Chapter Three

Section 3.1

1. Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r\,e^{rt}$. Direct substitution into the differential equation yields $(r^2 + 2r - 3)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3\,, 1$. Hence the general solution is $y = c_1 e^{t} + c_2 e^{-3t}$.

2. Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 3r + 2 = 0$. The roots of the equation are $r = -2\,, -1$. Hence the general solution is $y = c_1 e^{-t} + c_2 e^{-2t}$.

4. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $2r^2 - 3r + 1 = 0$. The roots of the equation are $r = 1/2\,, 1$. Hence the general solution is $y = c_1 e^{t/2} + c_2 e^{t}$.

6. The characteristic equation is $4r^2 - 9 = 0$, with roots $r = \pm 3/2$. Therefore the general solution is $y = c_1 e^{-3t/2} + c_2 e^{3t/2}$.

8. The characteristic equation is $r^2 - 2r - 2 = 0$, with roots $r = 1 \pm \sqrt{3}$. Hence the general solution is $y = c_1 exp\left(1 - \sqrt{3}\right)t + c_2 exp\left(1 + \sqrt{3}\right)t$.

9. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + r - 2 = 0$. The roots of the equation are $r = -2\,, 1$. Hence the general solution is $y = c_1 e^{-2t} + c_2 e^{t}$. Its derivative is $y' = -2 c_1 e^{-2t} + c_2 e^{t}$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-2c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^{t}$.

11. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $6r^2 - 5r + 1 = 0$. The roots of the equation are $r = 1/3\,, 1/2$. Hence the general solution is $y = c_1 e^{t/3} + c_2 e^{t/2}$. Its derivative is $y' = c_1 e^{t/3}/3 + c_2 e^{t/2}/2$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 4$. In order to satisfy the condition $y'(0) = 1$, we find that $c_1/3 + c_2/2 = 0$. Solving for the constants, $c_1 = 12$ and $c_2 = -8$. Hence the specific solution is $y(t) = 12\,e^{t/3} - 8\,e^{t/2}$.

12. The characteristic equation is $r^2 + 3r = 0$, with roots $r = -3\,, 0$. Therefore the general solution is $y = c_1 + c_2 e^{-3t}$, with derivative $y' = -3\,c_2 e^{-3t}$. In order to satisfy the initial conditions, we find that $c_1 + c_2 = -2$, and $-3\,c_2 = 3$. Hence the specific solution is $y(t) = -1 - e^{-3t}$.

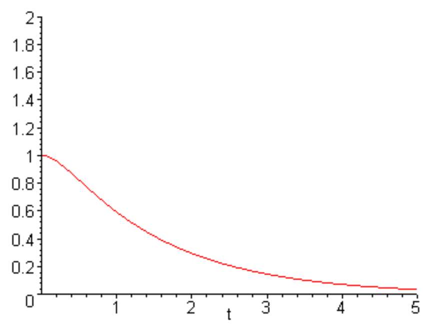
13. The characteristic equation is $r^2 + 5r + 3 = 0$, with roots

$$r_{1,2} = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}.$$

The general solution is $y = c_1 exp\left(-5-\sqrt{13}\right)t/2 + c_2 exp\left(-5+\sqrt{13}\right)t/2$, with derivative

$$y' = \frac{-5-\sqrt{13}}{2}c_1 exp\left(-5-\sqrt{13}\right)t/2 + \frac{-5+\sqrt{13}}{2}c_2 exp\left(-5+\sqrt{13}\right)t/2.$$

In order to satisfy the initial conditions, we require that $c_1 + c_2 = 1$, and $\frac{-5-\sqrt{13}}{2}c_1 + \frac{-5+\sqrt{13}}{2}c_2 = 0$. Solving for the coefficients, $c_1 = \left(1 - 5/\sqrt{13}\right)/2$ and $c_2 = \left(1 + 5/\sqrt{13}\right)/2$.



