Section 6.4

2. Let $h(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + 2\,Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 2s\,Y(s) + 2\,Y(s) - 1 = \mathcal{L}[h(t)].$$

The forcing function can be written as $h(t) = u_\pi(t) - u_{2\pi}(t)$. Its transform is

$$\mathcal{L}[h(t)] = \frac{e^{-\pi s} - e^{-2\pi s}}{s}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+2s+2)}.$$

First note that

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}.$$

Using partial fractions,

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{(s+1)+1}{(s+1)^2+1}.$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}\left[\frac{1}{s^2+2s+2}\right] = \mathcal{L}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t} sin\, t.$$

Now let

$$G(s) = \frac{1}{s(s^2+2s+2)}.$$

Then

$$\mathcal{L}^{-1}[G(s)] = \frac{1}{2} - \frac{1}{2}e^{-t} cos\, t - \frac{1}{2}e^{-t} sin\, t.$$

Using Theorem 6.3.1 ,

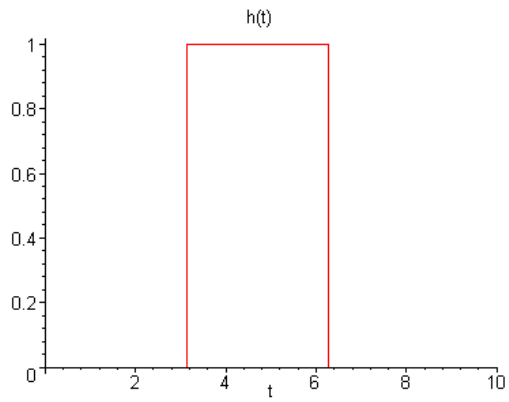
$$\mathcal{L}^{-1}[e^{-cs}G(s)] = \frac{1}{2}u_c(t) - \frac{1}{2}e^{-(t-c)}[cos(t-c) + sin(t-c)]u_c(t).$$

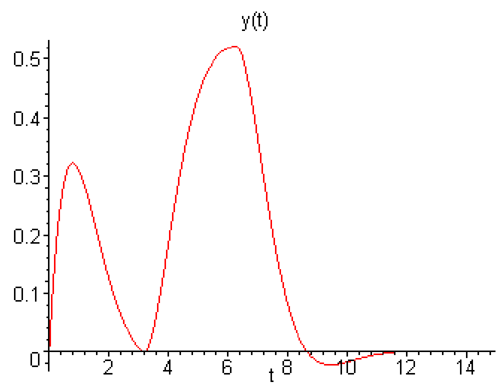
Hence the solution of the IVP is

$$\begin{aligned} y(t) \;&= e^{-t} sin\, t + \frac{1}{2} u_{\pi}(t) - \frac{1}{2} e^{-(t-\pi)} [cos(t-\pi) + sin(t-\pi)] u_{\pi}(t) - \\ &- \frac{1}{2} u_{2\pi}(t) + \frac{1}{2} e^{-(t-2\pi)} [cos(t-2\pi) + sin(t-2\pi)] u_{2\pi}(t) \,. \end{aligned}$$

That is,

$$\begin{aligned} y(t) \;&= e^{-t} sin\, t + \frac{1}{2} [u_{\pi}(t) - u_{2\pi}(t)] + \frac{1}{2} e^{-(t-\pi)} [cos\, t + sin\, t]\, u_{\pi}(t) + \\ &+ \frac{1}{2} e^{-(t-2\pi)} [cos\, t + sin\, t] u_{2\pi}(t) \,. \end{aligned}$$





The solution starts out as free oscillation, due to the initial conditions. The amplitude increases, as long as the forcing is present. Thereafter, the solution rapidly decays.

4. Let $h(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 4\, Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,

$$s^2\,Y(s) + 4\,Y(s) = \mathcal{L}[h(t)].$$

The transform of the forcing function is

$$\mathcal{L}[h(t)] = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{(s^2+4)(s^2+1)} + \frac{e^{-\pi s}}{(s^2+4)(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

It follows that

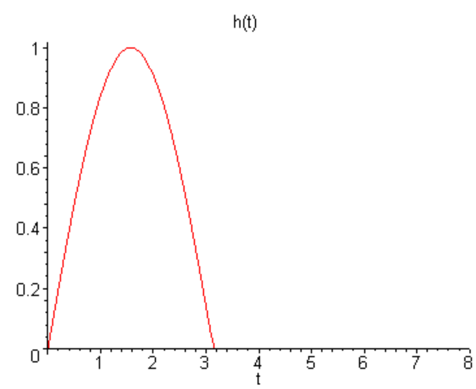
$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[sin\,t - \frac{1}{2}\,sin\,2t\right].$$

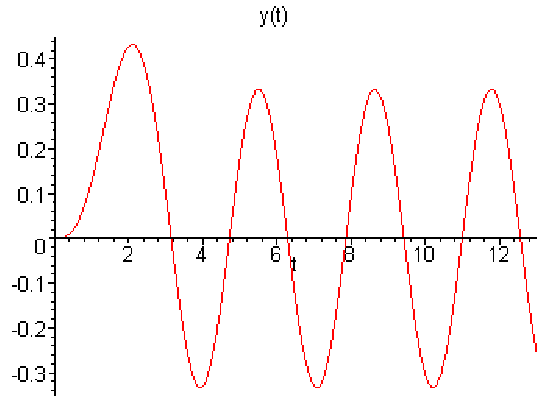
Based on Theorem 6.3.1 ,

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[sin(t-\pi) - \frac{1}{2}\,sin(2t-2\pi)\right]u_\pi(t)\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{3}\left[sin\,t - \frac{1}{2}\,sin\,2t\right] - \frac{1}{3}\left[sin\,t + \frac{1}{2}\,sin\,2t\right]u_\pi(t)\,.$$





Since there is no *damping term*, the solution follows the forcing function, after which the response is a steady oscillation about $y = 0$.

5. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2\,Y(s) = \mathcal{L}[f(t)].$$

Applying the initial conditions,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) = \mathcal{L}[f(t)].$$

The transform of the forcing function is

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}\,.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right].$$

Hence

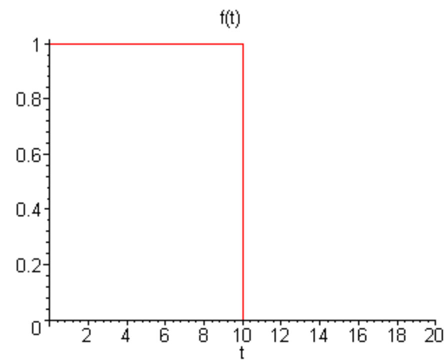
$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}\,.$$

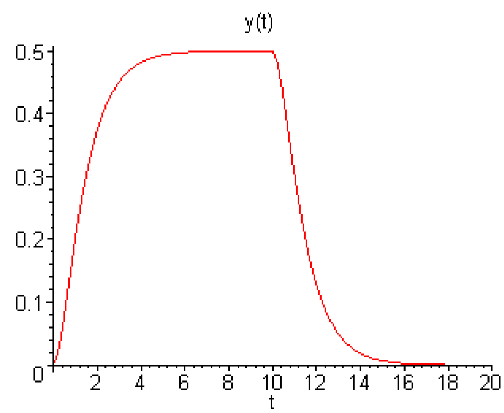
Based on Theorem 6.3.1 ,

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \frac{1}{2}\left[1 + e^{-2(t-10)} - 2e^{-(t-10)}\right]u_{10}(t)\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{2}[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}\left[e^{-(2t-20)} - 2e^{-(t-10)}\right]u_{10}(t)\,.$$





The solution increases to a *temporary* steady value of $y = 1/2$. After the forcing ceases, the response decays exponentially to $y = 0$.

6. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 3[s\, Y(s) - \, y(0)] + 2\, Y(s) = \frac{e^{-2s}}{s}.$$

Applying the initial conditions,

$$s^2\, Y(s) + 3s\, Y(s) + 2\, Y(s) - 1 = \frac{e^{-2s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2 + 3s + 2} + \frac{e^{-2s}}{s(s^2 + 3s + 2)}\,.$$

Using partial fractions,

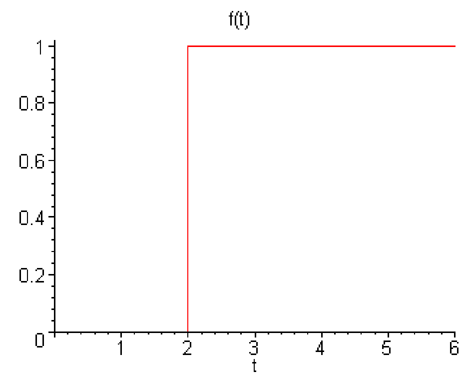
$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$$

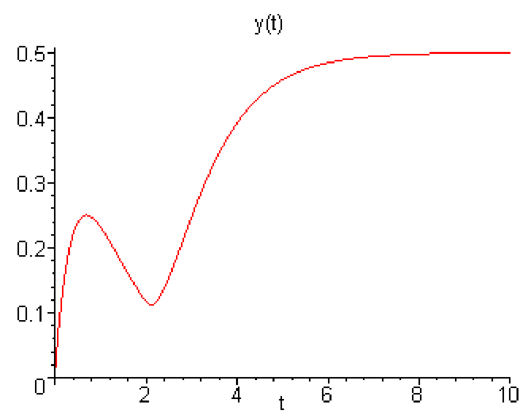
and

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right].$$

Taking the inverse transform. term-by-term, the solution of the IVP is

$$y(t) = e^{-t} - e^{-2t} + \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u_2(t)\,.$$





Due to the initial conditions, the response has a transient *overshoot*, followed by an exponential convergence to a steady value of $y_s = 1/2$.

7. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + Y(s) - s = \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{s^2+1} + \frac{e^{-3\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

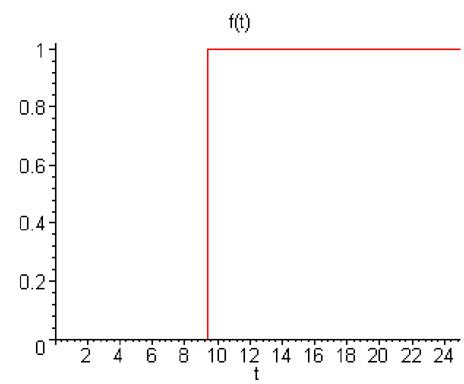
$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

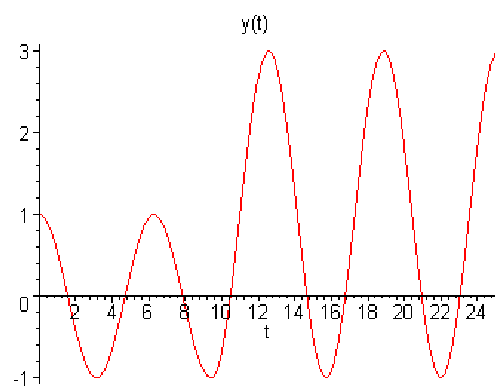
Hence

$$Y(s) = \frac{s}{s^2+1} + e^{-3\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right].$$

Taking the inverse transform, the solution of the IVP is

$$\begin{aligned} y(t) &= cos\,t + [1 - cos(t-3\pi)]u_{3\pi}(t) \\ &= cos\,t + [1 + cos\,t\,]u_{3\pi}(t)\,. \end{aligned}$$





Due to initial conditions, the solution temporarily oscillates about $y = 0$. After the forcing is applied, the response is a steady oscillation about $y_m = 1$.

