Section 9.2

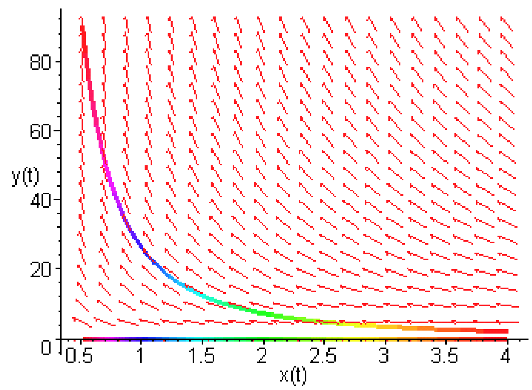
2. The differential equations can be combined to obtain a related ODE

$$\frac{dy}{dx} = -\frac{2y}{x}.$$

The equation is *separable*, with

$$\frac{dy}{y} = -\frac{2\,dx}{x}.$$

The solution is given by $y = C\,x^{-2}$. Note that the system is *uncoupled*, and hence we also have $x = x_0 e^{-t}$ and $y = y_0 e^{2t}$.



In order to determine the direction of motion along the trajectories, observe that for *positive* initial conditions, x will *decrease*, whereas y will *increase*.

4. The trajectories of the system satisfy the ODE

$$\frac{dy}{dx} = -\frac{bx}{ay}.$$

The equation is *separable*, with

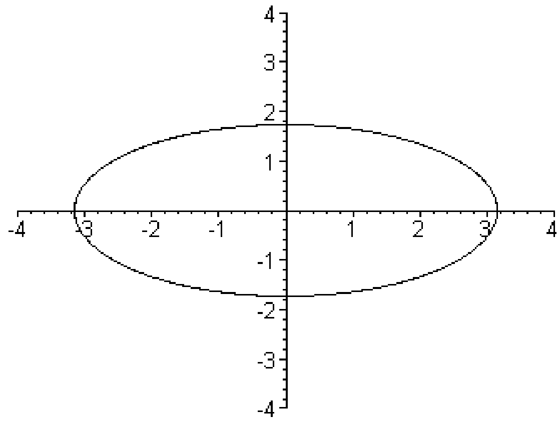
$$ay\,dy = -\,bx\,dx.$$

Hence the trajectories are given by $b\,x^2 + a\,y^2 = C^2$, in which C is arbitrary. Evidently, the trajectories are *ellipses*. Invoking the initial condition, we find that $C^2 = ab$. The system of ODEs can also be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & a \\ -\,b & 0 \end{pmatrix}\mathbf{x}.$$

Using the methods in Chapter 7, it is easy to show that

$$x = \sqrt{a}\cos\sqrt{ab}\,t$$
$$y = -\sqrt{b}\sin\sqrt{ab}\,t.$$



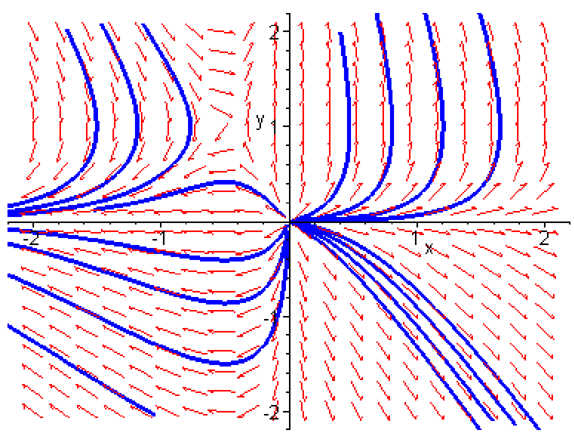
Note that for *positive* initial conditions, x will $increase$, whereas y will $decrease$.

$5(a)$. The critical points are given by the solution set of the equations

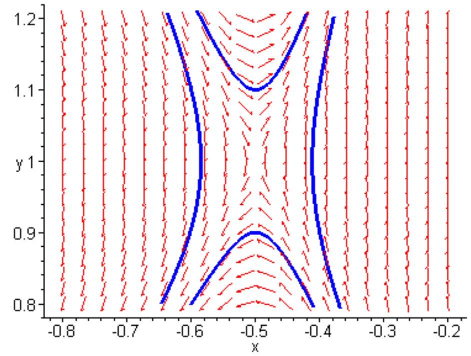
$$x(1-y)=0$$
$$y(1+2x)=0\,.$$

Clearly, $(0\,,0)$ is a solution. If $x\neq 0$, then $y=1$ and $x=-1/2$. Hence the critical points are $(0\,,0)$ and $(-1/2\,,1)$.

