Section 7.9

5. As shown in Prob. 2 , Section 7.8 , the general solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}t\\2t-\frac{1}{2}\end{pmatrix}.$$

An associated fundamental matrix is

$$\boldsymbol{\Psi}(t) = \begin{pmatrix}1 & t\\2 & 2t-\frac{1}{2}\end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\boldsymbol{\Psi}^{-1}(t) = \begin{pmatrix}4t-3 & -2t+2\\8t-8 & -4t+5\end{pmatrix}.$$

We can now compute

$$\boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t) = -\frac{1}{t^3}\begin{pmatrix}2t^2+4t-1\\-2t-4\end{pmatrix},$$

and

$$\int\boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix}-\frac{1}{2}t^{-2}+4t^{-1}-2ln\,t\\-2t^{-2}-2t^{-1}\end{pmatrix}.$$

Finally,

$$\mathbf{v}(t) = \boldsymbol{\Psi}(t)\int\boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = -\frac{1}{2}t^{-2}+2t^{-1}-2\,ln\,t-2$$
$$v_2(t) = 5t^{-1}-4\,ln\,t-4\,.$$

Note that the vector $(2\,,4)^T$ is a multiple of one of the fundamental solutions. Hence we can write the general solution as

$$\mathbf{x} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}t\\2t-\frac{1}{2}\end{pmatrix} - \frac{1}{t^2}\begin{pmatrix}1/2\\0\end{pmatrix} + \frac{1}{t}\begin{pmatrix}2\\5\end{pmatrix} - 2ln\,t\begin{pmatrix}1\\2\end{pmatrix}.$$

6. The eigenvalues of the coefficient matrix are $r_1 = 0$ and $r_2 = -5$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}-2e^{-5t}\\e^{-5t}\end{pmatrix}.$$

The coefficient matrix is *symmetric*. Hence the system is diagonalizable. Using the *normalized* eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix},\ \mathbf{T}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}.$$

Setting $\mathbf{x} = \mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = \frac{5+8t}{\sqrt{5}\,t}$$
$$y_2' = -5y_2 + \frac{4}{\sqrt{5}}.$$

The solutions are readily obtained as

$$y_1(t) = \sqrt{5}\,ln\,t + \frac{4}{\sqrt{5}}\,t + c_1 \ \text{ and } \ y_2(t) = c_2\,e^{-5t} + \frac{4}{5\sqrt{5}}.$$

Transforming back to the original variables, we have $\mathbf{x} = \mathbf{T}\mathbf{y}$, with

$$\begin{aligned}\mathbf{x} &= \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}y_1(t)\\y_2(t)\end{pmatrix}\\ &= \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}y_1(t) + \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}y_2(t).\end{aligned}$$

Hence the general solution is,

$$\mathbf{x} = k_1\begin{pmatrix}1\\2\end{pmatrix} + k_2\begin{pmatrix}-2e^{-5t}\\e^{-5t}\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}ln\,t + \frac{4}{5}\begin{pmatrix}1\\2\end{pmatrix}t + \frac{4}{25}\begin{pmatrix}-2\\1\end{pmatrix}.$$

7. The solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix}e^{-t}\\-2e^{-t}\end{pmatrix} + c_2\begin{pmatrix}e^{3t}\\2e^{3t}\end{pmatrix}.$$

Based on the simple form of the right hand side, we use the method of *undetermined coefficients*. Set $\mathbf{v} = \mathbf{a}\,e^t$. Substitution into the ODE yields

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}e^t = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}e^t + \begin{pmatrix}2\\-1\end{pmatrix}e^t.$$

In scalar form, after canceling the exponential, we have

$$
\begin{aligned}
a_1 &= a_1 + a_2 + 2 \\
a_2 &= 4a_1 + a_2 - 1\,,
\end{aligned}
$$

with $a_1 = 1/4$ and $a_2 = -2$. Hence the particular solution is

$$\mathbf{v} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t,$$

so that the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + \frac{1}{4} \begin{pmatrix} e^t \\ -8e^t \end{pmatrix}.$$

8. The eigenvalues of the coefficient matrix are $r_1 = 1$ and $r_2 = -1$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Use the method of *undetermined coefficients*. Since the right hand side is related to one of the fundamental solutions, set $\mathbf{v} = \mathbf{a}\, te^t + \mathbf{b}\, e^t$. Substitution into the ODE yields

$$
\begin{aligned}
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \left(e^t + te^t\right) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t &= \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} te^t + \\
&\quad + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.
\end{aligned}
$$

In scalar form, we have

$$
\begin{aligned}
(a_1 + b_1)e^t + a_1 te^t &= (2a_1 - a_2)te^t + (2b_1 - b_2)e^t + e^t \\
(a_2 + b_2)e^t + a_2 te^t &= (3a_1 - 2a_2)te^t + (3b_1 - 2b_2)e^t - e^t\,.
\end{aligned}
$$