9. Let $g(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \mathcal{L}[g(t)].$$

Applying the initial conditions,

$$s^2\,Y(s) + Y(s) - 1 = \mathcal{L}[g(t)].$$

The forcing function can be written as

$$\begin{aligned} g(t) &= \frac{t}{2}[1 - u_6(t)] + 3\,u_6(t) \\ &= \frac{t}{2} - \frac{1}{2}(t-6)u_6(t) \end{aligned}$$

with Laplace transform

$$\mathcal{L}[g(t)] = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}\,.$$

Solving for the transform,

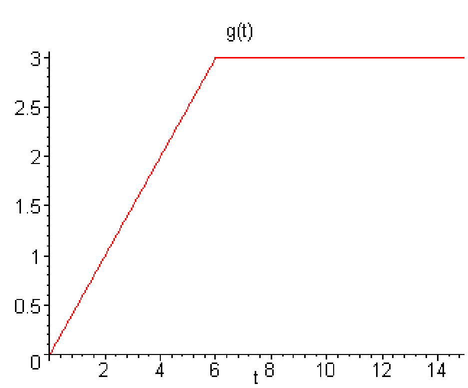
$$Y(s) = \frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}\,.$$

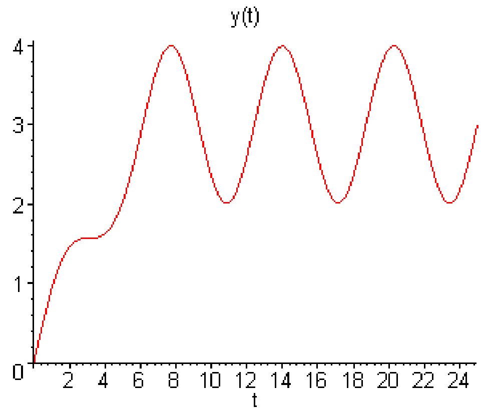
Using partial fractions,

$$\frac{1}{2s^2(s^2+1)} = \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right].$$

Taking the inverse transform, and using Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) &= sin\,t + \frac{1}{2}[t - sin\,t] - \frac{1}{2}[(t-6) - sin(t-6)]u_6(t) \\ &= \frac{1}{2}[t + sin\,t] - \frac{1}{2}[(t-6) - sin(t-6)]u_6(t). \end{aligned}$$





The solution increases, in response to the *ramp input*, and thereafter oscillates about a mean value of $y_m = 3$.

11. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 4\,Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

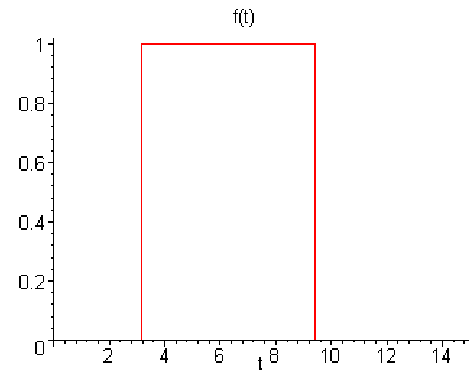
$$Y(s) = \frac{e^{-\pi s}}{s(s^2+4)} - \frac{e^{-3\pi s}}{s(s^2+4)}.$$

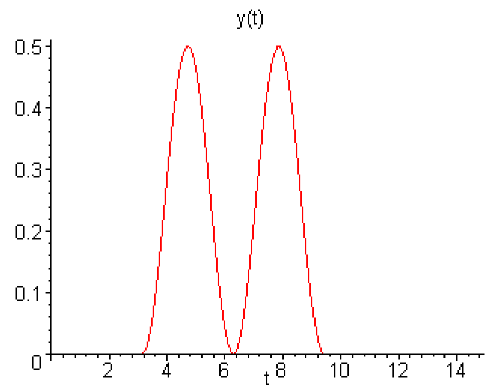
Using partial fractions,

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$$

Taking the inverse transform, and applying Theorem 6.3.1 ,

$$\begin{aligned} y(t) &= \frac{1}{4}[1 - cos(2t-2\pi)]u_\pi(t) - \frac{1}{4}[1 - cos(2t-6\pi)]u_{3\pi}(t) \\ &= \frac{1}{4}[u_\pi(t) - u_{3\pi}(t)] - \frac{1}{4}cos\,2t\cdot[u_\pi(t) - u_{3\pi}(t)]\,. \end{aligned}$$





Since there is no damping term, the solution responds immediately to the forcing input. There is a temporary oscillation about $y = 1/4$.

12. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Applying the *initial conditions*,

$$s^4 Y(s) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{e^{-s}}{s(s^4-1)} - \frac{e^{-2s}}{s(s^4-1)}\,.$$

Using partial fractions,

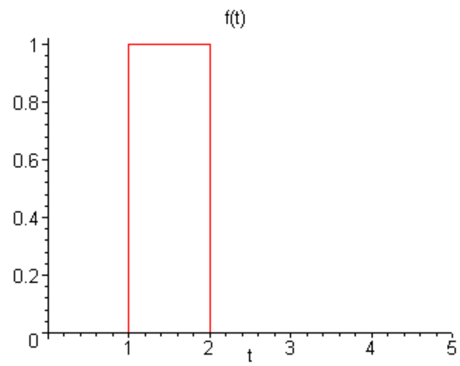
$$\frac{1}{s(s^4-1)} = \frac{1}{4}\left[-\frac{4}{s} + \frac{1}{s+1} + \frac{1}{s-1} + \frac{2s}{s^2+1}\right].$$