14. The characteristic equation is $2r^2 + r - 4 = 0$, with roots

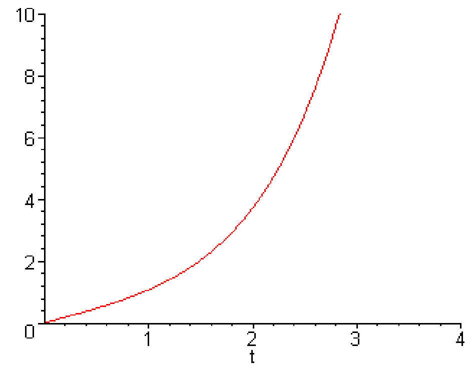
$$r_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}.$$

The general solution is $y = c_1 exp\left(-1-\sqrt{33}\right)t/4 + c_2 exp\left(-1+\sqrt{33}\right)t/4$, with derivative

$$y' = \frac{-1-\sqrt{33}}{4}c_1 exp\left(-1-\sqrt{33}\right)t/4 + \frac{-1+\sqrt{33}}{4}c_2 exp\left(-1+\sqrt{33}\right)t/4.$$

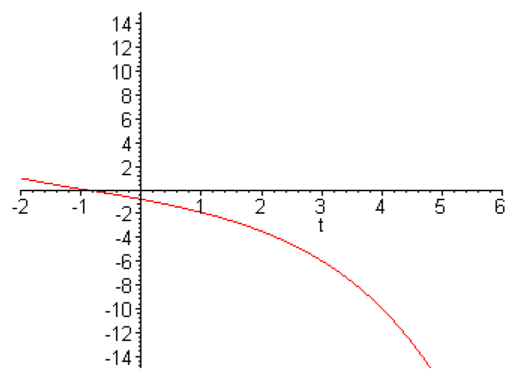
In order to satisfy the initial conditions, we require that $c_1 + c_2 = 0$, and $\frac{-1-\sqrt{33}}{4}c_1 + \frac{-1+\sqrt{33}}{4}c_2 = 1$. Solving for the coefficients, $c_1 = -2/\sqrt{33}$ and $c_2 = 2/\sqrt{33}$. The specific solution is

$$y(t) = -2\left[exp\left(-1-\sqrt{33}\right)t/4 - exp\left(-1+\sqrt{33}\right)t/4\right]/\sqrt{33}.$$



16. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Therefore the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$. Since the initial conditions are specified at $t = -2$, is more convenient to write $y = d_1 e^{-(t+2)/2} + d_2 e^{(t+2)/2}$. The derivative is given by $y' = -\left[d_1 e^{-(t+2)/2}\right]/2 + \left[d_2 e^{(t+2)/2}\right]/2$. In order to satisfy the initial conditions, we find that $d_1 + d_2 = 1$, and $-d_1/2 + d_2/2 = -1$. Solving for the coefficients, $d_1 = 3/2$, and $d_2 = -1/2$. The specific solution is

$$\begin{aligned} y(t) &= \frac{3}{2} e^{-(t+2)/2} - \frac{1}{2} e^{(t+2)/2} \\ &= \frac{3}{2e} e^{-t/2} - \frac{e}{2} e^{t/2} . \end{aligned}$$



18. An algebraic equation with roots -2 and $-1/2$ is $2r^2 + 5r + 2 = 0$. This is the characteristic equation for the ODE $2y'' + 5y' + 2y = 0$.

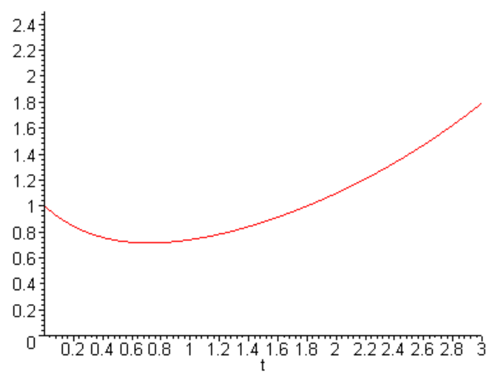
20. The characteristic equation is $2r^2 - 3r + 1 = 0$, with roots $r = 1/2, 1$. Therefore the general solution is $y = c_1 e^{t/2} + c_2 e^t$, with derivative $y' = c_1 e^{t/2}/2 + c_2 e^t$. In order to satisfy the initial conditions, we require $c_1 + c_2 = 2$ and $c_1/2 + c_2 = 1/2$. Solving for the coefficients, $c_1 = 3$, and $c_2 = -1$. The specific solution is $y(t) = 3e^{t/2} - e^t$. To find the *stationary point*, set $y' = 3e^{t/2}/2 - e^t = 0$. There is a unique solution, with $t_1 = ln(9/4)$. The maximum value is then $y(t_1) = 9/4$. To find

the *x-intercept*, solve the equation $3e^{t/2} - e^t = 0$. The solution is readily found to be $t_2 = ln9 \approx 2.1972$.

22. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Hence the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$, with derivative $y' = -c_1 e^{-t/2}/2 + c_2 e^{t/2}/2$. Invoking the initial conditions, we require that $c_1 + c_2 = 2$ and $-c_1 + c_2 = \beta$. The specific solution is $y(t) = (1-\beta)e^{-t/2} + (1+\beta)e^{t/2}$. Based on the form of the solution, it is evident that as $t \to \infty$, $y(t) \to 0$ as long as $\beta = -1$.

23. The characteristic equation is $r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$. Examining the coefficients, the roots are $r = \alpha$, $\alpha - 1$. Hence the general solution of the differential equation is $y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$. Assuming $\alpha \in \mathbb{R}$, all solutions will tend to *zero* as long as $\alpha < 0$. On the other hand, all solutions will become unbounded as long as $\alpha - 1 > 0$, that is, $\alpha > 1$.

25. $y(t) = 2\,e^{t/2}/5 + 3\,e^{-2t}/5$.



The minimum occurs at $(t_0\,, y_0) = (0.7167\,, 0.7155)$.

26(a). The characteristic roots are $r = -3\,, -2$. The solution of the initial value problem is $y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$.

(b). The maximum point has coordinates $t_0 = ln\left[\frac{3(4+\beta)}{2(6+\beta)}\right]$, $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2}$.

(c). $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2} \geq 4$, as long as $\beta \geq 6 + 6\sqrt{3}$.

(d). $\lim_{\beta\to\infty} t_0 = ln\,\frac{3}{2}$. $\lim_{\beta\to\infty} y_0 = \infty$.

29. Set $v = y'$ and $v' = y''$. Substitution into the ODE results in the first order equation $t\,v' + v = 1$. The equation is *linear*, and can be written as $(t\,v)' = 1$. Hence the general solution is $v = 1 + c_1/t$. Hence $y' = 1 + c_1/t$, and $y = t + c_1 ln\,t + c_2$.

31. Setting $v = y'$ and $v' = y''$, the transformed equation is $2t^2 v' + v^3 = 2t\,v$. This

is a *Bernoulli* equation, with $n = 3$. Let $w = v^{-2}$. Substitution of the new dependent variable yields $-t^2 w' + 1 = 2t\,w$, or $t^2 w' + 2t\,w = \; = 1$. Integrating, we find that $w = (t + c_1)/t^2$. Hence $v = \pm t/\sqrt{t + c_1}$, that is, $y' = \pm t/\sqrt{t + c_1}$. Integrating one more time results in $y(t) = \pm\frac{2}{3}(t - 2c_1)\sqrt{t + c_1} + c_2$. (Note that $v = 0$ is also a solution of the transformed equation).

32. Setting $v = y'$ and $v' = y''$, the transformed equation is $v' + v = e^{-t}$. This ODE is *linear*, with integrating factor $\mu(t) = e^t$. Hence $v = y' = (t + c_1)e^{-t}$. Integrating, we obtain $y(t) = -(t + c_1)e^{-t} + c_2$.

33. Set $v = y'$ and $v' = y''$. The resulting equation is $t^2 v' = v^2$. This equation is *separable*, with solution $v = y' = t/(1 + c_1 t)$. Integrating, the general solution is

$$y(t) = t/c_1 - c_1^{-2} ln|1 + c_1 t| + c_2\,,$$

as long as $c_1 \neq 0$. For $c_1 = 0$, the solution is $y(t) = t^2/2 + c_2$. Note that $v = 0$ is also a solution of the transformed equation.

