Section 5.3

2. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -(sin\,x)y' - (cos\,x)y\,.$$

Differentiating twice,

$$\begin{aligned} y''' &= -(sin\,x)y'' - 2(cos\,x)y' + (sin\,x)y \\ y^{iv} &= -(sin\,x)y''' - 3(cos\,x)y'' + 3(sin\,x)y' + (cos\,x)y\,. \end{aligned}$$

Given that $\phi(0) = 0$ and $\phi'(0) = 1$, the *first* equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -2$ and $\phi^{iv}(0) = 0$.

3. Let $y = \phi(x)$ be a solution of the initial value problem. First write

$$y'' = -\frac{1+x}{x^2}\,y' - \frac{3\,ln\,x}{x^2}\,y\,.$$

Differentiating twice,

$$y''' = \frac{-1}{x^3}\left[\left(x+x^2\right)y'' + (3x\,ln\,x - x - 2)y' + (3 - 6\,ln\,x)y\right].$$

$$\begin{aligned} y^{iv} = \frac{-1}{x^4}\Big[&\left(x^2+x^3\right)y''' + \left(3x^2ln\,x - 2x^2 - 4x\right)y'' + \\ &+ (6 + 8x - 12x\,ln\,x)y' + (18\,ln\,x - 15)y\Big]. \end{aligned}$$

Given that $\phi(1) = 2$ and $\phi'(1) = 0$, the *first* equation gives $\phi''(1) = 0$ and the last two equations give $\phi'''(0) = -6$ and $\phi^{iv}(0) = 42$.

4. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -x^2\,y' - (sin\,x)y\,.$$

Differentiating twice,

$$\begin{aligned} y''' &= -x^2\,y'' - (2x + sin\,x)y' - (cos\,x)y \\ y^{iv} &= -x^2\,y''' - (4x + sin\,x)y'' - (2 + 2cos\,x)y' + (sin\,x)y\,. \end{aligned}$$

Given that $\phi(0) = a_0$ and $\phi'(0) = a_1$, the *first* equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -a_0$ and $\phi^{iv}(0) = -4a_1$.

5. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all x. Hence the series solutions converge *everywhere*.

7. The *zeroes* of $P(x) = 1 + x^3$ are the *three* cube roots of -1. They all lie on the unit circle in the complex plane. So for $x_0 = 0$, $\rho_{min} = 1$. For $x_0 = 2$, the *nearest*

root is $e^{i\pi/3} = \left(1 + i\sqrt{3}\right)/2$, hence $\rho_{min} = \sqrt{3}$.

8. The only root of $P(x) = x$ is *zero*. Hence $\rho_{min} = 1$.

9(b). $p(x) = -x$ and $q(x) = -1$ are analytic for all x.
(c). $p(x) = -x$ and $q(x) = -1$ are analytic for all x.
(d). $p(x) = 0$ and $q(x) = kx^2$ are analytic for all x.
(e). The only root of $P(x) = 1 - x$ is 1. Hence $\rho_{min} = 1$.
(g). $p(x) = x$ and $q(x) = 2$ are analytic for all x.
(i). The *zeroes* of $P(x) = 1 + x^2$ are $\pm i$. Hence $\rho_{min} = 1$.
(j). The *zeroes* of $P(x) = 4 - x^2$ are ± 2. Hence $\rho_{min} = 2$.
(k). The *zeroes* of $P(x) = 3 - x^2$ are $\pm\sqrt{3}$. Hence $\rho_{min} = \sqrt{3}$.
(l). The only root of $P(x) = 1 - x$ is 1. Hence $\rho_{min} = 1$.
(m). $p(x) = x/2$ and $q(x) = 3/2$ are analytic for all x.
(n). $p(x) = (1 + x)/2$ and $q(x) = 3/2$ are analytic for all x.

12. The Taylor series expansion of e^x, about $x_0 = 0$, is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n\right] + x\sum_{n=0}^{\infty} a_n x^n = 0.$$

First note that

$$x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{n-1} x^n + \cdots.$$

The coefficient of x^n in the *product* of the two series is

$$c_n = 2a_2 \frac{1}{n!} + 6a_3 \frac{1}{(n-1)!} + 12a_4 \frac{1}{(n-2)!} + \cdots + (n+1)n\, a_{n+1} + (n+2)(n+1)a_{n+2}.$$

Expanding the individual series, it follows that

$$2a_2 + (2a_2 + 6a_3)x + (a_2 + 6a_3 + 12a_4)x^2 + (a_2 + 6a_3 + 12a_4 + 20a_5)x^3 + \cdots + \\ + a_0 x + a_1 x^2 + a_2 x^3 + \cdots = 0.$$