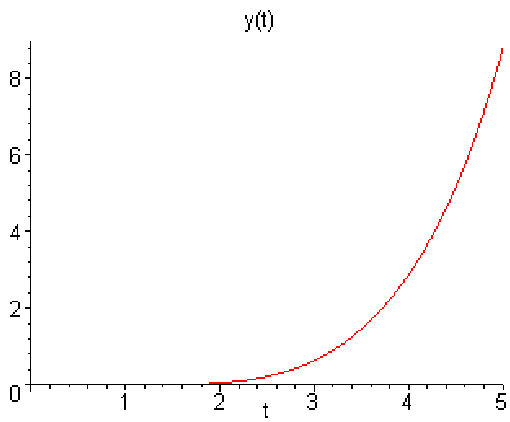
It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^4-1)}\right] = \frac{1}{4}\left[-4+e^{-t}+e^{t}+2\,cos\,t\right].$$

Based on Theorem 6.3.1 , the solution of the IVP is

$$y(t) \;=\; -\left[u_1(t)-u_2(t)\right]+\frac{1}{4}\left[e^{-(t-1)}+e^{(t-1)}+2\,cos(t-1)\right]u_1(t)-$$
$$-\frac{1}{4}\left[e^{-(t-2)}+e^{(t-2)}+2\,cos(t-2)\right]u_2(t)\,.$$





The solution increases without bound, exponentially.

13. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s)-s^3y(0)-s^2y'(0)-s\,y''(0)-y'''(0)+$$
$$+\,5\left[s^2Y(s)-s\,y(0)-y'(0)\right]+4\,Y(s)=\frac{1}{s}-\frac{e^{-\pi s}}{s}.$$

Applying the *initial conditions*,

$$s^4Y(s)+5s^2Y(s)+\;4Y(s)=\frac{1}{s}-\frac{e^{-\pi s}}{s}\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s(s^4+5s^2+4)} - \frac{e^{-\pi s}}{s(s^4+5s^2+4)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^4+5s^2+4)} = \frac{1}{12}\left[\frac{3}{s} + \frac{s}{s^2+4} - \frac{4s}{s^2+1}\right].$$

It follows that

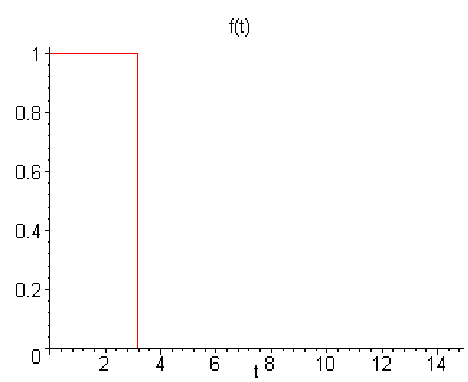
$$\mathcal{L}^{-1}\left[\frac{1}{s(s^4+5s^2+4)}\right] = \frac{1}{12}[3 + cos\,2t - 4\,cos\,t].$$

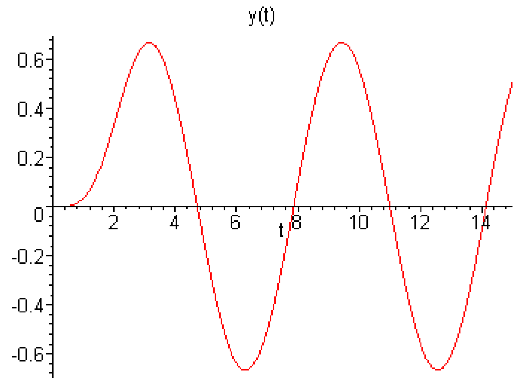
Based on Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) \ &= \frac{1}{4}[1-u_\pi(t)] + \frac{1}{12}[cos\,2t - 4\,cos\,t] - \\ &- \frac{1}{12}[cos\,2(t-\pi) - 4\,cos(t-\pi)]u_\pi(t)\,. \end{aligned}$$

That is,

$$\begin{aligned} y(t) \ &= \frac{1}{4}[1-u_\pi(t)] + \frac{1}{12}[cos\,2t - 4\,cos\,t] - \\ &- \frac{1}{12}[cos\,2t + 4\,cos\,t]u_\pi(t)\,. \end{aligned}$$





After an initial transient, the solution oscillates about $y_m = 0$.

14. The specified function is defined by

$$f(t) = \begin{cases} 0, & 0 \le t < t_0 \\ \frac{h}{k}(t - t_0), & t_0 \le t < t_0 + k \\ h, & t \ge t_0 + k \end{cases}$$

which can conveniently be expressed as

$$f(t) = \frac{h}{k}(t - t_0)\, u_{t_0}(t) - \frac{h}{k}(t - t_0 - k)\, u_{t_0+k}(t)\,.$$

15. The function is defined by

$$g(t) = \begin{cases} 0, & 0 \le t < t_0 \\ \frac{h}{k}(t - t_0), & t_0 \le t < t_0 + k \\ -\frac{h}{k}(t - t_0 - 2k), & t_0 + k \le t < t_0 + 2k \\ 0, & t \ge t_0 + 2k \end{cases}$$

which can also be written as

$$g(t) = \frac{h}{k}(t - t_0)\, u_{t_0}(t) - \frac{2h}{k}(t - t_0 - k)\, u_{t_0+k}(t) + \frac{h}{k}(t - t_0 - 2k)\, u_{t_0+2k}(t).$$

16(d). From Part (c), the solution is

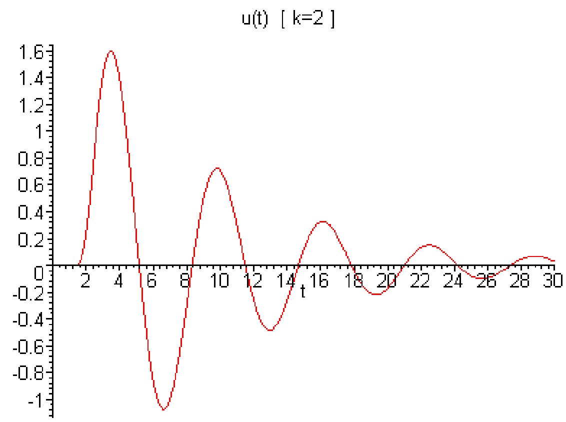
$$u(t) = 4k\, u_{3/2}(t)\, h\left(t - \frac{3}{2}\right) - 4k\, u_{5/2}(t)\, h\left(t - \frac{5}{2}\right),$$

where

$$h(t) = \frac{1}{4} - \frac{\sqrt{7}}{84}\, e^{-t/8} sin\left(\frac{3\sqrt{7}\, t}{8}\right) - \frac{1}{4}\, e^{-t/8} cos\left(\frac{3\sqrt{7}\, t}{8}\right).$$

Due to the *damping term*, the solution will decay to *zero*. The maximum will occur shortly after the forcing ceases. By plotting the various solutions, it appears that the solution will reach a value of $y = 2$, as long as $k > 2.51$.