35. Let $y' = v$ and $y'' = v\,dv/dy$. Then $v\,dv/dy + y = 0$ is the transformed equation for $v = v(y)$. This equation is *separable*, with $v\,dv = -y\,dy$. The solution is given by $v^2 = -y^2 + c_1$. Substituting for v, we find that $y' = \pm\sqrt{c_1 - y^2}$. This equation is *also*
separable, with solution $arcsin\big(y/\sqrt{c_1}\big) = \pm\, t + c_2$, or $y(t) = d_1 sin(t + d_2)$.

36. Let $y' = v$ and $y'' = v\,dv/dy$. It follows that $vdv/dy + yv^3 = 0$ is the differential equation for $v = v(y)$. This equation is *separable*, with $v^{-2}\,dv = -y\,dy$. The solution is given by $v = [y^2/2 + c_1]^{-1}$. Substituting for v, we find that $y' = [y^2/2 + c_1]^{-1}$. This equation is *also* separable, with $(y^2/2 + c_1)dy = dt$. The solution is defined *implicitly* by $y^3/6 + c_1 y + c_2 = t$.

38. Setting $y' = v$ and $y'' = vdv/dy$, the transformed equation is $y\,vdv/dy - v^3 = 0$. This equation is *separable*, with $v^{-2}\,dv = dy/y$. The solution is $v(y) = [c_1 - ln|y|]^{-1}$. Substituting for v, we obtain a *separable* equation, $(c_1 - ln|y|)dy = dx$. The solution is given *implicitly* by $c_2 y - y\,ln|y| + c_3 = t$.

39. Let $y' = v$ and $y'' = vdv/dy$. It follows that $vdv/dy + v^2 = 2e^{-y}$ is the equation for $v = v(y)$. Inspection of the left hand side suggests a substitution $w = v^2$. The resulting
equation is $dw/dy + 2w = 4e^{-y}$. This equation is *linear*, with integrating factor $\mu = e^{2y}$.
We obtain $d(e^{2y}\,w)/dy = 4\,e^{y}$, which upon integration yields $w(y) = 4\,e^{-y} + c_1 e^{-2y}$. Converting back to the original dependent variable, $y' = \pm e^{-y}\sqrt{4\,e^{y} + c_1}$. Separating variables, $e^{y}(4\,e^{y} + c_1)^{-1/2}dy = \pm\,dt$. Integration yields $\sqrt{4\,e^{y} + c_1} = \pm\,2t + c_2$.

41. Setting $y' = v$ and $y'' = vdv/dy$, the transformed equation is $vdv/dy - 3y^2 = 0$.

This equation is *separable*, with $vdv = 3y^2dy$. The solution is $y' = v = \sqrt{2y^3 + c_1}$. The *positive* root is chosen based on the initial conditions. Furthermore, when $t = 0$, $y = 2$, and $y' = v = 4$. The initial conditions require that $c_1 = 0$. It follows that $y' = \sqrt{2y^3}$. Separating variables and integrating, $1/\sqrt{y} = -t/\sqrt{2} + c_2$. Hence the solution is $y(t) = 2/(1 - t)^2$.

42. Setting $v = y'$ and $v' = y''$, the transformed equation is $(1 + t^2)v' + 2t\,v =$ $= -3t^{-2}$. Rewrite the equation as $v' + 2t\,v/(1 + t^2) = -3t^{-2}/(1 + t^2)$. This equation is *linear*, with integrating factor $\mu = 1 + t^2$. Hence we have

$$\left[(1 + t^2)v\right]' = -3t^{-2}.$$

Integrating both sides, $v = 3t^{-1}/(1 + t^2) + c_1/(1 + t^2)$. Invoking the initial condition $v(1) = -1$, we require that $c_1 = -5$. Hence $y' = (3 - 5t)/(t + t^3)$. Integrating, we obtain $y(t) = \frac{3}{2}ln[t^2/(1 + t^2)] - 5\,arctan(t) + c_2$. Based on the initial condition $y(1) = 2$, we find that $c_2 = \frac{3}{2}ln\,2 + \frac{5}{4}\pi + 2$.