Section 6.6

$1(a)$. The *convolution integral* is defined as

$$f * g\,(t) = \int_0^t f(t-\tau)g(\tau)d\tau\,.$$

Consider the change of variable $u = t - \tau$. It follows that

$$\begin{aligned}\int_0^t f(t-\tau)g(\tau)d\tau &= \int_t^0 f(u)g(t-u)(-du)\\ &= \int_0^t g(t-u)f(u)du\\ &= g * f\,(t)\,.\end{aligned}$$

(b). Based on the distributive property of the *real numbers*, the convolution is also distributive.

(c). By definition,

$$\begin{aligned}f*(g*h)(t) &= \int_0^t f(t-\tau)[g*h\,(\tau)]d\tau\\ &= \int_0^t f(t-\tau)\left[\int_0^\tau g(\tau-\eta)h(\eta)d\eta\right]d\tau\\ &= \int_0^t\int_0^\tau f(t-\tau)g(\tau-\eta)h(\eta)\,d\eta d\tau\,.\end{aligned}$$

The region of integration, in the double integral is the area between the straight lines $\eta = 0\,, \eta = \tau$ and $\tau = t$. Interchanging the order of integration,

$$\begin{aligned}\int_0^t\int_0^\tau f(t-\tau)g(\tau-\eta)h(\eta)\,d\eta d\tau &= \int_0^t\int_\eta^t f(t-\tau)g(\tau-\eta)h(\eta)\,d\tau d\eta\\ &= \int_0^t\left[\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau\right]h(\eta)\,d\eta\,.\end{aligned}$$

Now let $\tau - \eta = u$. Then

$$\begin{aligned}\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau &= \int_0^{t-\eta} f(t-\eta-u)g(u)du\\ &= f*g\,(t-\eta)\,.\end{aligned}$$

Hence

$$\int_0^t f(t-\tau)[g*h\,(\tau)]d\tau = \int_0^t [f*g\,(t-\tau)]h(\tau)\,d\tau\,.$$

2. Let $f(t) = e^t$. Then

$$\begin{aligned} f * 1\,(t) &= \int_0^t e^{t-\tau} \cdot 1\, d\tau \\ &= e^t \int_0^t e^{-\tau} d\tau \\ &= e^t - 1\,. \end{aligned}$$

3. It follows directly that

$$\begin{aligned} f * f\,(t) &= \int_0^t sin(t-\tau)\, sin(\tau)\, d\tau \\ &= \frac{1}{2}\int_0^t [cos(t-2\tau) - cos(t)] d\tau \\ &= \frac{1}{2}[sin(t) - t\, cos(t)]. \end{aligned}$$

The *range* of the resulting function is $\mathbb{R}$.

5. We have $\mathcal{L}[e^{-t}] = 1/(s+1)$ and $\mathcal{L}[sin\, t] = 1/(s^2+1)$. Based on Theorem 6.6.1,

$$\begin{aligned} \mathcal{L}\left[\int_0^t e^{-(t-\tau)} sin(\tau)\, d\tau\right] &= \frac{1}{s+1} \cdot \frac{1}{s^2+1} \\ &= \frac{1}{(s+1)(s^2+1)}\,. \end{aligned}$$

6. Let $g(t) = t$ and $h(t) = e^t$. Then $f(t) = g * h\,(t)$. Applying Theorem 6.6.1 ,

$$\begin{aligned} \mathcal{L}\left[\int_0^t g(t-\tau)h(\tau)\, d\tau\right] &= \frac{1}{s^2} \cdot \frac{1}{s-1} \\ &= \frac{1}{s^2(s-1)}\,. \end{aligned}$$

7. We have $f(t) = g * h\,(t)$, in which $g(t) = sin\, t$ and $h(t) = cos\, t$. The transform of the convolution integral is

$$\begin{aligned} \mathcal{L}\left[\int_0^t g(t-\tau)h(\tau)\, d\tau\right] &= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} \\ &= \frac{s}{(s^2+1)^2}\,. \end{aligned}$$

9. It is easy to see that

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = cos\, 2t\,.$$

Applying Theorem 6.6.1 ,

$$\mathcal{L}^{-1}\left[\frac{s}{(s+1)(s^2+4)}\right] = \int_0^t e^{-(t-\tau)} cos\, 2\tau\, d\tau\,.$$

10. We first note that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = t\, e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2} sin\, 2t\,.$$