Setting the coefficients equal to *zero*, we obtain the system $2a_2 = 0$, $2a_2 + 6a_3 + a_0 = 0$, $a_2 + 6a_3 + 12a_4 + a_1 = 0$, $a_2 + 6a_3 + 12a_4 + 20a_5 + a_2 = 0$, $\cdots$. Hence the general solution is

$$y(x) = a_0 + a_1 x - a_0\frac{x^3}{6} + (a_0 - a_1)\frac{x^4}{12} + (2a_1 - a_0)\frac{x^5}{40} + \left(\tfrac{4}{3}a_0 - 2a_1\right)\frac{x^6}{120} + \cdots .$$

We find that two linearly independent solutions are

$$y_1(x) = 1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \cdots$$

$$y_2(x) = x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \cdots$$

Since $p(x) = 0$ and $q(x) = xe^{-x}$ converge everywhere, $\rho = \infty$.

13. The Taylor series expansion of $cos\, x$, about $x_0 = 0$, is

$$cos\, x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)\,!} .$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)\,!}\right]\left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n\right] + \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 .$$

The coefficient of x^n in the *product* of the two series is

$$c_n = 2a_2 b_n + 6a_3 b_{n-1} + 12a_4 b_{n-2} + \cdots + (n+1)n\, a_{n+1} b_1 + (n+2)(n+1)a_{n+2} b_0 ,$$

in which $cos\, x = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$. It follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty}(n-2)a_n x^n = 0 .$$

Expanding the product of the series, it follows that

$$\begin{gathered} 2a_2 - 2a_0 + 6a_3 x + (-a_2 + 12a_4)x^2 + (-3a_3 + 20a_5)x^3 + \cdots - \\ - a_1 x + a_3 x^3 + 2a_4 x^4 + \cdots = 0 . \end{gathered}$$

Setting the coefficients equal to *zero*, $a_2 - a_0 = 0$, $6a_3 - a_1 = 0$, $-a_2 + 12a_4 = 0$, $-3a_3 + 20a_5 + a_3 = 0$, $\cdots$. Hence the general solution is

$$y(x) = a_0 + a_1 x + a_0 x^2 + a_1\frac{x^3}{6} + a_0\frac{x^4}{12} + a_1\frac{x^5}{60} + a_0\frac{x^6}{120} + a_1\frac{x^7}{560} + \cdots .$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \cdots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \cdots$$

The *nearest* zero of $P(x) = cos\, x$ is at $x = \pm\pi/2$. Hence $\rho_{min} = \pi/2$.

14. The Taylor series expansion of $ln(1+x)$, about $x_0 = 0$, is

$$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right] \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n +$$
$$+ \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}\right] \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0.$$

The *first* product is the series

$$2a_2 + (-2a_2 + 6a_3)x + (a_2 - 6a_3 + 12a_4)x^2 + (-a_2 + 6a_3 - 12a_4 + 20a_5)x^3 + \cdots.$$

The *second* product is the series

$$a_1 x + (2a_2 - a_1/2)x^2 + (3a_3 - a_2 + a_1/3)x^3 + (4a_4 - 3a_3/2 + 2a_2/3 - a_1/4)x^3 + \cdots.$$

Combining the series and equating the coefficients to *zero*, we obtain

$$\begin{aligned} 2a_2 &= 0 \\ -2a_2 + 6a_3 + a_1 - a_0 &= 0 \\ 12a_4 - 6a_3 + 3a_2 - 3a_1/2 &= 0 \\ 20a_5 - 12a_4 + 9a_3 - 3a_2 + a_1/3 &= 0 \\ &\vdots \end{aligned}$$

Hence the general solution is

$$y(x) = a_0 + a_1 x + (a_0 - a_1)\frac{x^3}{6} + (2a_0 + a_1)\frac{x^4}{24} + a_1 \frac{7x^5}{120} + \left(\frac{5}{3}a_1 - a_0\right)\frac{x^6}{120} + \cdots.$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{120} + \cdots$$

$$y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + \cdots$$

The coefficient $p(x) = e^x ln(1+x)$ is analytic at $x_0 = 0$, but its power series has a radius of convergence $\rho = 1$.

15. If $y_1 = x$ and $y_2 = x^2$ are solutions, then substituting y_2 into the ODE results in

$$2\,P(x) + 2x\,Q(x) + x^2 R(x) = 0\,.$$

Setting $x = 0$, we find that $P(0) = 0$. Similarly, substituting y_1 into the ODE results in $Q(0) = 0$. Therefore $P(x)/Q(x)$ and $R(x)/P(x)$ may not be analytic. If they were, Theorem 3.2.1 would guarantee that y_1 and y_2 were the *only* two solutions. But note that an *arbitrary* value of $y(0)$ cannot be a linear combination of $y_1(0)$ and $y_2(0)$. Hence $x_0 = 0$ must be a singular point.

16. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}\left[(n+1)a_{n+1} - a_n\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain

$$a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, \cdots$. It is easy to see that $a_n = a_0/(n\,!)$. Therefore the general solution is

$$\begin{aligned} y(x) &= a_0\left[1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots\right] \\ &= a_0 e^x. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

17. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - x\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

Combining the series, we have

$$a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - a_{n-1}]\, x^n = 0\,.$$

Setting the coefficient equal to *zero*, $a_1 = 0$ and $a_{n+1} = a_{n-1}/(n+1)$ for $n = 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \cdots = \frac{a_0}{2 \cdot 4 \cdots (2k)}$$

and

$$a_{2k+1} = 0\,.$$

Hence the general solution is

$$\begin{aligned} y(x) &= a_0\left[1 + \frac{x^2}{2} + \frac{x^4}{2^2\,2\,!} + \frac{x^6}{2^3\,3\,!} + \cdots + \frac{x^{2n}}{2^n\,n\,!} + \cdots\right] \\ &= a_0 exp\big(x^2/2\big). \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

19. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$(1-x)\sum_{n=0}^{\infty}(n+1)a_{n+1}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\, x^n - \sum_{n=1}^{\infty} n\, a_n\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Combining the series, we have

$$a_1 - a_0 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - n\, a_n - a_n]\, x^n = 0\,.$$

Setting the coefficients equal to *zero*, $a_1 = a_0$ and $a_{n+1} = a_n$ for $n = 0, 1, 2, \cdots$. Hence the general solution is

$$\begin{aligned} y(x) &= a_0\big[1 + x + x^2 + x^3 + \cdots + x^n + \cdots\big] \\ &= a_0 \frac{1}{1-x}\,. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + x\sum_{n=0}^{\infty}a_n x^n = 1 + x\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty}a_{n-1}x^n = 1 + x\,.$$

Combining the series, and the nonhomogeneous terms, we have

$$(a_1 - 1) + (2a_2 + a_0 - 1)x + \sum_{n=2}^{\infty}\left[(n+1)a_{n+1} + a_{n-1}\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain $a_1 = 1$, $2a_2 + a_0 - 1 = 0$, and

$$a_n = -\,\frac{a_{n-2}}{n}\,,\quad n = 3, 4, \cdots.$$

The indices differ by *two*, so for $k = 2, 3, \cdots$

$$a_{2k} = -\,\frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \cdots = \frac{(-1)^{k-1}a_2}{4\cdot 6\cdots(2k)} = \frac{(-1)^k(a_0-1)}{2\cdot 4\cdot 6\cdots(2k)}\,,$$

and for $k = 1, 2, \cdots$

$$a_{2k+1} = \,=\, -\,\frac{a_{2k-1}}{(2k+1)} = \frac{a_{2k-3}}{(2k-1)(2k+1)} = \cdots = \frac{(-1)^k}{3\cdot 5\cdots(2k+1)}\,.$$

Hence the general solution is

$$y(x) = a_0 + x + \frac{1-a_0}{2}x^2 - \frac{x^3}{3} + a_0\frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} - a_0\frac{x^6}{2^3\,3\,!} - \cdots$$

Collecting the terms containing a_0,

$$\begin{aligned} y(x) &= a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2\,2\,!} - \frac{x^6}{2^3\,3\,!} + \cdots\right] + \\ &\quad + \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right]. \end{aligned}$$

Upon inspection, we find that

$$y(x) = a_0 exp(-x^2/2) + \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right].$$

Note that the given ODE is *first order linear*, with integrating factor $\mu(t) = e^{x^2/2}$. The general solution is given by

$$y(x) = e^{-x^2/2}\int_0^x e^{u^2/2}du + (y(0)-1)e^{-x^2/2} + 1\,.$$

23. If $\alpha = 0$, then $y_1(x) = 1$. If $\alpha = 2n$, then $a_{2m} = 0$ for $m \geq n+1$. As a result,

$$y_1(x) = 1 + \sum_{m=1}^{n}(-1)^m \frac{2^m n(n-1)\cdots(n-m+1)(2n+1)(2n+3)\cdots(2n+2m-1)}{(2m)!}\,x^{2m}.$$

$\alpha = 0$	1
$\alpha = 2$	$1-3x^2$
$\alpha = 4$	$1-10x^2+\frac{35}{3}x^4$

If $\alpha = 2n+1$, then $a_{2m+1} = 0$ for $m \geq n+1$. As a result,

$$y_2(x) = x + \sum_{m=1}^{n}(-1)^m \frac{2^m n(n-1)\cdots(n-m+1)(2n+3)(2n+5)\cdots(2n+2m+1)}{(2m+1)!}\,x^{2m+1}.$$

$\alpha = 1$	x
$\alpha = 3$	$x-\frac{5}{3}x^3$
$\alpha = 5$	$x-\frac{14}{3}x^3+\frac{21}{5}x^5$

