Section 10.5

1. We consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substitution into the partial differential equation results in

$$xX''T + XT' = 0\,.$$

Divide both sides of the differential equation by the product XT to obtain

$$x\frac{X''}{X} + \frac{T'}{T} = 0\,,$$

so that

$$x\frac{X''}{X} = -\,\frac{T'}{T}\,.$$

Since both sides of the resulting equation are functions of different variables, each must be equal to a constant, say λ. We obtain the ordinary differential equations

$$xX'' - \lambda X = 0 \text{ and } T' + \lambda T = 0\,.$$

2. In order to apply the method of separation of variables, we consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substituting the assumed form of the solution into the partial differential equation, we obtain

$$t\,X''T + x\,XT' = 0\,.$$

Divide both sides of the differential equation by the product $xtXT$ to obtain

$$\frac{X''}{xX} + \frac{T'}{t\,T} = 0\,,$$

so that

$$\frac{X''}{xX} = -\,\frac{T'}{t\,T}\,.$$

Since both sides of the resulting equation are functions of *different* variables, it follows that

$$\frac{X''}{xX} = -\,\frac{T'}{t\,T} = \lambda\,.$$

Therefore $X(x)$ and $T(t)$ are solutions of the ordinary differential equations

$$X'' - \lambda x\,X = 0 \text{ and } T' + \lambda t\,T = 0\,.$$

4. Assume that the solution of the PDE has the form $u(x\,,t) = X(x)T(t)$. Substitution into the partial differential equation results in

$$[p(x)X']'T - r(x)X\,T'' = 0\,.$$

Divide both sides of the differential equation by the product $r(x)XT$ to obtain

$$\frac{[p(x)X']'}{r(x)X} - \frac{T''}{T} = 0\,,$$

that is,

$$\frac{[p(x)X']'}{r(x)X} = \frac{T''}{T}\,.$$

Since both sides of the resulting equation are functions of different variables, each must be equal to a constant, say $-\lambda$. We obtain the ordinary differential equations

$$[p(x)X']' + \lambda r(x)X = 0 \text{ and } T'' + \lambda T = 0\,.$$

6. We consider solutions of the form $u(x\,,y) = X(x)Y(y)$. Substitution into the partial differential equation results in

$$X''Y + XY'' + xXY = 0\,.$$

Divide both sides of the differential equation by the product XY to obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + x = 0\,,$$

that is,

$$\frac{X''}{X} + x = -\frac{Y''}{Y}\,.$$

Since both sides of the resulting equation are functions of *different* variables, it follows that

$$\frac{X''}{X} + x = -\frac{Y''}{Y} = -\lambda\,.$$

We obtain the ordinary differential equations

$$X'' + (x + \lambda)X = 0 \text{ and } Y'' - \lambda Y = 0\,.$$

7. The heat conduction equation, $100\,u_{xx} = u_t$, and the given boundary conditions are homogeneous. We consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substitution into
the partial differential equation results in

$$100\,X''T = XT'.$$

Divide both sides of the differential equation by the product XT to obtain

$$\frac{X''}{X} = \frac{T'}{100\,T}\,.$$

Since both sides of the resulting equation are functions of *different* variables, it follows that

$$\frac{X''}{X} = \frac{T'}{100\,T} \;=\; -\,\lambda\,.$$

Therefore $X(x)$ and $T(t)$ are solutions of the ordinary differential equations

$$X'' + \lambda\,X = 0 \text{ and } T' + 100\lambda\,T = 0\,.$$

The general solution of the *spatial* equation is $X = c_1 cos\,\lambda^{1/2}x + c_2\,sin\,\lambda^{1/2}x$. In order to satisfy the homogeneous boundary conditions, we require that $c_1 = 0$, and

$$\lambda^{1/2} = n\pi\,.$$

Hence the eigenfunctions are $X_n = sin\,n\pi x$, with associated eigenvalues $\lambda_n = n^2\pi^2$.

