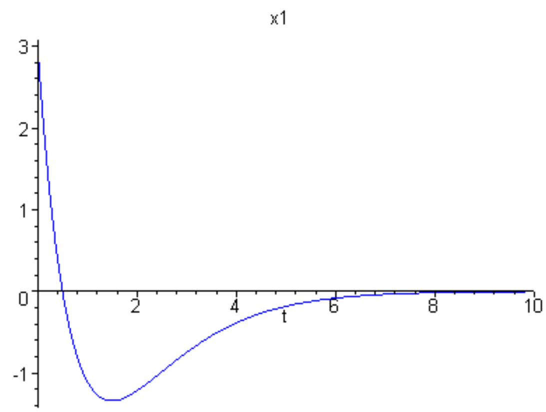
Imposing the initial conditions, find that

$$c_1 = 3$$
$$c_1 + \frac{2}{3}c_2 = -1\,,$$

so that $c_1 = 3$ and $c_2 = -6$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - \begin{pmatrix} 6 \\ 6 \end{pmatrix} te^{-t}.$$





10. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 9 \\ -1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with a single root $r = 0$. Setting $r = 0$, the two equations reduce to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (-3\,,1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

which is a constant vector. A second linearly independent solution is obtained from the system

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $\eta_1 + 3\eta_2 = -1$. Let $\eta_2 = k$. We obtain $\eta_1 = -1 - 3k$, and a second linearly independent solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1-3k \\ k \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + k\begin{pmatrix} -3 \\ 1 \end{pmatrix}. \end{aligned}$$

Dropping the last term, the general solution is

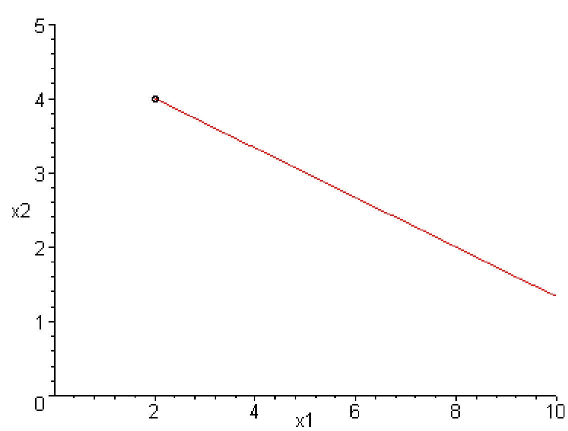
$$\mathbf{x} = c_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2\left[\begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right].$$

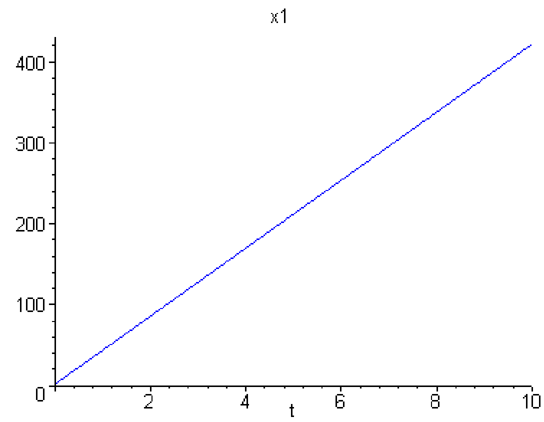
Imposing the initial conditions, we require that

$$\begin{aligned} -3c_1 - c_2 &= 2 \\ c_1 &= 4, \end{aligned}$$

which results in $c_1 = 4$ and $c_2 = -14$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 14\begin{pmatrix} -3 \\ 1 \end{pmatrix} t.$$





12. The characteristic equation of the system is $8\,r^3+60\,r^2+126\,r+49=0$. The eigenvalues are $r_1=-1/2$ and $r_{2,3}=-7/2$. The eigenvector associated with r_1 is $\boldsymbol{\xi}^{(1)}=(1\,,1\,,1)^T$. Setting $r=-7/2$, the components of the eigenvectors must satisfy the relation

$$\xi_1+\xi_2+\xi_3=0\,.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(2)}=(1\,,0\,,-1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r=-7/2$) is $\boldsymbol{\xi}^{(3)}=(0\,,1\,,-1)^T$. Therefore the general solution may be written as

$$\mathbf{x}=c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{-t/2}+c_2\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-7t/2}+c_3\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{-7t/2}.$$