24(a). Based on Prob. 23,

$\alpha = 0$	1	$y_1(1) = 1$
$\alpha = 2$	$1-3x^2$	$y_1(1) = -2$
$\alpha = 4$	$1-10x^2+\frac{35}{3}x^4$	$y_1(1) = \frac{8}{3}$

Normalizing the polynomials, we obtain

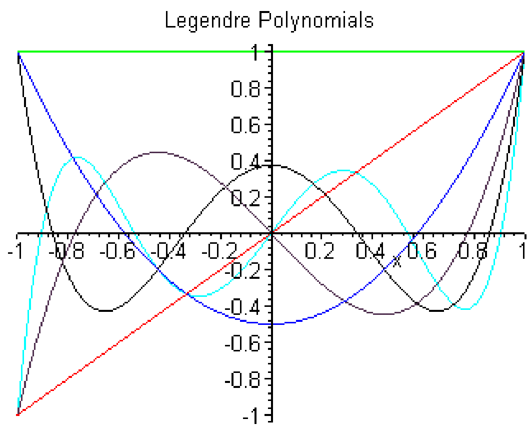
$$P_0(x) = 1$$
$$P_2(x) = -\frac{1}{2}+\frac{3}{2}x^2$$
$$P_4(x) = \frac{3}{8}-\frac{15}{4}x^2+\frac{35}{8}x^4$$

$\alpha = 1$	x	$y_2(1) = 1$
$\alpha = 3$	$x-\frac{5}{3}x^3$	$y_2(1) = -\frac{2}{3}$
$\alpha = 5$	$x-\frac{14}{3}x^3+\frac{21}{5}x^5$	$y_2(1) = \frac{8}{15}$

Similarly,

$$
\begin{aligned}
P_1(x) &= x \\
P_3(x) &= -\frac{3}{2}x + \frac{5}{2}x^3 \\
P_5(x) &= \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5
\end{aligned}
$$

(b).



(c). $P_0(x)$ has no roots. $P_1(x)$ has one root at $x = 0$. The *zeros* of $P_2(x)$ are at $x = \pm 1/\sqrt{3}$. The *zeros* of $P_3(x)$ are $x = 0, \pm\sqrt{3/5}$. The roots of $P_4(x)$ are given by $x^2 = \left(15 + 2\sqrt{30}\right)/35$, $\left(15 - 2\sqrt{30}\right)/35$. The roots of $P_5(x)$ are given by $x = 0$ and $x^2 = \left(35 + 2\sqrt{70}\right)/63$, $\left(35 - 2\sqrt{70}\right)/63$.

25. Observe that

$$
\begin{aligned}
P_n(-1) &= \frac{(-1)^n}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} \\
&= (-1)^n P_n(1).
\end{aligned}
$$

But $P_n(1) = 1$ for *all* nonnegative integers n.

27. We have

$$
(x^2 - 1)^n = \sum_{k=0}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!}\, x^{2k},
$$

which is a polynomial of *degree* $2n$. Differentiating *n times*,

$$
\frac{d^n}{dx^n}(x^2 - 1)^n = \sum_{k=\mu}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!}(2k)(2k-1)\cdots(2k-n+1)x^{2k-n},
$$

in which the *lower index* is $\mu = [n/2] + 1$. Note that if $n = 2m + 1$, then $\mu = m + 1$.

Now shift the index, by setting

$$k = n - j\,.$$

Hence

$$\begin{aligned}\frac{d^n}{dx^n}\left(x^2-1\right)^n &= \sum_{j=0}^{[n/2]}\frac{(-1)^j\, n\,!}{(n-j)!j\,!}(2n-2j)(2n-2j-1)\cdots(n-2j+1)x^{n-2j} \\ &= n!\sum_{j=0}^{[n/2]}\frac{(-1)^j(2n-2j)!}{(n-j)!j\,!(n-2j)!}\,x^{n-2j}.\end{aligned}$$

Based on Prob. 25,

$$\frac{d^n}{dx^n}\left(x^2-1\right)^n = n!\,2^n P_n(x).$$

29. Since the $n+1$ polynomials $P_0, P_1, \cdots, P_n$ are *linearly independent*, and the *degree* of P_k is k , any polynomial, f , of degree n can be expressed as a linear combination

$$f(x) = \sum_{k=0}^{n} a_k P_k(x)\,.$$

Multiplying both sides by P_m and integrating,

$$\int_{-1}^{1} f(x)P_m(x)dx = \sum_{k=0}^{n} a_k \int_{-1}^{1} P_k(x)P_m(x)dx\,.$$

Based on Prob. 28,

$$\int_{-1}^{1} P_k(x)P_m(x)dx = \frac{2}{2m+1}\delta_{km}.$$

Hence

$$\int_{-1}^{1} f(x)P_m(x)dx = \frac{2}{2m+1}\,a_m\,.$$