(b).



(c). Based on the phase portrait, all trajectories starting near the origin *diverge*. Hence the critical point $(0\,,0)$ is *unstable*. Examining the phase curves near the critical point $(-1/2\,,1)$,



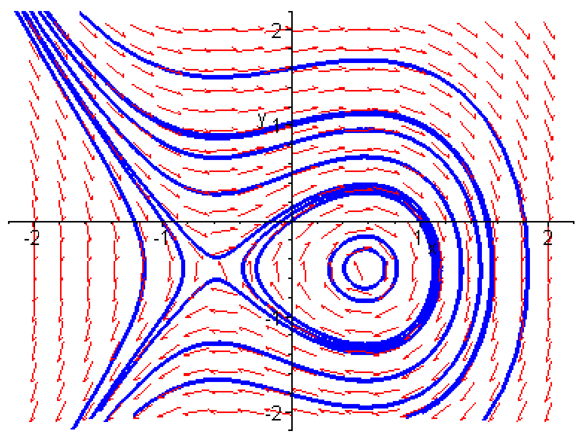
the equilibrium point has the properties of a *saddle*, and hence it is *unstable*.

$6(a)$. The critical points are solutions of the equations

$$1+2y=0$$
$$1-3x^2=0\,.$$

There are two equilibrium points, $\left(-1/\sqrt{3}\,,\,-1/2\right)$ and $\left(1/\sqrt{3}\,,\,-1/2\right)$.

(b).



(c). Locally, the trajectories near the point $\left(-1/\sqrt{3}\,,\,-1/2\right)$ resemble the behavior near a *saddle*. Hence the critical point is *unstable*. Near the point $\left(1/\sqrt{3}\,,\,-1/2\right)$, the solutions are *periodic*. Therefore the second critical point is *stable*.

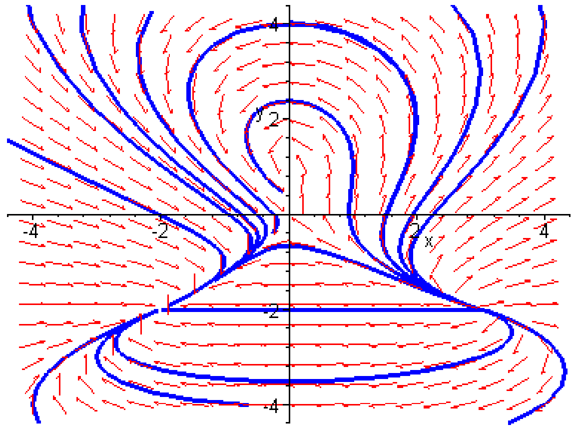
$8(a)$. The critical points are solutions of the equations

$$-(x-y)(1-x-y)=0$$
$$x(2+y)=0\,.$$

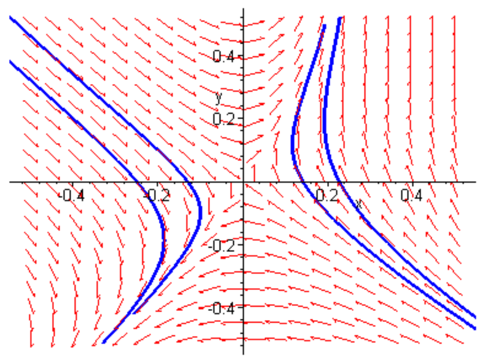
If $x=y$, then $x=y=0$ or $x=y=-2$. If $x=1-y$, then $x=0$ and $y=1$, or $x=3$ and $y=-2$. It follows that the critical points are $(0\,,0)$, $(-2\,,-2)$, $(0\,,1)$

and $(3\,,\,-2)$.

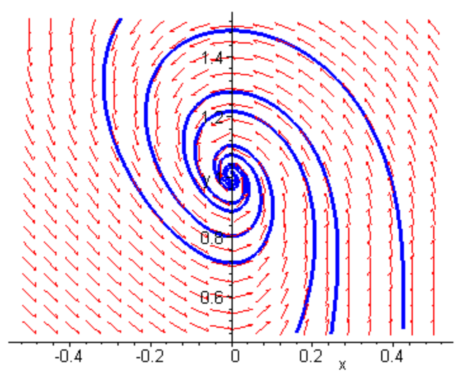
(b).



(c). Near the origin, the trajectories resemble those of a *saddle*, and hence it is *unstable*.