(e).



Based on the graph, and numerical calculation, $|u(t)| < 0.1$ for $t > 25.6773$.

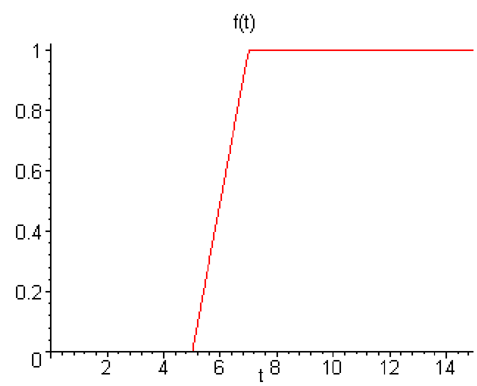
17. We consider the initial value problem

$$y'' + 4y = \frac{1}{k}[(t-5)\,u_5(t) - (t-5-k)\,u_{5+k}(t)],$$

with $y(0) = y'(0) = 0$.

(a). The specified function is defined by

$$f(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{1}{k}(t-5), & 5 \le t < 5+k \\ 1, & t \ge 5+k \end{cases}$$



(b). Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Applying the initial conditions,

$$s^2\,Y(s) + 4\,Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-5s}}{ks^2(s^2+4)} - \frac{e^{-(5+k)s}}{ks^2(s^2+4)}\,.$$

Using partial fractions,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}t - \frac{1}{8}sin\,2t\,.$$

Using Theorem 6.3.1 , the solution of the IVP is

$$y(t) = \frac{1}{k}[h(t-5)\,u_5(t) - h(t-5-k)\,u_{5+k}(t)]\,,$$

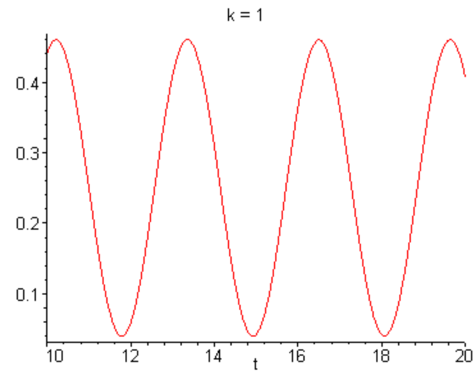
in which $h(t) = \frac{1}{4}t - \frac{1}{8}sin\,2t$.

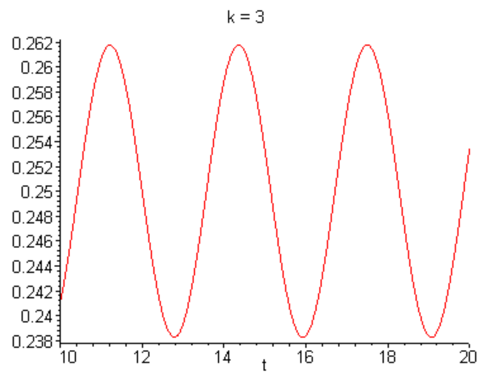
(c). Note that for $t > 5+k$, the solution is given by

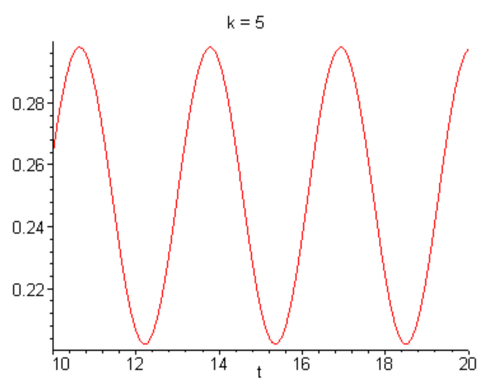
$$\begin{aligned} y(t) &= \frac{1}{4} - \frac{1}{8k}sin(2t-10) + \frac{1}{8k}\,sin(2t-10-2k) \\ &= \frac{1}{4} - \frac{sin\,k}{4k}\,cos(2t-10-k)\,. \end{aligned}$$

So for $t > 5+k$, the solution oscillates about $y_m = 1/4$, with an amplitude of

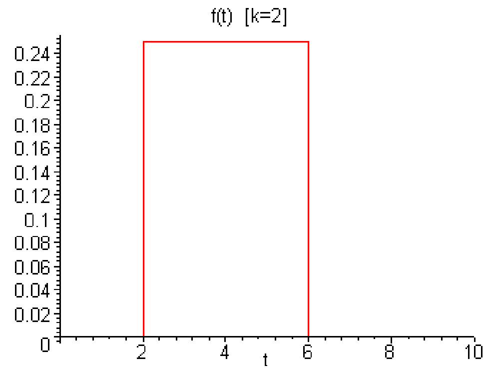
$$A = \frac{|sin(k)|}{4k}\,.$$


k = 1
0.4
0.3
0.2
0.1
10
12
14
t
16
18
20


k = 3
0.262
0.26
0.258
0.256
0.254
0.252
0.25
0.248
0.246
0.244
0.242
0.24
0.238
10
12
14
t
16
18
20


k = 5
0.28
0.26
0.24
0.22
10
12
14
t
16
18
20

$18(a)$.