We thus obtain the family of equations $T' + 100\lambda_n\,T = 0$. Solution are given by

$$T_n = e^{-100\lambda_n t}.$$

Hence the fundamental solutions of the PDE are

$$u_n(x\,,t) = e^{-100n^2\pi^2 t} sin\,n\pi x\,,$$

which yield the general solution

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\,e^{-100n^2\pi^2 t} sin\,n\pi x\,.$$

Finally, the initial condition $u(x\,,0) = sin\,2\pi x - sin\,5\pi x$ must be satisfied. Therefore is it necessary that

$$\sum_{n=1}^{\infty} c_n\,sin\,n\pi x = sin\,2\pi x - sin\,5\pi x\,.$$

It follows from the *othogonality* conditions that $c_2 = -\,c_5 = 1$, with all other $c_n = 0$. Therefore the solution of the given heat conduction problem is

$$u(x\,,t) = e^{-400\pi^2 t} sin\,2\pi x - e^{-2500\pi^2 t} sin\,5\pi x\,.$$

9. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx} &= u_t\,, & 0 < x < 40\,,\ t > 0\,; \\ u(0\,,t) &= 0\,, & u(40\,,t) = 0\,,\ t > 0\,; \\ u(x\,,0) &= 50\,, & 0 < x < 40\,. \end{aligned}$$

Assume a solution of the form $u(x\,,t) = X(x)T(t)$. Following the procedure in this section, we obtain the eigenfunctions $X_n = sin\,n\pi x/40$, with associated eigenvalues $\lambda_n = n^2\pi^2/1600$. The solutions of the *temporal* equations are

$$T_n = e^{-\lambda_n t}.$$

Hence the general solution of the given problem is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

The coefficients c_n are the *Fourier sine* coefficients of $u(x\,,0) = 50$. That is,

$$\begin{aligned} c_n\ &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L}dx \\ &= \frac{5}{2}\int_0^{40} sin\frac{n\pi x}{40}dx \\ &= 100\,\frac{1-\ cos\,n\pi}{n\pi}\,. \end{aligned}$$

The sine series of the initial condition is

$$50 = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-cos\,n\pi}{n} sin\frac{n\pi x}{40}\,.$$

Therefore the solution of the given heat conduction problem is

$$u(x\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-cos\,n\pi}{n}\ e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

11. Refer to Prob. 9 for the formulation of the problem. In this case, the initial condition is given by

$$u(x\,,0) = \begin{cases} 0\,, & 0 \le x < 10\,, \\ 50\,, & 10 \le x \le 30\,, \\ 0\,, & 30 < x \le 40\,. \end{cases}$$

All other data being the same, the solution of the given problem is

$$u(x\,,t)=\sum_{n=1}^{\infty}c_n\,e^{-n^2\pi^2t/1600}sin\frac{n\pi x}{40}\,.$$

The coefficients c_n are the *Fourier sine* coefficients of $u(x\,,0)$. That is,

$$\begin{aligned}c_n\;&=\frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx\\&=\frac{5}{2}\int_{10}^{30}sin\frac{n\pi x}{40}dx\\&=100\,\frac{cos\frac{n\pi}{4}-cos\frac{3n\pi}{4}}{n\pi}\,.\end{aligned}$$

Therefore the solution of the given heat conduction problem is

$$u(x\,,t)=\frac{100}{\pi}\sum_{n=1}^{\infty}\frac{cos\frac{n\pi}{4}-cos\frac{3n\pi}{4}}{n}\;e^{-n^2\pi^2t/1600}sin\frac{n\pi x}{40}\,.$$

12. Refer to Prob. 9 for the formulation of the problem. In this case, the initial condition is given by

$$u(x\,,0)=x\,,\quad 0<x<40\,.$$

All other data being the same, the solution of the given problem is

$$u(x\,,t)=\sum_{n=1}^{\infty}c_n\,e^{-n^2\pi^2t/1600}sin\frac{n\pi x}{40}\,.$$

The coefficients c_n are the *Fourier sine* coefficients of $u(x\,,0)=x$. That is,

$$\begin{aligned}c_n\;&=\frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx\\&=\frac{1}{20}\int_0^{40}x\,sin\frac{n\pi x}{40}dx\\&=\;-\,80\,\frac{cos\,n\pi}{n\pi}\,.\end{aligned}$$

Therefore the solution of the given heat conduction problem is

$$u(x\,,t)=\frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\;e^{-n^2\pi^2t/1600}sin\frac{n\pi x}{40}\,.$$

13. Substituting $x = 20$, into the solution, we have

$$u(20\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-cos\,n\pi}{n}\;e^{-n^2\pi^2 t/1600}sin\frac{n\pi}{2}\,.$$

We can also write

$$u(20\,,t) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\;e^{-(2k-1)^2\pi^2 t/1600}\,.$$

Therefore,

$$u(20\,,5) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\;e^{-(2k-1)^2\pi^2/320}.$$

Let

$$A_k = \frac{(-1)^{n+1}200}{\pi(2k-1)}e^{-(2k-1)^2\pi^2/320}.$$

It follows that $|A_k| < 0.005$ for $k \geq 9$. So for $n = 2k-1 \geq 17$, the summation is unaffected by additional terms.