Near the critical point $(0\,,1)$, the trajectories resemble those of a stable *spiral*. Hence the equilibrium point is *asymptotically stable*.



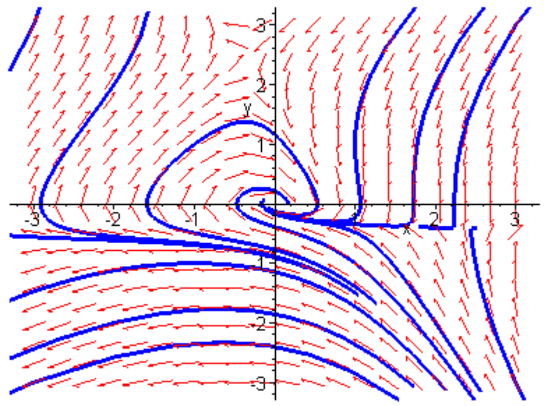
Based on the global phase portrait, it is evident that the other critical points are *nodes*. Closer examination reveals that the point $(-2\,,\,-2)$ is *asymptotically stable*, whereas the point $(3\,,\,-2)$ is *unstable*.

9(a). The critical points are given by the solution set of the equations

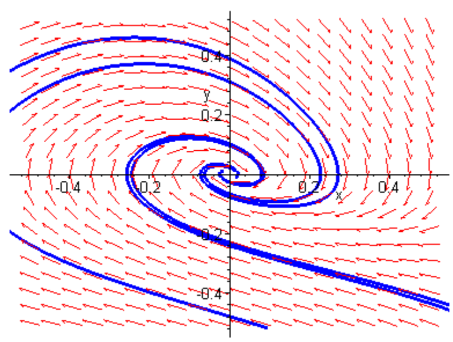
$$y(2-x-y)=0$$
$$-x-y-2xy=0\,.$$

Clearly, $(0\,,0)$ is a critical point. If $x=2-y$, then it follows that $y(y-2)=1$. The additional critical points are $\left(1-\sqrt{2}\,,1+\sqrt{2}\right)$ and $\left(1+\sqrt{2}\,,1-\sqrt{2}\right)$.

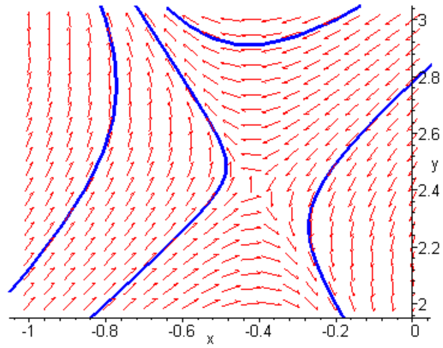
(b).



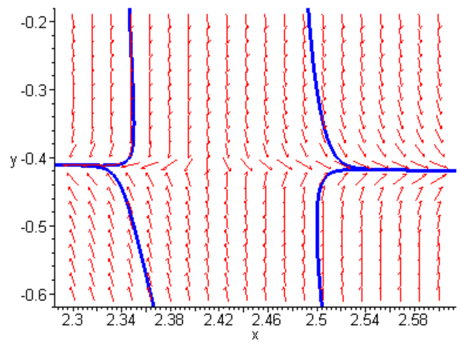
(c). The behavior near the origin is that of a *stable spiral*. Hence the point $(0\,,0)$ is *asymptotically stable*.



At the critical point $\left(1-\sqrt{2}\,,1+\sqrt{2}\right)$, the trajectories resemble those near a *saddle*. Hence the critical point is *unstable*.



Near the point $\left(1+\sqrt{2}\,,1-\sqrt{2}\right)$, the trajectories resemble those near a *saddle*. Hence the critical point is also *unstable*.

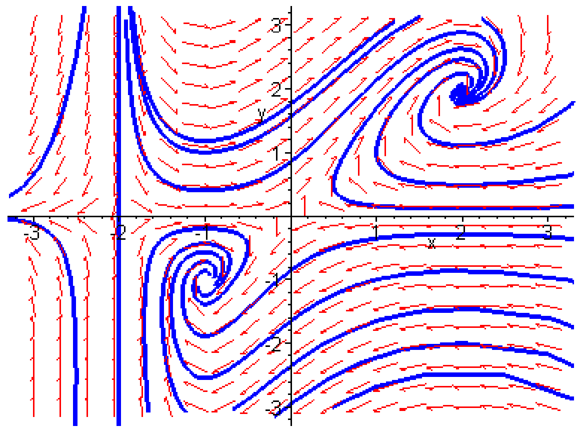


$10(a)$. The critical points are solutions of the equations

$$(2+x)(y-x)=0$$
$$y\left(2+x-x^2\right)=0\,.$$

The origin is evidently a critical point. If $x=\,-2$, then $y=0$. If $x=y$, then either $y=0$ or $x=y=\,-1$ or $x=y=2$. Hence the other critical points are $(\,-2\,,0)$, $(\,-1\,,\,-1)$ and $(2\,,2)$.

