Chapter Five

Section 5.1

1. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|(x-3)^{n+1}\right|}{\left|(x-3)^{n}\right|}=\lim_{n\to\infty}|x-3|=|x-3|.$$

Hence the series converges absolutely for $|x-3|<1$. The radius of convergence is $\rho=1$. The series diverges for $x=2$ and $x=4$, since the *n-th* term does not approach zero.

3. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!\,x^{2n+2}|}{|(n+1)!\,x^{2n}|}=\lim_{n\to\infty}\frac{x^2}{n+1}=0\,.$$

The series converges absolutely for *all* values of x . Thus the radius of convergence is $\rho=\infty$.

4. Apply the ratio test :

$$\lim_{n\to\infty}\frac{|2^{n+1}x^{n+1}|}{|2^n x^n|}=\lim_{n\to\infty}2|x|=2|x|.$$

Hence the series converges absolutely for $2|x|$, or $|x|<1/2$. The radius of convergence is $\rho=1/2$. The series diverges for $x=\pm1/2$, since the *n-th* term does not approach zero.

6. Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|n(x-x_0)^{n+1}\right|}{\left|(n+1)(x-x_0)^{n}\right|}=\lim_{n\to\infty}\frac{n}{n+1}|(x-x_0)|=|(x-x_0)|.$$

Hence the series converges absolutely for $|(x-x_0)|<1$. The radius of convergence is $\rho=1$. At $x=x_0+1$, we obtain the *harmonic series*, which is *divergent*. At the other endpoint, $x=x_0-1$, we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,,$$

which is *conditionally* convergent.

7. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|3^n(n+1)^2(x+2)^{n+1}\right|}{\left|3^{n+1}n^2(x+2)^n\right|}=\lim_{n\to\infty}\frac{(n+1)^2}{3\,n^2}|(x+2)|=\frac{1}{3}|(x+2)|.$$

Hence the series converges absolutely for $\frac{1}{3}|x+2|<1$, or $|x+2|<3$. The radius of convergence is $\rho=3$. At $x=-5$ and $x=+1$, the series diverges, since the *n-th* term does not approach zero.

8. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n^n(n+1)!\,x^{n+1}|}{\left|(n+1)^{n+1}n!\,x^n\right|}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}|x|=\frac{1}{e}|x|,$$

since

$$\lim_{n\to\infty}\frac{n^n}{(n+1)^n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}=e^{-1}.$$

Hence the series converges absolutely for $|x|<e$. The radius of convergence is $\rho=e$. At $x=\pm e$, the series *diverges*, since the *n-th* term does not approach zero. This follows from the fact that

$$\lim_{n\to\infty}\frac{n!\,e^n}{n^n\sqrt{2\pi n}}=1\,.$$

10. We have $f(x)=e^x$, with $f^{(n)}(x)=e^x$, for $n=1,2,\cdots$. Therefore $f^{(n)}(0)=1$. Hence the Taylor expansion about $x_0=0$ is

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\,.$$

Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!x^{n+1}|}{|(n+1)!\,x^n|}=\lim_{n\to\infty}\frac{1}{n+1}|x|=0.$$

The radius of convergence is $\rho=\infty$.

11. We have $f(x)=x$, with $f'(x)=1$ and $f^{(n)}(x)=0$, for $n=2,\cdots$. Clearly, $f(1)=1$ and $f'(1)=1$, with all other derivatives equal to *zero*. Hence the Taylor expansion about $x_0=1$ is

$$x=1+(x-1).$$

Since the series has only a finite number of terms, the converges absolutely for all x.

14. We have $f(x)=1/(1+x)$, $f'(x)=-1/(1+x)^2$, $f''(x)=2/(1+x)^3,\cdots$ with $f^{(n)}(x)=(-1)^n n!/(1+x)^{n+1}$, for $n\geq 1$. It follows that $f^{(n)}(0)=(-1)^n n!$

for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \to \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

15. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \cdots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = n!$, for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \to \infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

16. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \cdots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(2) = (-1)^{n+1} n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 2$ is

$$\frac{1}{1-x} = -\sum_{n=0}^{\infty} (-1)^n (x-2)^n.$$

Applying the ratio test,

$$\lim_{n \to \infty} \frac{|(x-2)^{n+1}|}{|(x-2)^n|} = \lim_{n \to \infty} |x-2| = |x-2|.$$

The series converges absolutely for $|x-2| < 1$, but diverges at $x = 1$ and $x = 3$.

17. Applying the ratio test,

$$\lim_{n \to \infty} \frac{|(n+1)x^{n+1}|}{|n\, x^n|} = \lim_{n \to \infty} \frac{n+1}{n}|x| = |x|.$$

The series converges absolutely for $|x| < 1$. Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16\,x^3 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n^2 (n-1)\,x^{n-2} = 4 + 18x + 48x^2 + 100x^3 + \cdots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty} (n+1)^2 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)^2 (n+1)\,x^n.$$

20. Shifting the index in the *second* series, that is, setting $n = k + 1$,

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1} x^n\,.$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1}) x^{k+1}.\end{aligned}$$

21. Shifting the index by 2, that is, setting $m = n - 2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2}\, x^m \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n\,.\end{aligned}$$

22. Shift the index *down* by 2, that is, set $m = n + 2$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^{n+2} &= \sum_{m=2}^{\infty} a_{m-2}\, x^m \\ &= \sum_{n=2}^{\infty} a_{n-2}\, x^n\,.\end{aligned}$$

24. Clearly,

$$\left(1-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

Shifting the index in the *first* series, that is, setting $k = n-2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\, x^k \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n.\end{aligned}$$

Hence

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n - \sum_{n=2}^{\infty} n(n-1)a_n\, x^n.$$

Note that when $n=0$ and $n=1$, the coefficients in the *second* series are *zero*. So that

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - n(n-1)a_n]x^n.$$

26. Clearly,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=1}^{\infty} na_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^{n+1}.$$

Shifting the index in the *first* series, that is, setting $k = n-1$,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k.$$

Shifting the index in the *second* series, that is, setting $k = n+1$,

$$\sum_{n=0}^{\infty} a_n\, x^{n+1} = \sum_{k=1}^{\infty} a_{k-1}x^k.$$

Combining the series, and starting the summation at $n=1$,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n = a_1 + \sum_{n=1}^{\infty}[(n+1)a_{n+1} + a_{n-1}]x^n.$$

27. We note that

$$x\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^n.$$

Shifting the index in the *first* series, that is, setting $k = n-1$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} &= \sum_{k=1}^{\infty} k(k+1)a_{k+1}x^k \\ &= \sum_{k=0}^{\infty} k(k+1)a_{k+1}x^k,\end{aligned}$$

since the coefficient of the term associated with $k = 0$ is *zero*. Combining the series,

$$x\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} [n(n+1)a_{n+1} + a_n]x^n.$$