Invoking the initial conditions, we require that

$$\begin{aligned}c_1+c_2&=2\\c_1+c_3&=3\\c_1-c_2-c_3&=-1\,.\end{aligned}$$

Hence the solution of the IVP is

$$\mathbf{x}=\frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}e^{-t/2}+\frac{2}{3}\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{-7t/2}+\frac{5}{3}\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{-7t/2}.$$

13. Setting $\mathbf{x}=\boldsymbol{\xi}\,t^r$ results in the algebraic equations

$$\begin{pmatrix}3-r & -4\\1 & -1-r\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

The characteristic equation is $r^2-2r+1=0$, with a single root of $r_{1,2}=1$. With

$r = 1$, the system reduces to a single equation $\xi_1 - 2\,\xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (2\,,1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,.$$

In order to find a second linearly independent solution, we search for a *generalized eigenvector* whose components satisfy

$$\begin{pmatrix} 2 & -\,4 \\ 1 & -\,2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These equations reduce to $\eta_1 - 2\,\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = 1 + 2k$. [Before proceeding, note that if we set $u = ln\,t$, the original equation is transformed into a constant coefficient equation with independent variable u. Recall that a second solution is obtained by multiplication of the first solution by the factor u. This implies that we must multiply first solution by a factor of $ln\,t$.] Hence a second linearly independent solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,ln\,t + \begin{pmatrix} 1 + 2k \\ k \end{pmatrix} t \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,ln\,t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + k \begin{pmatrix} 2 \\ 1 \end{pmatrix} t. \end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t\,ln\,t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right].$$

15. The characteristic equation is

$$r^2 - (a+d)r + ad - bc = 0\,.$$

Hence the eigenvalues are

$$r_{1,2} = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a+d)^2 - 4(ad-bc)}\;.$$

16(a). Using the result in Prob. 15, the eigenvalues are

$$r_{1,2} = -\,\frac{1}{2RC} \pm \frac{\sqrt{L^2 - 4R^2CL}}{2RCL}\,.$$

The discriminant vanishes when $L = 4R^2CL$.

(b). The system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ -1 & -1 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

The associated eigenvalue problem is

$$\begin{pmatrix} -r & \frac{1}{4} \\ -1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + r + 1/4 = 0$, with a single root of $r_{1,2} = -1/2$. Setting $r = -1/2$, the algebraic equations reduce to $2\xi_1 + \xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (1\,,\,-2)^T$. Hence one solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}.$$

A second solution is obtained from a generalized eigenvector whose components satisfy

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

It follows that $\eta_1 = k$ and $\eta_2 = 4 - 2k$. A second linearly independent solution is

$$\begin{aligned} \begin{pmatrix} I \\ V \end{pmatrix}^{(2)} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} k \\ 4-2k \end{pmatrix} e^{-t/2} \\ &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2} + k\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}. \end{aligned}$$

Dropping the last term, the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2} + c_2\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2}\right].$$

Imposing the initial conditions, we require that

$$\begin{aligned} c_1 &= 1 \\ -2c_1 + 4c_2 &= 2\,, \end{aligned}$$

which results in $c_1 = 1$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/2} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2}.$$

$18(a)$. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r^2 + 3r - 1 = 0$, with a single root of *multiplicity three*, $r = 1$. Setting $r = 1$, we have

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system of algebraic equations reduce to a single equation

$$4\xi_1 - 3\xi_2 - 2\xi_3 = 0\,.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,0\,,2)^T$. Since the last equation has two free variables, a second linearly independent eigenvector (associated with $r = 1$) is $\boldsymbol{\xi}^{(2)} = (0\,,2\,,\,-3)^T$. Therefore two solutions are obtained as

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^t.$$

(b). It follows directly that $\mathbf{x}' = \boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t$. Hence the coefficient vectors must satisfy $\boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t = \mathbf{A}\boldsymbol{\xi} te^t + \mathbf{A}\boldsymbol{\eta} e^t$. Rearranging the terms, we have

$$\boldsymbol{\xi} e^t = (\mathbf{A} - \mathbf{I})\boldsymbol{\xi} te^t + (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} e^t.$$

Given an eigenvector $\boldsymbol{\xi}$, it follows that $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$.