(b).



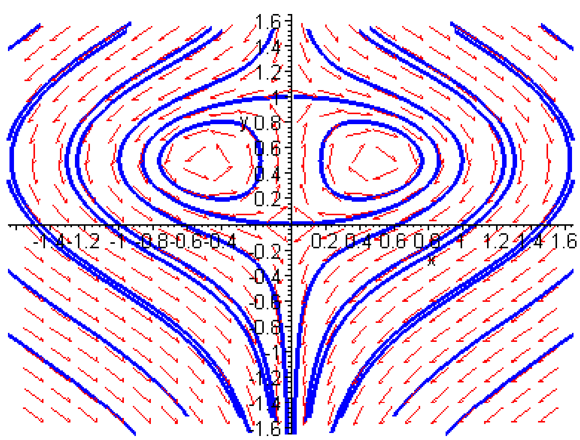
(c). Based on the global phase portrait, the critical points $(0\,,0)$ and $(\,-\,2\,,0)$ have the characteristics of a *saddle*. Hence these points are *unstable*. The behavior near the remaining two critical points resembles those near a *stable spiral*. Hence the critical points $(\,-\,1\,,\,-\,1)$ and $(2\,,2)$ are *asymptotically stable.*

$11(a)$. The critical points are given by the solution set of the equations

$$x(1-2y)=0$$
$$y-x^2-y^2=0\,.$$

If $x=0$, then either $y=0$ or $y=1$. If $y=1/2$, then $x=\pm 1/2$. Hence the critical points are at $(0\,,0)$, $(0\,,1)$, $(\,-\,1/2\,,1/2)$ and $(1/2\,,1/2)$.

(b).



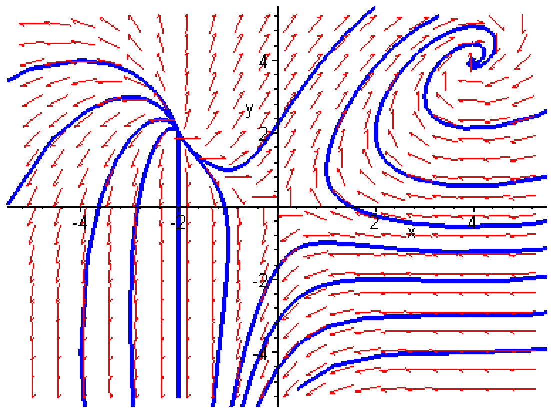
(c). The trajectories near the critical points $(\,-\,1/2\,,1/2)$ and $(1/2\,,1/2)$ are closed curves. Hence the critical points have the characteristics of a *center*, which is *stable*. The trajectories near the critical points $(0\,,0)$ and $(0,1)$ resemble those near a *saddle*. Hence these critical points are *unstable*.

$13(a)$. The critical points are solutions of the equations

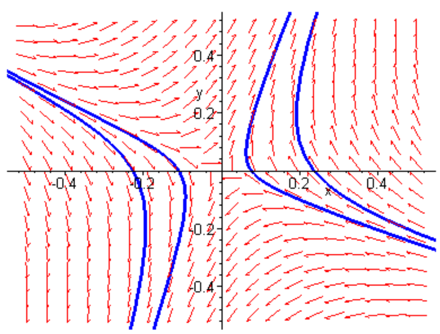
$$(2+x)(y-x)=0$$
$$(4-x)(y+x)=0\,.$$

If $y=x$, then either $x=y=0$ or $x=y=4$. If $x=-2$, then $y=2$. If $x=-y$, then $y=2$ or $y=0$. Hence the critical points are at $(0,0)$, $(4,4)$ and $(-2,2)$.

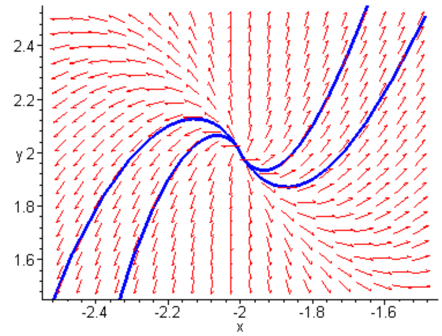
(b).