(b). The forcing function can be expressed as

$$f_k(t) = \frac{1}{2k}[u_{4-k}(t) - u_{4+k}(t)].$$

Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \frac{1}{3}[s\,Y(s) - \,y(0)] + 4\,Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Applying the initial conditions,

$$s^2\,Y(s) + \frac{1}{3}s\,Y(s) + 4\,Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Solving for the transform,

$$Y(s) = \frac{3\,e^{-(4-k)s}}{2ks(3s^2+s+12)} - \frac{3\,e^{-(4+k)s}}{2ks(3s^2+s+12)}\,.$$

Using partial fractions,

$$\begin{aligned}\frac{1}{s(3s^2+s+12)} &= \frac{1}{12}\left[\frac{1}{s} - \frac{1+3s}{3s^2+s+12}\right]\\ &= \frac{1}{12}\left[\frac{1}{s} - \frac{1}{6}\,\frac{1+6\left(s+\frac{1}{6}\right)}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right].\end{aligned}$$

Let

$$H(s) = \frac{1}{8k}\left[\frac{1}{s} - \frac{\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}} - \frac{s+\frac{1}{6}}{\left(s+\frac{1}{6}\right)^2+\frac{143}{36}}\right].$$

It follows that

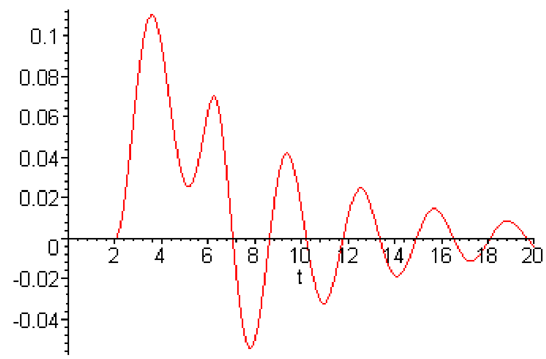
$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{1}{8k} - \frac{e^{-t/6}}{8k}\left[\frac{1}{\sqrt{143}}\, sin\left(\frac{\sqrt{143}\,t}{6}\right) + cos\left(\frac{\sqrt{143}\,t}{6}\right)\right].$$

Based on Theorem 6.3.1 , the solution of the IVP is

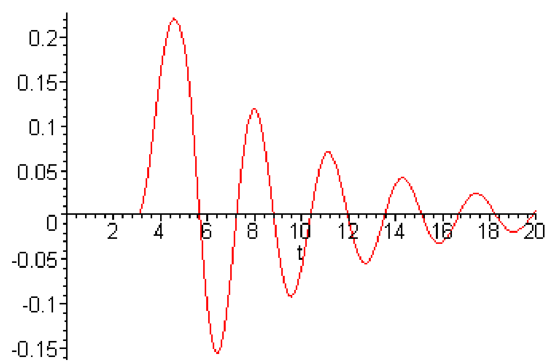
$$y(t) = h(t-4+k)\,u_{4-k}(t) - h(t-4-k)\,u_{4+k}(t)\,.$$

(c).

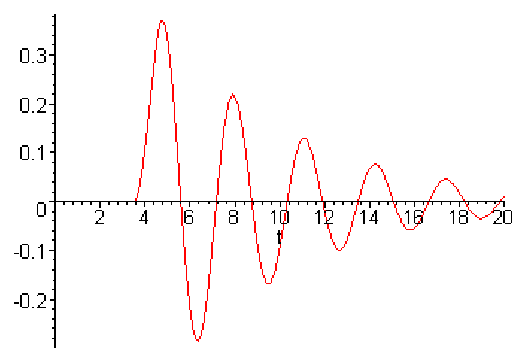
k = 2



k = 1



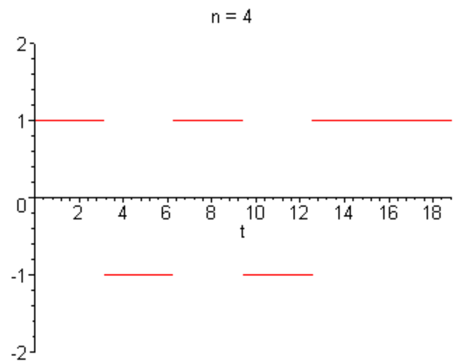
k = 1/2



As the parameter k decreases, the solution remains *null* for a longer period of time.

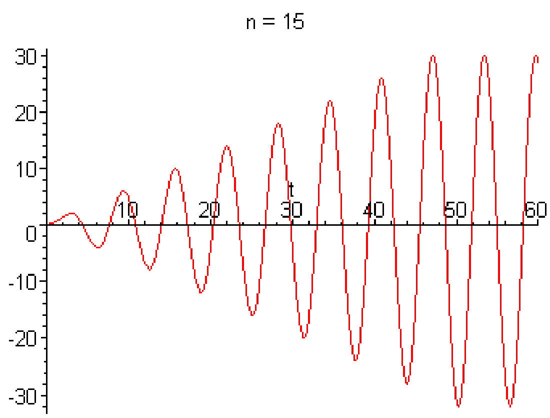
Since the *magnitude* of the impulsive force *increases*, the initial *overshoot* of the response also increases. The *duration* of the impulse decreases. All solutions eventually decay to $y = 0$.

$19(a)$.