(c). Note that a linear combination of two eigenvectors, associated with the *same* eigenvalue, is also an eigenvector. Consider the equation $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}$. The *augmented* matrix is

$$\left(\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 8 & -6 & -4 & 2c_2 \\ -4 & 3 & 2 & 2c_1 - 3c_2 \end{array}\right).$$

Using elementary row operations, we obtain

$$\left(\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 0 & 0 & 0 & -2c_1 + 2c_2 \\ 0 & 0 & 0 & 3c_1 - 3c_2 \end{array}\right).$$

It is evident that a solution exists provided $c_1 = c_2$.

(d). Let $c_1 = c_2 = 2$. The components of the generalized eigenvector must satisfy

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}.$$

Based on Part (c), the equations reduce to the single equation $4\eta_1 - 3\eta_2 - 2\eta_3 = 2$. Let $\eta_1 = \alpha$ and $\eta_2 = 2\beta$, where α and β are arbitrary constants. We then have

$$\eta_3 = -1 + 2\alpha - 3\beta,$$

so that

$$\boldsymbol{\eta} = \begin{pmatrix} \alpha \\ 2\beta \\ -1 + 2\alpha - 3\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \alpha\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}.$$

Observe that $\boldsymbol{\eta} = \alpha\,\boldsymbol{\xi}^{(1)} + \beta\,\boldsymbol{\xi}^{(2)}$. Hence a third linearly independent solution is

$$\mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} e^t.$$

(e). Given the three linearly independent solutions, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^t & 0 & 2t\,e^t \\ 0 & 2e^t & 4t\,e^t \\ 2e^t & -3e^t & -2t\,e^t - e^t \end{pmatrix}.$$

(f). We construct the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix},$$

with inverse

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 2 & -3/2 & -1 \end{pmatrix}.$$

The *Jordan form* of the matrix **A** is

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

20(a). Direct multiplication results in

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}, \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}, \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b). Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{J}^{n+1} &= \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda \cdot \lambda^n & 0 & 0 \\ 0 & \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} \\ 0 & 0 & \lambda \cdot \lambda^n \end{pmatrix}. \end{aligned}$$

Hence the result follows by mathematical induction.

(c). Note that **J** is *block diagonal*. Hence each *block* may be *exponentiated*. Using the result in Prob. (19),

$$exp(\mathbf{J}t) = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d). Setting $\lambda = 1$, and using the transformation matrix **T** in Prob. (18),

$$\begin{aligned} \mathbf{T}exp(\mathbf{J}t) &= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} \\ &= \begin{pmatrix} e^t & 2e^t & 2t\, e^t \\ 0 & 4e^t & 4t\, e^t \\ 2e^t & -2e^t & -2t\, e^t - e^t \end{pmatrix}. \end{aligned}$$

Based on the form of $\mathbf{J}$, $exp(\mathbf{J}t)$ is the fundamental matrix associated with the solutions

$$\mathbf{y}^{(1)} = \boldsymbol{\xi}^{(1)} e^t,\ \mathbf{y}^{(2)} = \big(2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)}\big)e^t \text{ and } \mathbf{y}^{(3)} = \big(2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)}\big)te^t + \boldsymbol{\eta}e^t.$$