(c). The critical point at $(4,4)$ is evidently a *stable spiral*, which is *asymptotically stable*. Closer examination of the critical point at $(0,0)$ reveals that it is a *saddle*, which is *unstable*.



The trajectories near the critical point $(-2,2)$ resemble those near an *unstable node*.

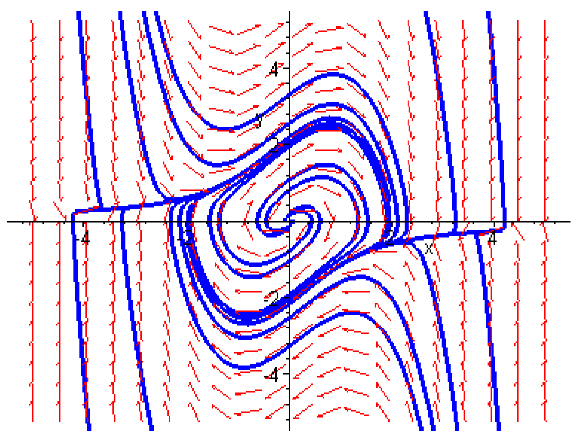


$14(a)$. The critical points consist of the solution set of the equations

$$y = 0$$
$$(1 - x^2)y - x = 0\,.$$

It is easy to see that the only critical point is at $(0\,,0)$.

(b).



(c). The origin is an *unstable spiral*.

$16(a)$. The trajectories are solutions of the differential equation

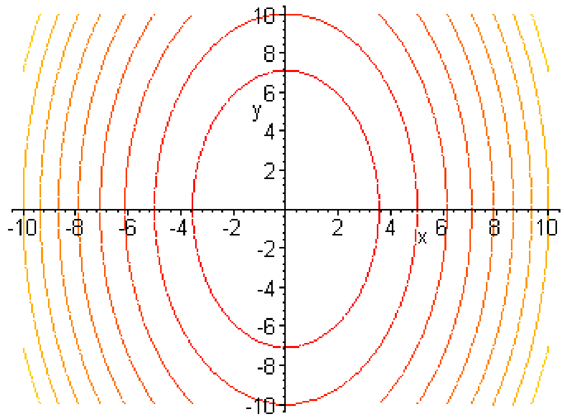
$$\frac{dy}{dx} = -\frac{4x}{y},$$

which can also be written as $4x\,dx + y\,dy = 0$. Integrating, we obtain

$$4x^2 + y^2 = C^2.$$

Hence the trajectories are ellipses.

(b).



Based on the differential equations, the direction of motion on each trajectory is *clockwise*.

$17(a)$. The trajectories of the system satisfy the ODE

$$\frac{dy}{dx} = \frac{2x+y}{y},$$

which can also be written as $(2x+y)dx - ydy = 0$. This differential equation is *homogeneous*. Setting $y = x\,v(x)$, we obtain

$$v + x\frac{dv}{dx} = \frac{2}{v} + 1,$$

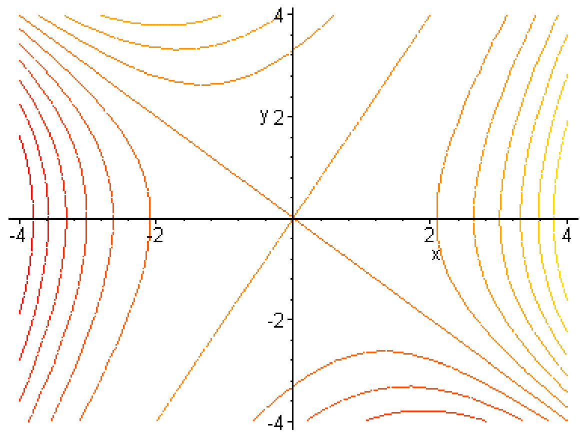
that is,

$$x\frac{dv}{dx} = \frac{2+v-v^2}{v}.$$

The resulting ODE is *separable*, with solution $x^3(v+1)(v-2)^2 = C$. Reverting back to the original variables, the trajectories are level curves of

$$H(x\,,y) = (x+y)(y-2x)^2.$$

(b).



The origin is a *saddle*. Along the line $y = 2x$, solutions increase without bound. Along the line $y = -x$, solutions converge toward the origin.