For $t = 20$,

$$u(20\,,20) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\;e^{-(2k-1)^2\pi^2/80}.$$

Let

$$A_k = \frac{(-1)^{n+1}200}{\pi(2k-1)}e^{-(2k-1)^2\pi^2/80}.$$

It follows that $|A_k| < 0.003$ for $k \geq 5$. So for $n = 2k-1 \geq 9$, the summation is unaffected by additional terms.

For $t = 80$,

$$u(20\,,80) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\;e^{-(2k-1)^2\pi^2/20}.$$

Let

$$A_k = \frac{(-1)^{n+1}200}{\pi(2k-1)}e^{-(2k-1)^2\pi^2/20}.$$

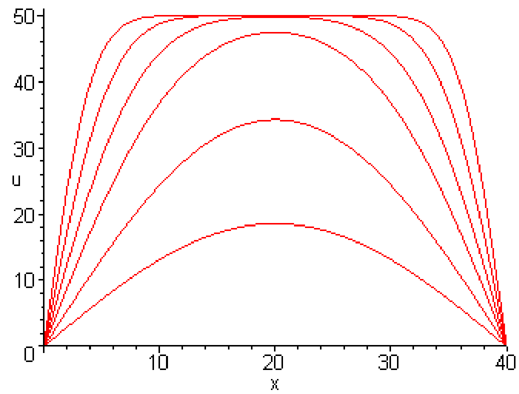
It follows that $|A_k| < 0.00005$ for $k \geq 3$. So for $n = 2k-1 \geq 5$, the summation is unaffected by additional terms.

The series solution converges *faster* as t increases.

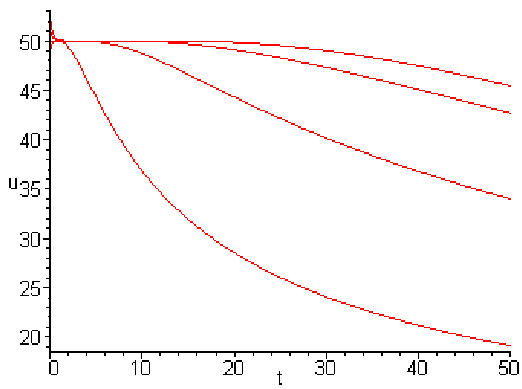
$14(a)$. The solution of the given heat conduction problem is

$$u(x\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\; e^{-n^2\pi^2 t/1600}\sin\frac{n\pi x}{40}\,.$$

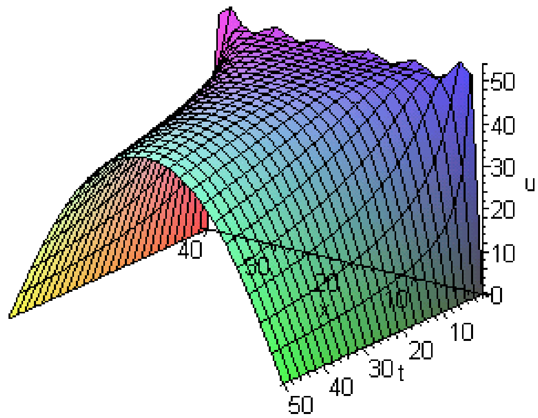
Setting $t = 5, 10, 20, 40, 100, 200$:



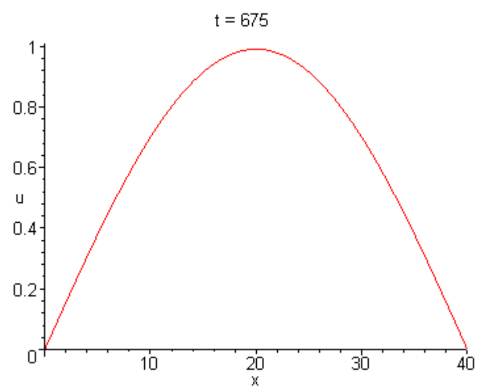
(b). Setting $x = 5, 10, 15, 20$:



(c). Surface plot of $u(x\,,t)$:



