Chapter Seven

Section 7.1

1. Introduce the variables $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$\begin{aligned} x_2' &= u'' \\ &= -2u - 0.5\,u'. \end{aligned}$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5\,x_2\,. \end{aligned}$$

3. First divide both sides of the equation by t^2, and write

$$u'' = -\frac{1}{t}\,u' - \left(1 - \frac{1}{4t^2}\right)u\,.$$

Set $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$\begin{aligned} x_2' &= u'' \\ &= -\frac{1}{t}\,u' - \left(1 - \frac{1}{4t^2}\right)u\,. \end{aligned}$$

We obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\left(1 - \frac{1}{4t^2}\right)x_1 - \frac{1}{t}\,x_2\,. \end{aligned}$$

6. One of the ways to transform the system is to assign the variables

$$y_1 = x_1\ ,\ \ y_2 = x_1'\ ,\ \ y_3 = x_2\ ,\ \ y_4 = x_2'\,.$$

Before proceeding, note that

$$\begin{aligned} x_1'' &= \frac{1}{m_1}[-(k_1 + k_2)x_1 + k_2x_2 + F_1(t)] \\ x_2'' &= \frac{1}{m_2}[k_2x_1 - (k_2 + k_3)x_2 + F_2(t)]\,. \end{aligned}$$

Differentiating the new variables, we obtain the system of four first order equations

$$\begin{aligned}
y_1' &= y_2 \\
y_2' &= \frac{1}{m_1}[-(k_1+k_2)y_1 + k_2 y_3 + F_1(t)] \\
y_3' &= y_4 \\
y_4' &= \frac{1}{m_2}[k_2 y_1 - (k_2+k_3)y_3 + F_2(t)]\,.
\end{aligned}$$

$7(a)$. Solving the *first* equation for x_2, we have $x_2 = x_1' + 2x_1$. Substitution into the second equation results in

$$(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)\,.$$

That is, $x_1'' + 4\,x_1' + 3\,x_1 = 0$. The resulting equation is a second order differential equation with *constant coefficients*. The general solution is

$$x_1(t) = c_1 e^{-t} + c_2 e^{-3t}\,.$$

With x_2 given in terms of x_1, it follows that

$$x_2(t) = c_1 e^{-t} - c_2 e^{-3t}\,.$$

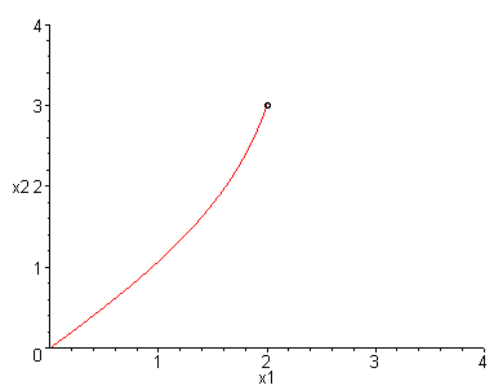
(b). Imposing the specified initial conditions, we obtain

$$\begin{aligned}
c_1 + c_2 &= 2 \\
c_1 - c_2 &= 3\,,
\end{aligned}$$

with solution $c_1 = 5/2$ and $c_2 = -1/2$. Hence

$$x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \text{ and } x_2(t) = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t}\,.$$

(c).



10. Solving the *first* equation for x_2, we obtain $x_2 = (x_1 - x_1')/2$. Substitution into

the second equation results in

$$(x_1 - x_1')'/2 = 3\,x_1 - 2(x_1 - x_1')\,.$$

Rearranging the terms, the single differential equation for x_1 is

$$x_1'' + 3\,x_1' + 2\,x_1 = 0\,.$$

The general solution is

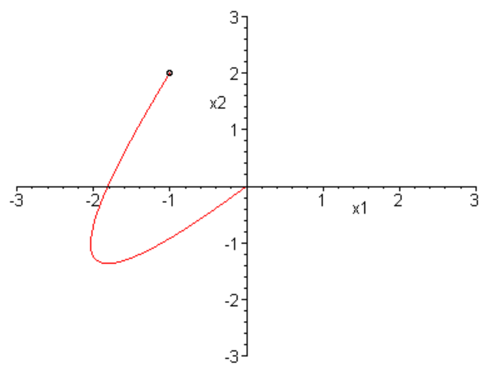
$$x_1(t) = c_1 e^{-t} + c_2 e^{-2t}\,.$$

With x_2 given in terms of x_1, it follows that

$$x_2(t) = c_1 e^{-t} + \frac{3}{2} c_2 e^{-3t}\,.$$

Invoking the specified *initial conditions*, $c_1 = -7$ and $c_2 = 6$. Hence

$$x_1(t) = -7e^{-t} + 6\,e^{-2t} \text{ and } x_2(t) = -7e^{-t} + 9\,e^{-3t}\,.$$