$18(a)$. The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x+y}{x-y},$$

which is *homogeneous*. Setting $y = x\,v(x)$, we obtain

$$v + x\frac{dv}{dx} = \frac{x+xv}{x-xv},$$

that is,

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}.$$

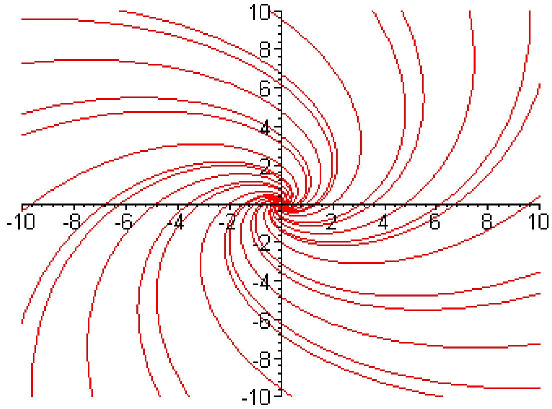
The resulting ODE is *separable*, with solution

$$arctan(v) = ln|x|\sqrt{1+v^2}.$$

Reverting back to the original variables, the trajectories are level curves of

$$H(x\,,y) = arctan(y/x) - ln\sqrt{x^2+y^2}.$$

(b).



The origin is a *stable spiral*.

$20(a)$. The trajectories are solutions of the differential equation

$$\frac{dy}{dx}=\frac{-2xy^2+6\,xy}{2x^2y-3x^2-4y},$$

which can also be written as $(2xy^2-6\,xy)dx+(2x^2y-3x^2-4y)dy=0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x}=2xy^2-6\,xy \text{ and } \frac{\partial H}{\partial y}=2x^2y-3x^2-4y\,.$$

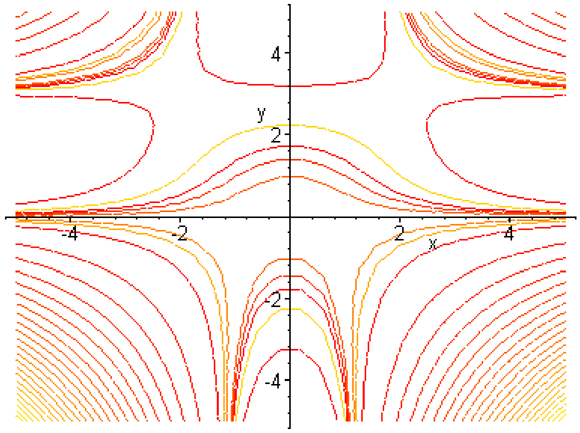
Integrating the first equation, we find that $H(x\,,y)=x^2y^2-3x^2y+f(y)$. It follows that

$$\frac{\partial H}{\partial y}=2x^2y-3x^2+f\,'(y).$$

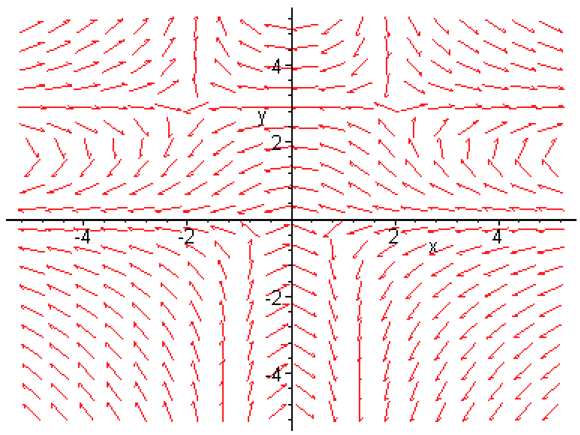
Comparing the two partial derivatives, we obtain $f(y)=\,-2y^2+c\,.$ Hence

$$H(x\,,y)=x^2y^2-3x^2y-2y^2.$$

(b).



The associated direction field shows the direction of motion along the trajectories.



$22(a)$. The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-6\,x + x^3}{6\,y},$$

which can also be written as $(6\,x - x^3)dx + 6\,ydy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = 6\,x - x^3 \text{ and } \frac{\partial H}{\partial y} = 6\,y\,.$$

Integrating the first equation, we have $H(x\,,y) = 3x^2 - x^4/4 + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f\,'(y).$$

Comparing the two partial derivatives, we conclude that $f(y) = 3y^2 + c$. Hence

$$H(x\,,y) = 3x^2 - \frac{x^4}{4} + 3y^2.$$

(b).