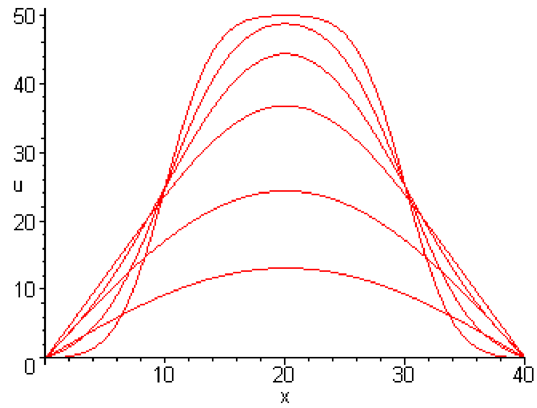
(d). $0 \le u(x\,,t) \le 1$ for $t \ge 675\,sec\,.$



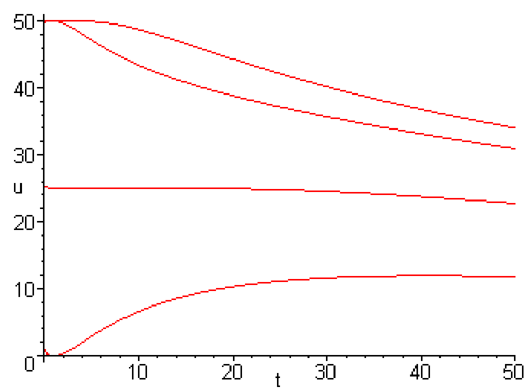
$16(a)$. The solution of the given heat conduction problem is

$$u(x\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{cos\frac{n\pi}{4} - cos\frac{3n\pi}{4}}{n}\; e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

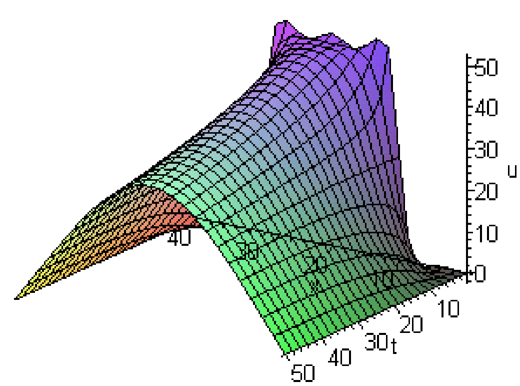
Setting $t = 5, 10, 20, 40, 100, 200$:



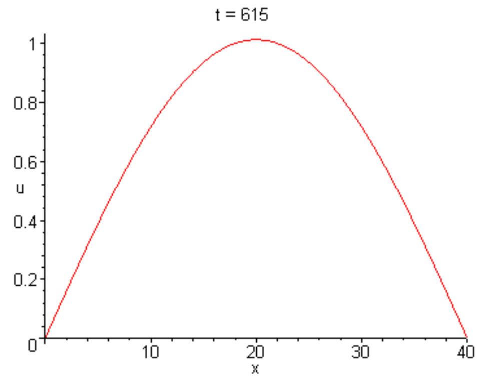
(b). Setting $x = 5, 10, 15, 20$:



(c). Surface plot of $u(x\,,t)$:



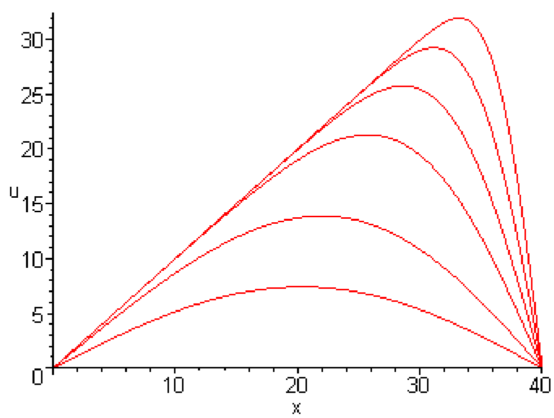
(d). $0 \le u(x\,,t) \le 1$ for $t \ge 615\,sec\,.$



$17(a)$. The solution of the given heat conduction problem is

$$u(x\,,t) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,e^{-n^2\pi^2 t/1600}sin\frac{n\pi x}{40}\,.$$