Based on the *convolution theorem* ,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] &= \frac{1}{2}\int_0^t (t-\tau)e^{-(t-\tau)} sin\, 2\tau\, d\tau \\ &= \frac{1}{2}\int_0^t \tau\, e^{-\tau} sin(2t-2\tau)\, d\tau\,.\end{aligned}$$

11. Let $g(t) = \mathcal{L}^{-1}[G(s)]$. Since $\mathcal{L}^{-1}[1/(s^2+1)] = sin\, t$, the inverse transform of the product is

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+1}\right] &= \int_0^t g(t-\tau)\, sin\, \tau\, d\tau \\ &= \int_0^t sin(t-\tau)\, g(\tau)\, d\tau\,.\end{aligned}$$

12. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\, Y(s) - 1 + \omega^2\, Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+\omega^2} + \frac{G(s)}{s^2+\omega^2}.$$

As shown in a related situation, Prob. 11 ,

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+\omega^2}\right] = \frac{1}{\omega}\int_0^t sin\, \omega(t-\tau)\, g(\tau)\, d\tau\,.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{\omega} \sin \omega t + \frac{1}{\omega}\int_0^t \sin \omega(t-\tau)\, g(\tau)\, d\tau\,.$$

14. The transform of the ODE (given the specified initial conditions) is

$$4s^2\, Y(s) + 4s\, Y(s) + 17\, Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)}{4s^2 + 4s + 17}.$$

First write

$$\frac{1}{4s^2 + 4s + 17} = \frac{\frac{1}{4}}{\left(s + \frac{1}{2}\right)^2 + 4}\,.$$

Based on the elementary properties of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{1}{4s^2 + 4s + 17}\right] = \frac{1}{8} e^{-t/2} \sin 2t\,.$$

Applying the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{1}{8}\int_0^t e^{-(t-\tau)/2} \sin 2(t-\tau)\, g(\tau)\, d\tau\,.$$

16. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\, Y(s) - 2s + 3 + 4[s\, Y(s) - 2] + 4\, Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{2s+5}{(s+2)^2} + \frac{G(s)}{(s+2)^2}.$$

We can write

$$\frac{2s+5}{(s+2)^2} = \frac{2}{s+2} + \frac{1}{(s+2)^2}\,.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{2}{s+2}\right] = 2e^{-2t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = t\, e^{-2t}\,.$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = 2e^{-2t} + t\,e^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}g(\tau)\,d\tau\,.$$

18. The transform of the ODE (given the specified initial conditions) is

$$s^4\,Y(s) - \,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \;\frac{G(s)}{s^4-1}.$$

First write

$$\frac{1}{s^4-1} = \frac{1}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^4-1}\right] = \frac{1}{2}[sinh\,t - sin\,t]\,.$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{1}{2}\int_0^t [sinh(t-\tau) - sin(t-\tau)]g(\tau)\,d\tau\,.$$

19. Taking the initial conditions into consideration, the transform of the ODE is

$$s^4\,Y(s) - s^3 + 5s^2\,Y(s) - 5s + 4\,Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^3+5s}{(s^2+1)(s^2+4)} + \frac{G(s)}{(s^2+1)(s^2+4)}.$$

Using partial fractions, we find that

$$\frac{s^3+5s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{4s}{s^2+1} - \frac{s}{s^2+4}\right],$$

and

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{s(s^2+5)}{(s^2+1)(s^2+4)}\right] = \frac{4}{3}\,cos\,t - \frac{1}{3}\,cos\,2t,$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\,sin\,t - \frac{1}{6}\,sin\,2t\,.$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{4}{3}\,cos\,t - \frac{1}{3}\,cos\,2t + \frac{1}{6}\int_0^t [2\,sin(t-\tau) - sin\,2(t-\tau)]g(\tau)\,d\tau\,.$$

$21(a)$. Let $\phi(t) = u''(t)$. Substitution into the *integral equation* results in

$$u''(t) + \int_0^t (t-\xi)\,u''(\xi)\,d\xi = sin\,2t\,.$$

Integrating by parts,

$$\begin{aligned}\int_0^t (t-\xi)\,u''(\xi)\,d\xi &= (t-\xi)\,u'(\xi)\Big|_{\xi=0}^{\xi=t} + \int_0^t u'(\xi)\,d\xi \\ &= -\,t\,u'(0) + u(t) - u(0)\,.\end{aligned}$$

Hence

$$u''(t) + u(t) - t\,u'(0) - u(0) = sin\,2t\,.$$

(b). Substituting the given *initial conditions* for the function $u(t)$,

$$u''(t) + u(t) = sin\,2t\,.$$

Hence the solution of the IVP is equivalent to solving the integral equation in Part (a).