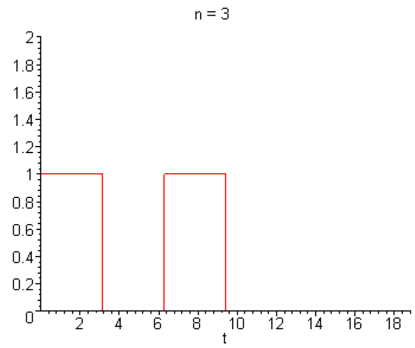


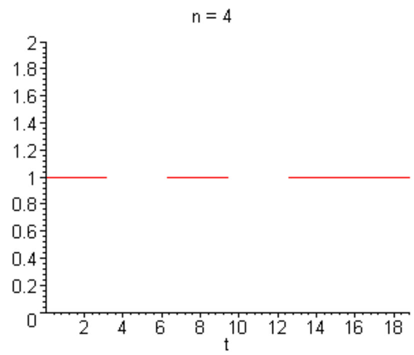
(c). From Part (b),

$$u(t) = 1 - cos\, t + 2\sum_{k=1}^{n}(-1)^k[1 - cos(t - k\pi)]u_{k\pi}(t)\,.$$



$21(a)$.





(b). Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,U(s) - s\,u(0) - u'(0) + U(s) = \frac{1}{s} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s}.$$

Applying the initial conditions,

$$s^2\,U(s) + U(s) = \frac{1}{s} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2+1)} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s(s^2+1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - cos\,t\,.$$

Applying Theorem 6.3.1 , term-by-term, the solution of the IVP is

$$u(t) = h(t) + \sum_{k=1}^{n}(-1)^k h(t-k\pi)\,u_{k\pi}(t)\,.$$

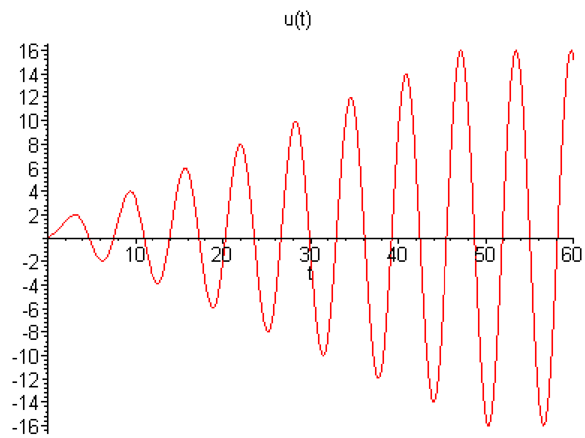
Note that

$$\begin{aligned} h(t-k\pi) &= u_0(t-k\pi) - cos(t-k\pi) \\ &= u_{k\pi}(t) - (-1)^k cos\,t\,. \end{aligned}$$

Hence

$$u(t) = 1 - cos\,t + \sum_{k=1}^{n}(-1)^k\,u_{k\pi}(t) - (cos\,t)\sum_{k=1}^{n}u_{k\pi}(t)\,.$$

(c).



The ODE has no *damping term*. Each interval of forcing adds to the energy of the system.
Hence the amplitude will increase. For $n = 15$, $g(t) = 0$ when $t > 15\pi$. Therefore the oscillation will eventually become *steady*, with an amplitude depending on the values of $u(15\pi)$ and $u'(15\pi)$.

(d). As n increases, the interval of forcing also increases. Hence the amplitude of the transient will increase with n . Eventually, the forcing function will be *constant*. In fact, for *large* values of t ,

$$g(t) = \begin{cases} 1\,, & n \text{ even} \\ 0\,, & n \text{ odd} \end{cases}$$

Further, for $t > n\pi$,

$$u(t) = 1 - \cos t - n \cos t - \frac{1 - (-1)^n}{2}\,.$$

Hence the steady state solution will oscillate about 0 or 1, depending on n, with an amplitude of $A = n + 1$.

In the limit, as $n \to \infty$, the forcing function will be a periodic function, with period 2π. From Prob. 27, in Section 6.3,

$$\mathcal{L}[g(t)] = \frac{1}{s(1 + e^{-s})}\,.$$

As n increases, the duration and magnitude of the transient will increase without bound.

$22(a)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,U(s) + 0.1\,s\,U(s) + U(s) = \frac{1}{s} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2 + 0.1s + 1)} + \sum_{k=1}^{n} \frac{(-1)^k e^{-k\pi s}}{s(s^2 + 0.1s + 1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 0.1s + 1)} = \frac{1}{s} - \frac{s + 0.1}{s^2 + 0.1s + 1}\,.$$

Since the denominator in the second term is irreducible, write

$$\frac{s + 0.1}{s^2 + 0.1s + 1} = \frac{(s + 0.05) + 0.05}{(s + 0.05)^2 + (399/400)}\,.$$

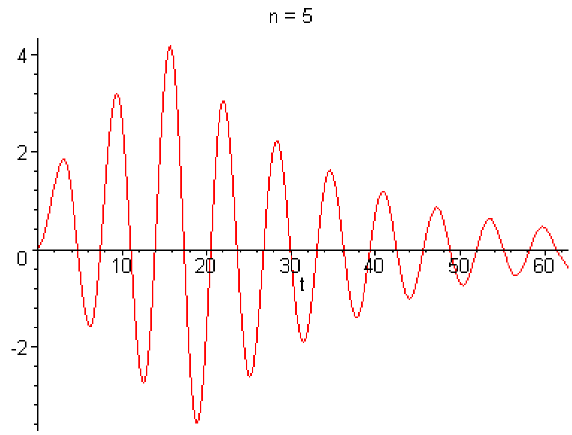
Let

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s + 0.05)}{(s + 0.05)^2 + (399/400)} - \frac{0.05}{(s + 0.05)^2 + (399/400)}\right] \\ &= 1 - e^{-t/20}\left[\cos\left(\frac{\sqrt{399}}{20}\,t\right) + \frac{1}{\sqrt{399}}\sin\left(\frac{\sqrt{399}}{20}\,t\right)\right]. \end{aligned}$$