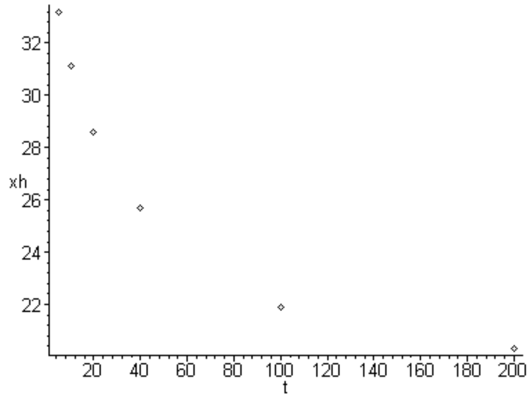
Setting $t = 5, 10, 20, 40, 100, 200$:



(b). Analyzing the individual plots, we find that the 'hot spot' varies with time:

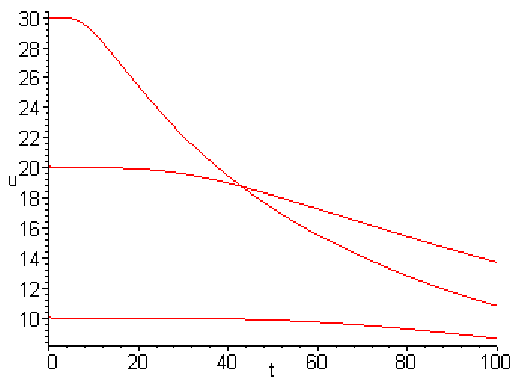
t	5	10	20	40	100	200
x_h	33	31	29	26	22	21

Location of the 'hot spot', x_h , versus *time* :

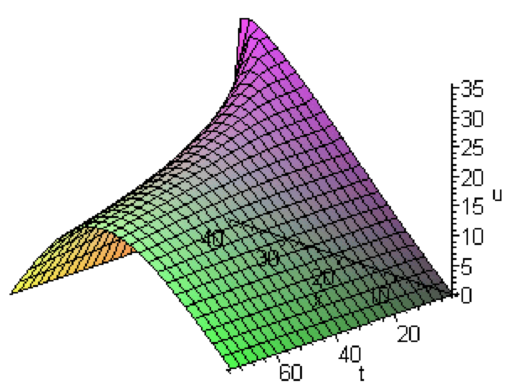


Evidently, the location of the greatest temperature migrates to the center of the rod.

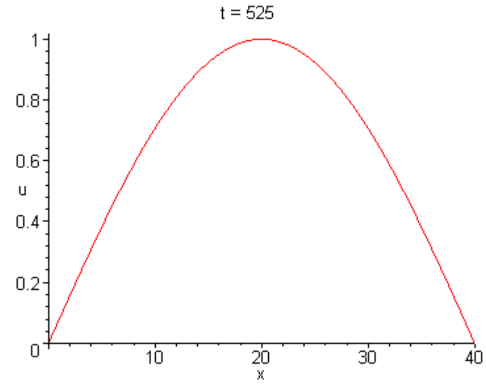
(c). Setting $x = 5, 10, 15, 20$:



(d). Surface plot of $u(x\,,t)$:



(e). $0 \leq u(x\,,t) \leq 1$ for $t \geq 525\,sec$.



19. The solution of the given heat conduction problem is

$$u(x\,,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1 - cos\,n\pi}{n}\,e^{-n^2\pi^2\alpha^2 t/400}sin\frac{n\pi x}{20}\,.$$

Setting $x = 10\,cm$,

$$u(10\,,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1 - cos\,n\pi}{n}\,e^{-n^2\pi^2\alpha^2 t/400}sin\frac{n\pi}{2}\,.$$

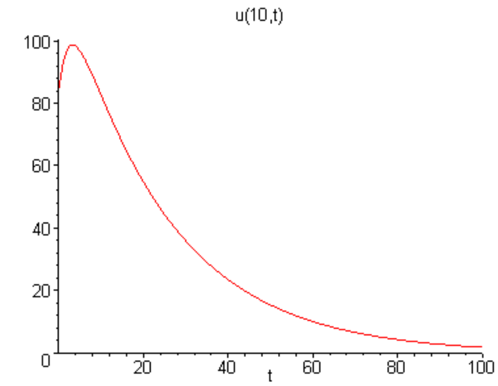
A *two-term* approximation is given by

$$u(10\,,t) \approx \frac{400}{3\pi}\left[3\,e^{-\pi^2\alpha^2 t/400} - e^{-9\pi^2\alpha^2 t/400}\right].$$