11. Solving the *first* equation for x_2, we have $x_2 = x_1'/2$. Substitution into the second equation results in

$$x_1''/2 = -2\,x_1\,.$$

The resulting equation is $x_1'' + 4\,x_1 = 0$, with general solution

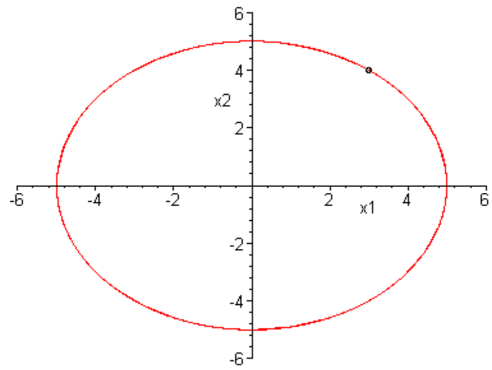
$$x_1(t) = c_1 cos\,2t + c_2 sin\,2t\,.$$

With x_2 given in terms of x_1, it follows that

$$x_2(t) = -c_1 sin\,2t + c_2\,cos\,2t\,.$$

Imposing the specified initial conditions, we obtain $c_1 = 3$ and $c_2 = 4$. Hence

$$x_1(t) = 3\,cos\,2t + 4sin\,2t \text{ and } x_2(t) = -3sin\,2t + 4cos\,2t\,.$$



12. Solving the *first* equation for x_2, we obtain $x_2 = x_1'/2 + x_1/4$. Substitution into the second equation results in

$$x_1''/2 + x_1'/4 = -2x_1 - (x_1'/2 + x_1/4)/2.$$

Rearranging the terms, the single differential equation for x_1 is

$$x_1'' + x_1' + \frac{17}{4}x_1 = 0.$$

The general solution is

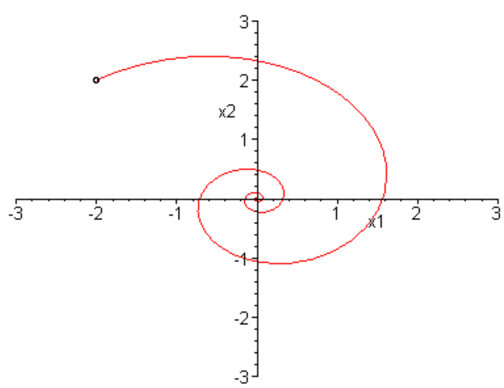
$$x_1(t) = e^{-t/2}[c_1 cos\, 2t + c_2 sin\, 2t].$$

With x_2 given in terms of x_1, it follows that

$$x_2(t) = e^{-t/2}[-c_1 cos\, 2t + c_2 sin\, 2t].$$

Imposing the specified initial conditions, we obtain $c_1 = -2$ and $c_2 = 2$. Hence

$$x_1(t) = e^{-t/2}[-2cos\, 2t + 2sin\, 2t] \text{ and } x_2(t) = e^{-t/2}[2cos\, 2t + 2sin\, 2t].$$



13. Solving the *first* equation for V, we obtain $V = L \cdot I'$. Substitution into the second equation results in

$$L \cdot I'' = -\frac{I}{C} - \frac{L}{RC} I' .$$

Rearranging the terms, the single differential equation for I is

$$LRC \cdot I'' + L \cdot I' + R \cdot I = 0 .$$

15. Direct substitution results in

$$\begin{aligned}(c_1 x_1(t) + c_2 x_2(t))' &= p_{11}(t)[c_1 x_1(t) + c_2 x_2(t)] + p_{12}(t)[c_1 y_1(t) + c_2 y_2(t)] \\ (c_1 y_1(t) + c_2 y_2(t))' &= p_{21}(t)[c_1 x_1(t) + c_2 x_2(t)] + p_{22}(t)[c_1 y_1(t) + c_2 y_2(t)] .\end{aligned}$$

Expanding the left-hand-side of the *first* equation,

$$\begin{aligned}c_1 x_1'(t) + c_2 x_2'(t) &= c_1[p_{11}(t) x_1(t) + p_{12}(t) y_1(t)] + \\ &+ c_2[p_{11}(t) x_2(t) + p_{12}(t) y_2(t)] .\end{aligned}$$

Repeat with the second equation to show that the system of ODEs is identically satisfied.