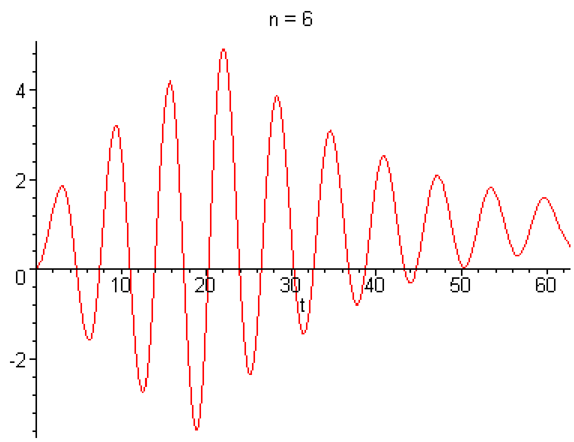
Applying Theorem 6.3.1, term-by-term, the solution of the IVP is

$$u(t) = h(t) + \sum_{k=1}^{n} (-1)^k h(t - k\pi)\, u_{k\pi}(t)\,.$$

For *odd* values of n, the solution approaches $y = 0$.



For *even* values of n, the solution approaches $y = 1$.



(b). The solution is a sum of *damped sinusoids*, each of frequency $\omega = \sqrt{399}/20 \approx 1$. Each term has an 'initial' amplitude of approximately 1. For any given n, the solution contains $n+1$ such terms. Although the amplitude will *increase* with n, the amplitude will also be bounded by $n+1$.

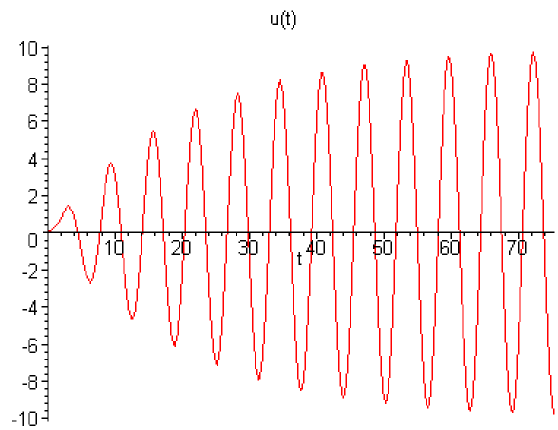
(c). Suppose that the forcing function is replaced by $g(t) = sin\, t$. Based on the methods
in Chapter 3, the general solution of the differential equation is

$$u(t) = e^{-t/20}\left[c_1\, cos\left(\frac{\sqrt{399}}{20}\, t\right) + c_2\, sin\left(\frac{\sqrt{399}}{20}\, t\right)\right] + u_p(t)\,.$$

Note that $u_p(t) = A\, cos\, t + B\, sin\, t$. Using the method of *undetermined coefficients*, $A = -10$ and $B = 0$. Based on the initial conditions, the solution of the IVP is

$$u(t) = 10\,e^{-t/20}\left[cos\left(\frac{\sqrt{399}}{20}\,t\right) + \frac{1}{\sqrt{399}}sin\left(\frac{\sqrt{399}}{20}\,t\right)\right] - 10\,cos\,t\,.$$

Observe that both solutions have the same frequency, $\omega = \sqrt{399}/20 \approx 1$.



$23(a)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,U(s) + U(s) = \frac{1}{s} + 2\sum_{k=1}^{n}\frac{(-1)^k e^{-(11k/4)s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n}\frac{(-1)^k e^{-(11k/4)s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - cos\,t\,.$$

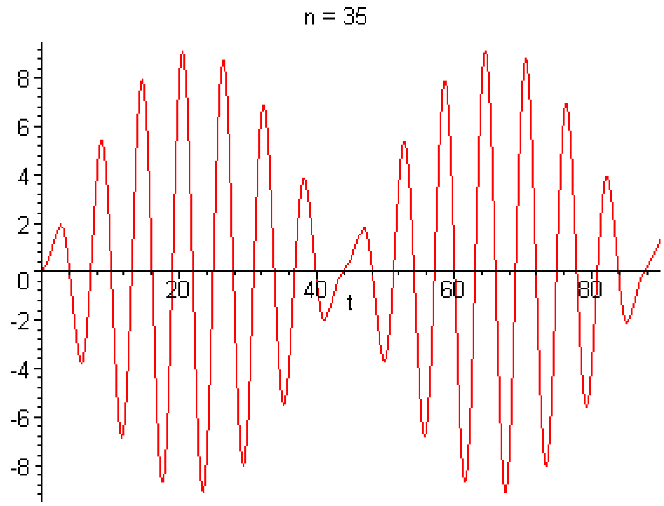
Applying Theorem 6.3.1 , term-by-term, the solution of the IVP is

$$u(t) = h(t) + 2\sum_{k=1}^{n}(-1)^k h\left(t - \frac{11k}{4}\right)u_{11k/4}(t)\,.$$

That is,

$$u(t) = 1 - cos\,t + 2\sum_{k=1}^{n}(-1)^k\left[1 - cos\left(t - \frac{11k}{4}\right)\right]u_{11k/4}(t)\,.$$

(b).



(c). Based on the plot, the '*slow period*' appears to be 88. The '*fast period*' appears to be about 6. These values correspond to a '*slow frequency*' of $\omega_s = 0.0714$ and a '*fast frequency*' $\omega_f = 1.0472$.

(d). The natural frequency of the system is $\omega_0 = 1$. The forcing function is initially periodic, with period $T = 11/2 = 5.5$. Hence the corresponding forcing frequency is $w = 1.1424$. Using the results in Section 3.9, the '*slow frequency*' is given by

$$\omega_s = \frac{|\omega - \omega_0|}{2} = 0.0712$$

and the '*fast frequency*' is given by

$$\omega_f = \frac{|\omega + \omega_0|}{2} = 1.0712\,.$$

Based on theses values, the '*slow period*' is predicted as 88.247 and the '*fast period*' is given as 5.8656.