(c). Taking the Laplace transform of the integral equation, with $\Phi(s) = \mathcal{L}[\phi(t)]$,

$$\Phi(s) + \frac{1}{s^2}\cdot\Phi(s) = \frac{2}{s^2+4}\,.$$

Note that the *convolution theorem* was applied. Solving for the transform $\Phi(s)$,

$$\Phi(s) = \frac{2s^2}{(s^2+1)(s^2+4)}\,.$$

Using partial fractions, we can write

$$\frac{2s^2}{(s^2+1)(s^2+4)} = \frac{2}{3}\left[\frac{4}{s^2+4} - \frac{1}{s^2+1}\right].$$

Therefore the solution of the *integral equation* is

$$\phi(t) = \frac{4}{3}\,sin\,2t - \frac{2}{3}\,sin\,t\,.$$

(d). Taking the Laplace transform of the ODE, with $U(s) = \mathcal{L}[u(t)]$,

$$s^2U(s) + U(s) = \frac{2}{s^2+4}\,.$$

Solving for the transform of the solution,

$$U(s) = \frac{2}{(s^2+1)(s^2+4)}\,.$$

Using partial fractions, we can write

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{2}{s^2+1} - \frac{2}{s^2+4}\right].$$

It follows that the solution of the IVP is

$$u(t) = \frac{2}{3}sin\,t - \frac{1}{3}\,sin\,2t\,.$$

We find that $u''(t) = -\frac{2}{3}sin\,t + \frac{4}{3}\,sin\,2t\,.$

$22(a)$. First note that

$$\int_0^b \frac{f(y)}{\sqrt{b-y}}dy = \left(\frac{1}{\sqrt{y}} * f\right)(b).$$

Take the Laplace transformation of both sides of the equation. Using the *convolution theorem*, with $F(s) = \mathcal{L}[f(y)]$,

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}}F(s)\cdot\mathcal{L}\left[\frac{1}{\sqrt{y}}\right].$$

It was shown in Prob. $27(c)$, Section 6.1, that

$$\mathcal{L}\left[\frac{1}{\sqrt{y}}\right] = \sqrt{\frac{\pi}{s}}\,.$$

Hence

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}} F(s) \cdot \sqrt{\frac{\pi}{s}} \, ,$$

with

$$F(s) = \sqrt{\frac{2g}{\pi}} \cdot \frac{T_0}{\sqrt{s}} \, .$$

Taking the inverse transform, we obtain

$$f(y) = \frac{T_0}{\pi} \sqrt{\frac{2g}{y}} \, .$$

(b). Combining equations (i) and (iv),

$$\frac{2g\,T_0^2}{\pi^2\,y} = 1 + \left(\frac{dx}{dy}\right)^2 .$$

Solving for the derivative dx/dy,

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}} \, ,$$

in which $\alpha = gT_0^2/\pi^2$.

(c). Consider the *change of variable* $y = 2\alpha\, sin^2(\theta/2)$. Using the chain rule,

$$\frac{dy}{dx} = 2\alpha\, sin(\theta/2) cos(\theta/2) \cdot \frac{d\theta}{dx}$$

and

$$\frac{dx}{dy} = \frac{1}{2\alpha\, sin(\theta/2) cos(\theta/2)} \cdot \frac{dx}{d\theta} \, .$$

It follows that

$$\begin{aligned} \frac{dx}{d\theta} &= 2\alpha\, sin(\theta/2) cos(\theta/2) \sqrt{\frac{cos^2(\theta/2)}{sin^2(\theta/2)}} \\ &= 2\alpha\, cos^2(\theta/2) \\ &= \alpha + \alpha\, cos\,\theta \, . \end{aligned}$$

Direct integration results in

$$x(\theta) = \alpha\,\theta + \alpha\, sin\,\theta + C \, .$$

Since the curve passes through the *origin*, we require $y(0) = x(0) = 0$. Hence $C = 0$, and $x(\theta) = \alpha\,\theta + \alpha\, sin\,\theta$. We also have

$$
\begin{aligned}
y(\theta) &= 2\alpha\, sin^2(\theta/2) \\
&= \alpha - \alpha\, cos\,\theta\,.
\end{aligned}
$$