16. Based on the hypothesis,

$$\begin{aligned}x_1'(t) &= p_{11}(t) x_1(t) + p_{12}(t) y_1(t) + g_1(t) \\ x_2'(t) &= p_{11}(t) x_2(t) + p_{12}(t) y_2(t) + g_1(t) .\end{aligned}$$

Subtracting the two equations,

$$x_1'(t) - x_2'(t) = p_{11}(t)[x_1'(t) - x_2'(t)] + p_{12}(t)[y_1'(t) - y_2'(t)] .$$

Similarly,

$$y_1'(t) - y_2'(t) = p_{21}(t)[x_1'(t) - x_2'(t)] + p_{22}(t)[y_1'(t) - y_2'(t)] .$$

Hence the *difference* of the two solutions satisfies the *homogeneous* ODE.

17. For *rectilinear motion* in one dimension, Newton's second law can be stated as

$$\sum F = m\, x'' .$$

The $resisting$ force exerted by a linear spring is given by $F_s = k\,\delta$, in which δ is the *displacement* of the end of a spring from its equilibrium configuration. Hence, with $0 < x_1 < x_2$, the first two springs are in *tension*, and the last spring is in *compression*. The *sum* of the spring forces on m_1 is

$$F_s^1 = -k_1 x_1 - k_2(x_2 - x_1) .$$

The *total* force on m_1 is

$$\sum F^1 = -k_1 x_1 + k_2(x_2 - x_1) + F_1(t)\,.$$

Similarly, the *total* force on m_2 is

$$\sum F^2 = -k_2(x_2 - x_1) - k_3 x_2 + F_2(t)\,.$$

$18(a)$. Taking a *clockwise* loop around each of the paths, it is easy to see that voltage drops are given by $V_1 - V_2 = 0$, and $V_2 - V_3 = 0$.

(b). Consider the *right node*. The *current in* is given by $I_1 + I_2$. The current *leaving* the node is $-I_3$. Hence the current passing through the node is $(I_1 + I_2) - (-I_3)$. Based on Kirchhoff's first law, $I_1 + I_2 + I_3 = 0$.

(c). In the capacitor,

$$C\,V_1{}' = I_1.$$

In the resistor,

$$V_2 = R\,I_2\,.$$

In the inductor,

$$L\,I_3' = V_3\,.$$

(d). Based on part (a), $V_3 = V_2 = V_1$. Based on part (b),

$$C\,V_1{}' + \frac{1}{R}V_2 + I_3 = 0\,.$$

It follows that

$$C\,V_1{}' = -\frac{1}{R}V_1 - I_3 \text{ and } L\,I_3' = V_1.$$

20. Let $I_1, I_2, I_3,$ and I_4 be the current through the resistors, inductor, and capacitor, respectively. Assign $V_1, V_2, V_3,$ and V_4 as the respective voltage drops. Based on Kirchhoff's second law, the net voltage drops, around each loop, satisfy

$$V_1 + V_3 + V_4 = 0\,,\; V_1 + V_3 + V_2 = 0 \text{ and } V_4 - V_2 = 0\,.$$

Applying Kirchhoff's first law to the upper-right node,

$$I_3 - (I_2 + I_4) = 0\,.$$

Likewise, in the remaining nodes,

$$I_1 - I_3 = 0 \text{ and } I_2 + I_4 - I_1 = 0\,.$$

That is,

$$V_4 - V_2 = 0\,,\; V_1 + V_3 + V_4 = 0 \text{ and } I_2 + I_4 - I_3 = 0\,.$$

Using the current-voltage relations,

$$V_1 = R_1 I_1,\; V_2 = R_2 I_2,\; L\,I_3' = V_3\,,\; C\,V_4' = I_4.$$

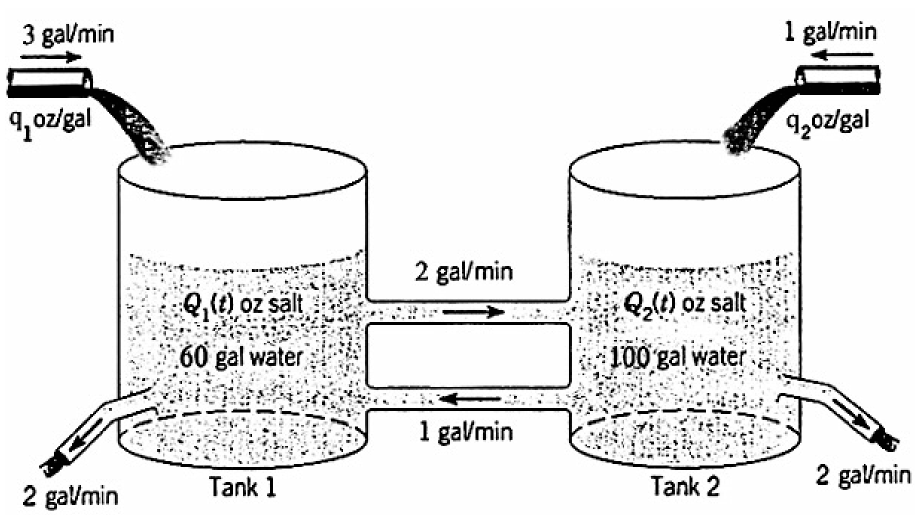
Combining these equations,

$$R_1 I_3 + L\,I_3' + V_4 = 0 \text{ and } C\,V_4' = I_3 - \frac{V_4}{R_2}.$$

Now set $I_3 = I$ and $V_4 = V$, to obtain the system of equations

$$L\,I' = -R_1 I - V \quad \text{and} \quad C\,V' = I - \frac{V}{R_2}.$$

$22(a)$.