Equating the coefficients in these two equations, we find that

$$
\begin{aligned}
a_1 &= 2a_1 - a_2 \\
a_1 + b_1 &= 2b_1 - b_2 + 1 \\
a_2 &= 3a_1 - 2a_2 \\
a_2 + b_2 &= 3b_1 - 2b_2 - 1\,.
\end{aligned}
$$

It follows that $a_1 = a_2$. Setting $a_1 = a_2 = a$, the equations reduce to

$$
\begin{aligned}
b_1 - b_2 &= a - 1 \\
3b_1 - 3b_2 &= 1 + a\,.
\end{aligned}
$$

Combining these equations, it is necessary that $a = 2$. As a result, $b_1 = b_2 + 1$. Choosing $a_1 = a_2 = 2$, and $b_2 = k$, some arbitrary constant, a particular solution is

$$\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} k+1 \\ k \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

Since the *second* vector is a fundamental solution, the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

9. Note that the coefficient matrix is *symmetric*. Hence the system is diagonalizable. The eigenvalues and eigenvectors are given by

$$r_1 = -\frac{1}{2}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } r_2 = -2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Using the *normalized* eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ \mathbf{T}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Setting $\mathbf{x} = \mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = -\frac{1}{2} y_1 + \sqrt{2}\,t + \frac{1}{\sqrt{2}} e^t$$
$$y_2' = -2y_2 + \sqrt{2}\,t - \frac{1}{\sqrt{2}} e^t\,.$$

Using any elementary method for first order linear equations, the solutions are

$$y_1(t) = k_1 e^{-t/2} + \frac{\sqrt{2}}{3} e^t - 4\sqrt{2} + 2\sqrt{2}\,t$$
$$y_2(t) = k_2 e^{-2t} - \frac{1}{3\sqrt{2}} e^t - \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\,t\,.$$

Transforming back to the original variables, $\mathbf{x} = \mathbf{T}\mathbf{y}$, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} - \frac{1}{4} \begin{pmatrix} 17 \\ 15 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 5 \\ 3 \end{pmatrix} t + \frac{1}{6} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t.$$

10. Since the coefficient matrix is *symmetric*, the differential equations can be decoupled.
The eigenvalues and eigenvectors are given by

$$r_1 = -4\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \text{ and } r_2 = -1\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

Using the *normalized* eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix}\ ,\ \mathbf{T}^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

Setting $\mathbf{x} = \mathbf{Ty}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = -4y_1 + \frac{1}{\sqrt{3}}\left(1+\sqrt{2}\right)e^{-t}$$
$$y_2' = -y_2 + \frac{1}{\sqrt{3}}\left(1-\sqrt{2}\right)e^{-t}\,.$$

The solutions are easily obtained as

$$y_1(t) = k_1e^{-4t} + \frac{1}{3\sqrt{3}}\left(1+\sqrt{2}\right)e^{-t}$$
$$y_2(t) = k_2e^{-t} + \frac{1}{\sqrt{3}}\left(1-\sqrt{2}\right)te^{-t}.$$

Transforming back to the original variables, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} + \frac{1}{9}\begin{pmatrix} 2+\sqrt{2}+3\sqrt{3} \\ 3\sqrt{6}-\sqrt{2}-1 \end{pmatrix}e^{-t} + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix}te^{-t}.$$

Note that

$$\begin{pmatrix} 2+\sqrt{2}+3\sqrt{3} \\ 3\sqrt{6}-\sqrt{2}-1 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} \\ -\sqrt{2}-1 \end{pmatrix} + 3\sqrt{3}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The *second* vector is an *eigenvector*, hence the solution may be written as

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} + \frac{1}{9}\begin{pmatrix} 2+\sqrt{2} \\ -\sqrt{2}-1 \end{pmatrix}e^{-t} + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix}te^{-t}.$$

11. Based on the solution of Prob. 3 of Section 7.6, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & -\cos t + 2\sin t \end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{5}\begin{pmatrix} cos\,t - 2sin\,t & 5sin\,t \\ 2cos\,t + sin\,t & -5cos\,t \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} cos\,t\,sin\,t \\ -cos^2 t \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \frac{1}{2}sin^2 t \\ -\frac{1}{2}cos\,t\,sin\,t - \frac{1}{2}t \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = \frac{5}{2}cos\,t\,sin\,t - cos^2 t + \frac{5}{2}t + 1$$
$$v_2(t) = cos\,t\,sin\,t - \frac{1}{2}cos^2 t + t + \frac{1}{2}\,.$$

Hence the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 5\,cos\,t \\ 2\,cos\,t + sin\,t \end{pmatrix} + c_2\begin{pmatrix} 5\,sin\,t \\ -cos\,t + 2\,sin\,t \end{pmatrix} - $$
$$- t\,sin\,t\begin{pmatrix} 5/2 \\ 1 \end{pmatrix} + (t\,cos\,t + sin\,t)\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