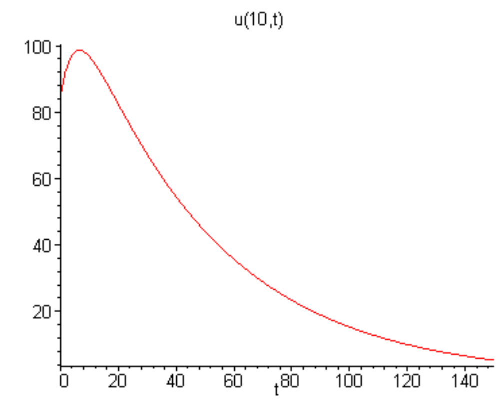
From Table 10.5.1 :

	α^2
silver	1.71
aluminum	0.86
cast iron	0.12

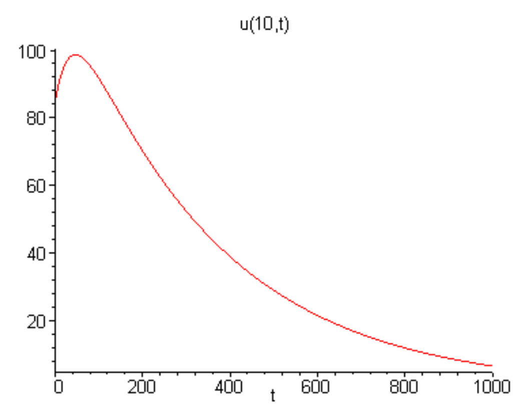
$(a).\ \alpha^2 = 1.71:$



$(b).\ \alpha^2 = 0.86:$



$(c).\ \alpha^2 = 0.12:$



$21(a)$. Given the partial differential equation

$$a\,u_{xx} - b\,u_t + c\,u = 0\,,$$

in which a , b , and c are constants, set $u(x\,,t) = e^{\delta t}w(x\,,t)$. Substitution into the PDE results in

$$a\,e^{\delta t}w_{xx} - b\big(\delta\,e^{\delta t}w + e^{\delta t}w_t\big) + c\,e^{\delta t}w = 0\,.$$

Dividing both sides of the equation by $e^{\delta t}$, we obtain

$$a\,w_{xx} - b\,w_t + (c - b\delta)\,w = 0\,.$$

As long as $b \neq 0$, choosing $\delta = c/b$ yields

$$\frac{a}{b}\,w_{xx} - w_t = 0\,,$$

which is the *heat conduction equation* with dependent variable w .

23. The heat conduction equation in *polar coordinates* is given by

$$\alpha^2\left[u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\theta\theta}\right] = u_t\,.$$

We consider solutions of the form $u(r\,,\theta\,,t) = R(r)\Theta(\theta)T(t)$. Substitution into the PDE
results in

$$\alpha^2\left[R''\Theta T + \frac{1}{r}\,R'\Theta T + \frac{1}{r^2}\,R\Theta'' T\right] = R\Theta T'\,.$$

Dividing both sides of the equation by the factor $R\Theta T$, we obtain

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = \frac{T'}{\alpha^2 T}\,.$$

Since both sides of the resulting differential equation depend on *different* variables, each side must be equal to a constant, say $-\lambda$. That is,

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = \frac{T'}{\alpha^2 T} = -\lambda^2\,.$$

It follows that $T' + \alpha^2\lambda^2 T = 0$, and

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = -\lambda^2\,.$$

Multiplying both sides of this differential equation by r^2, we find that

$$r^2\frac{R''}{R} + r\,\frac{R'}{R} + \frac{\Theta''}{\Theta} = -\lambda^2 r^2\,,$$

which can be written as

$$r^2 \frac{R''}{R} + r\,\frac{R'}{R} + \lambda^2 r^2 = \,-\,\frac{\Theta''}{\Theta}\,.$$

Once again, since both sides of the resulting differential equation depend on *different* variables, each side must be equal to a constant. Hence

$$r^2 \frac{R''}{R} + r\,\frac{R'}{R} + \lambda^2 r^2 = \mu^2 \;\text{ and }\; -\,\frac{\Theta''}{\Theta} = \mu^2.$$

The resulting ordinary equations are

$$\begin{aligned} r^2 R'' + rR' + \big(\lambda^2 r^2 - \mu^2\big)R &= 0 \\ \Theta'' + \mu^2\Theta &= 0 \\ T' + \alpha^2\lambda^2 T &= 0\,. \end{aligned}$$