Hence the resulting matrix is the fundamental matrix associated with the solution set

$$\left\{\boldsymbol{\xi}^{(1)}e^t\,,\,\left(2\boldsymbol{\xi}^{(1)}+2\boldsymbol{\xi}^{(2)}\right)e^t,\,\left(2\boldsymbol{\xi}^{(1)}+2\boldsymbol{\xi}^{(2)}\right)te^t+\boldsymbol{\eta}e^t\right\},$$

as opposed to the solution set in Prob. (18), given by

$$\left\{\boldsymbol{\xi}^{(1)}e^t\,,\boldsymbol{\xi}^{(2)}e^t,\,\left(2\boldsymbol{\xi}^{(1)}+2\boldsymbol{\xi}^{(2)}\right)te^t+\boldsymbol{\eta}e^t\right\}.$$

$21(a)$. Direct multiplication results in

$$\mathbf{J}^2=\begin{pmatrix}\lambda^2 & 2\lambda & 1\\ 0 & \lambda^2 & 2\lambda\\ 0 & 0 & \lambda^2\end{pmatrix},\ \mathbf{J}^3=\begin{pmatrix}\lambda^3 & 3\lambda^2 & 3\lambda\\ 0 & \lambda^3 & 3\lambda^2\\ 0 & 0 & \lambda^3\end{pmatrix},\ \mathbf{J}^4=\begin{pmatrix}\lambda^4 & 4\lambda^3 & 6\lambda^2\\ 0 & \lambda^4 & 4\lambda^3\\ 0 & 0 & \lambda^4\end{pmatrix}.$$

(b). Suppose that

$$\mathbf{J}^n=\begin{pmatrix}\lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2}\\ 0 & \lambda^n & n\lambda^{n-1}\\ 0 & 0 & \lambda^n\end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{J}^{n+1}&=\begin{pmatrix}\lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2}\\ 0 & \lambda^n & n\lambda^{n-1}\\ 0 & 0 & \lambda^n\end{pmatrix}\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}\\ &=\begin{pmatrix}\lambda\cdot\lambda^n & \lambda^n+n\lambda\cdot\lambda^{n-1} & n\lambda^{n-1}+\frac{n(n-1)}{2}\lambda\cdot\lambda^{n-2}\\ 0 & \lambda\cdot\lambda^n & \lambda^n+n\lambda\cdot\lambda^{n-1}\\ 0 & 0 & \lambda\cdot\lambda^n\end{pmatrix}.\end{aligned}$$

The result follows by noting that

$$\begin{aligned}n\lambda^{n-1}+\frac{n(n-1)}{2}\lambda\cdot\lambda^{n-2}&=\left[n+\frac{n(n-1)}{2}\right]\lambda^{n-1}\\ &=\frac{n^2+n}{2}\lambda^{n-1}.\end{aligned}$$

(c). We first observe that

$$\sum_{n=0}^{\infty}\lambda^n\frac{t^n}{n!}=e^{\lambda t}$$
$$\sum_{n=0}^{\infty}n\lambda^{n-1}\frac{t^n}{n!}=t\sum_{n=1}^{\infty}\lambda^{n-1}\frac{t^{n-1}}{(n-1)!}=t\,e^{\lambda t}$$
$$\sum_{n=0}^{\infty}\frac{n(n-1)}{2}\lambda^{n-2}\frac{t^n}{n!}=\frac{t^2}{2}\sum_{n=2}^{\infty}\lambda^{n-2}\frac{t^{n-2}}{(n-2)!}=\frac{t^2}{2}e^{\lambda t}.$$

Therefore

$$exp(\mathbf{J}t)=\begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t}\end{pmatrix}.$$

(d). Setting $\lambda=2$, and using the transformation matrix $\mathbf{T}$ in Prob. (17),

$$\begin{aligned}\mathbf{T}exp(\mathbf{J}t) &= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3\end{pmatrix}\begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t}\end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{2t} & te^{2t}+2e^{2t} \\ e^{2t} & te^{2t}+e^{2t} & \frac{t^2}{2}e^{2t}+te^{2t} \\ -e^{2t} & -te^{2t} & -\frac{t^2}{2}e^{2t}+3e^{2t}\end{pmatrix}.\end{aligned}$$