$13(a)$. As shown in Prob. 25 of Section 7.6 , the solution of the homogeneous system is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} cos(t/2) \\ 4\,sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} sin(t/2) \\ -4\,cos(t/2) \end{pmatrix}.$$

Therefore the associated fundamental matrix is given by

$$\mathbf{\Psi}(t) = e^{-t/2}\begin{pmatrix} cos(t/2) & sin(t/2) \\ 4\,sin(t/2) & -4\,cos(t/2) \end{pmatrix}.$$

(b). The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{e^{t/2}}{4}\begin{pmatrix} 4\,cos(t/2) & sin(t/2) \\ 4\,sin(t/2) & -cos(t/2) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \frac{1}{2}\begin{pmatrix} cos(t/2) \\ sin(t/2) \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} sin(t/2) \\ -\,cos(t/2) \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = 0$$
$$v_2(t) = 4\,e^{-t/2}\,.$$

Hence the general solution is

$$\mathbf{x} = c_1 e^{-t/2}\begin{pmatrix} cos(t/2) \\ 4\,sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} sin(t/2) \\ -\,4\,cos(t/2) \end{pmatrix} + 4\,e^{-t/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Imposing the initial conditions, we require that

$$c_1 = 0$$
$$-\,4c_2 + 4 = 0\,,$$

which results in $c_1 = 0$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\mathbf{x} = e^{-t/2}\begin{pmatrix} sin(t/2) \\ 4 - 4\,cos(t/2) \end{pmatrix}.$$

15. The general solution of the homogeneous problem is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}t^{-1} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}t^2,$$

which can be verified by substitution into the system of ODEs. Since the vectors are linearly independent, a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} t^{-1} & 2t^2 \\ 2t^{-1} & t^2 \end{pmatrix}.$$

The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{3}\begin{pmatrix} -t & 2t \\ 2t^{-2} & -t^{-2} \end{pmatrix}.$$

Dividing both equations by t, we obtain

$$\mathbf{g}(t) = \begin{pmatrix} -2 \\ t^3 - t^{-1} \end{pmatrix}.$$

Proceeding with the method of *variation of parameters*,

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} \frac{2}{3}t^4 + \frac{2}{3}t - \frac{2}{3} \\ -\frac{1}{3}t - \frac{4}{3}t^{-2} + \frac{1}{3}t^{-3} \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \frac{2}{15}t^5 + \frac{1}{3}t^2 - \frac{2}{3}t \\ -\frac{1}{6}t^2 + \frac{4}{3}t^{-1} - \frac{1}{6}t^{-2} \end{pmatrix}.$$

Hence a particular solution is obtained as

$$\mathbf{v} = \begin{pmatrix} -\frac{1}{5}t^4 + 3t - 1 \\ \frac{1}{10}t^4 + 2t - \frac{3}{2} \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = \; = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}t^{-1} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}t^2 + \frac{1}{10}\begin{pmatrix} -2 \\ 1 \end{pmatrix}t^4 + \begin{pmatrix} 3 \\ 2 \end{pmatrix}t - \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}.$$

16. Based on the hypotheses,

$$\phi'(t) = \mathbf{P}(t)\phi(t) + \mathbf{g}(t) \;\text{ and }\; \mathbf{v}'(t) = \mathbf{P}(t)\mathbf{v}(t) + \mathbf{g}(t)\,.$$

Subtracting the two equations results in

$$\phi'(t) - \mathbf{v}'(t) = \mathbf{P}(t)\phi(t) - \mathbf{P}(t)\mathbf{v}(t)\,,$$

that is,

$$[\phi(t) - \mathbf{v}(t)]' = \mathbf{P}(t)[\phi(t) - \mathbf{v}(t)].$$

It follows that $\phi(t) - \mathbf{v}(t)$ is a solution of the *homogeneous equation*. According to Theorem 7.4.2,

$$\phi(t) - \mathbf{v}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t).$$

Hence

$$\phi(t) = \mathbf{u}(t) + \mathbf{v}(t),$$

in which $\mathbf{u}(t)$ is the general solution of the homogeneous problem.

$17(a)$. Setting $t_0 = 0$ in Eq. (34),

$$\begin{aligned}\mathbf{x} &= \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds \\ &= \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds\,.\end{aligned}$$

It was shown in Prob. $15(c)$ in Section 7.7 that $\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s) = \mathbf{\Phi}(t-s)$. Therefore

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t-s)\mathbf{g}(s)ds\,.$$

(b). The *principal* fundamental matrix is identified as $\mathbf{\Phi}(t) = exp(\mathbf{A}t)$. Hence

$$\mathbf{x} = exp(\mathbf{A}t)\mathbf{x}^0 + \int_0^t exp[\mathbf{A}(t-s)]\mathbf{g}(s)ds\,.$$

In Prob. 26 of Section 3.7, the particular solution is given as

$$Y(t) = \int_{t_0}^t K(t-s)g(s)ds\,,$$

in which the kernel $K(t)$ depends on the nature of the fundamental solutions.