Let $Q_1(t)$ and $Q_2(t)$ be the *amount* of salt in the respective tanks at time t. Note that the *volume* of each tank remains constant. Based on conservation of mass, the *rate of increase* of salt, in any given tank, is given by

$$rate\,of\;increase = rate\,in\; - \;rate\,out\,.$$

For Tank 1, the rate of salt flowing *into* Tank 1 is

$$\begin{aligned} r_{in} &= \left[q_1 \frac{oz}{gal}\right]\left[3\frac{gal}{min}\right] + \left[\frac{Q_2}{100}\frac{oz}{gal}\right]\left[1\frac{gal}{min}\right] \\ &= 3\,q_1 + \frac{Q_2}{100}\ \frac{oz}{min}\,. \end{aligned}$$

The rate at which salt flow *out* of Tank 1 is

$$r_{out} = \left[\frac{Q_1}{60}\frac{oz}{gal}\right]\left[4\frac{gal}{min}\right] = \frac{Q_1}{15}\frac{oz}{min}\,.$$

Hence

$$\frac{dQ_1}{dt} = 3\,q_1 + \frac{Q_2}{100} - \frac{Q_1}{15}\,.$$

Similarly, for Tank 2,

$$\frac{dQ_2}{dt} = q_2 + \frac{Q_1}{30} - \frac{3Q_2}{100}\,.$$

The process is modeled by the system of equations

$$\begin{aligned} Q_1' &= -\frac{Q_1}{15} + \frac{Q_2}{100} + 3\,q_1 \\ Q_2' &= \frac{Q_1}{30} - \frac{3Q_2}{100} + q_2\,. \end{aligned}$$

The initial conditions are $Q_1(0) = Q_1^0$ and $Q_2(0) = Q_2^0$.

(b). The *equilibrium values* are obtain by solving the system

$$\begin{aligned} -\frac{Q_1}{15} + \frac{Q_2}{100} + 3\,q_1 &= 0 \\ \frac{Q_1}{30} - \frac{3Q_2}{100} + q_2 &= 0\,. \end{aligned}$$

Its solution leads to $Q_1^E = 54\,q_1 + 6\,q_2$ and $Q_2^E = 60\,q_1 + 40\,q_2$.

(c). The question refers to possible solution of the system

$$\begin{aligned} 54\,q_1 + 6\,q_2 &= 60 \\ 60\,q_1 + 40\,q_2 &= 50\,. \end{aligned}$$

It is possible for formally solve the system of equations, but the unique solution gives

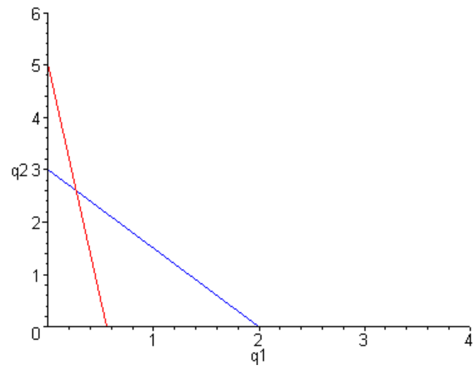
$$q_1 = \frac{7}{6}\,\frac{oz}{gal} \ \text{ and } \ q_2 = -\frac{1}{2}\,\frac{oz}{gal}\,,$$

which is *not* physically possible.

(d). We can write

$$q_2 = -9\,q_1 + \frac{Q_1^E}{6}$$
$$q_2 = -\frac{3}{2}\,q_1 + \frac{Q_2^E}{40}\,,$$

which are the equations of two lines in the $q_1 q_2$-plane:



The intercepts of the *first* line are $Q_1^E/54$ and $Q_1^E/6$. The intercepts of the *second* line are $Q_2^E/60$ and $Q_2^E/40$. Therefore the system will have a unique solution, in the *first quadrant*, as long as $Q_1^E/54 \le Q_2^E/60$ or $Q_2^E/40 \le Q_1^E/6$. That is,

$$\frac{10}{9} \le \frac{Q_2^E}{Q_1^E} \le \frac{20}{3}\,.$$