Section 8.4

$1(a)$. Using the notation $f_n = f(t_n\,, y_n)$, the *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\, f_n - 59\, f_{n-1} + 37\, f_{n-2} - 9\, f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\ .$$

We use the starting values generated by the Runge-Kutta method :

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.1	0.2	0.3
y_n	1.0	1.19516	1.38127	1.55918

	$n = 4(pre)$	$n = 4(cor)$	$n = 5(pre)$	$n = 5(cor)$
t_n	0.4	0.4	0.5	0.5
y_n	1.72967690	1.72986801	1.89346436	1.89346973

(b). With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\ .$$

In this problem, $f_{n+1} = 3 + t_{n+1} - y_{n+1}$. Since the ODE is *linear*, we can solve for

$$y_{n+1} = \frac{1}{24 + 9\, h}[24\, y_n + 27\, h + 9h\, t_{n+1} + h(19\, f_n - 5\, f_{n-1} + f_{n-2})]\ .$$

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	1.7296800	1.8934695

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h\, f_{n+1}].$$

In this problem, $f_{n+1} = 3 + t_{n+1} - y_{n+1}$. Since the ODE is *linear*, we can solve for

$$y_{n+1} = \frac{1}{25 + 12\, h}[36\, h + 12h\, t_{n+1} + 48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3}]\ .$$

	$n=4$	$n=5$
t_n	0.4	0.5
y_n	1.7296805	1.8934711

The exact solution of the IVP is $y(t) = 2 + t - e^{-t}$.

$2(a)$. Using the notation $f_n = f(t_n\,, y_n)$, the *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\ .$$

We use the starting values generated by the Runge-Kutta method :

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.1	0.2	0.3
y_n	2.0	1.62231	1.33362	1.12686

	$n=4(pre)$	$n=4(cor)$	$n=5(pre)$	$n=5(cor)$
t_n	0.4	0.4	0.5	0.5
y_n	0.993751	0.993852	0.925469	0.925764

(b). With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\ .$$

In this problem, $f_{n+1} = 5t_{n+1} - 3\sqrt{y_{n+1}}$. Since the ODE is *nonlinear*, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[\,45t_{n+1} - 27\sqrt{y_{n+1}} + 19\,f_n - 5\,f_{n-1} + f_{n-2}\right]$$

at each time step. We obtain the approximate values:

	$n=4$	$n=5$
t_n	0.4	0.5
y_n	0.993847	0.925746

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h f_{n+1}].$$

Since the ODE is *nonlinear*, an equation solver is used to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h\left(5t_{n+1} - 3\sqrt{y_{n+1}}\,\right)\right]$$

at each time step.

	$n=4$	$n=5$
t_n	0.4	0.5
y_n	0.993869	0.925837

The exact solution of the IVP is given *implicitly* by

$$\frac{1}{\left(2\sqrt{y}+5t\right)^5\left(t-\sqrt{y}\right)^2} = \frac{\sqrt{2}}{512}\,.$$

$3(a)$. The *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\,.$$

Using the starting values generated by the Runge-Kutta method :

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.1	0.2	0.3
y_n	1.0	1.205350	1.422954	1.655527

	$n=4(pre)$	$n=4(cor)$	$n=5(pre)$	$n=5(cor)$
t_n	0.4	0.4	0.5	0.5
y_n	1.906340	1.906382	2.179455	2.179567

(b). With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\,.$$

In this problem, $f_{n+1} = 2\,y_{n+1} - 3\,t_{n+1}$. Since the ODE is *linear*, we can solve for

$$y_{n+1} = \frac{1}{24 - 18\,h}[24\,y_n - 27h\,t_{n+1} + h(19\,f_n - 5\,f_{n-1} + f_{n-2})]\,.$$

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	1.906385	2.179576

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h f_{n+1}].$$

In this problem, $f_{n+1} = 2\,y_{n+1} - 3\,t_{n+1}$. Since the ODE is *linear*, we can solve for

$$y_{n+1} = \frac{1}{25 - 24\,h}[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} - 36h\,t_{n+1}]\,.$$

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	1.906395	2.179611

The exact solution of the IVP is $y(t) = e^{2t}/4 + 3t/2 + 3/4$.

$5(a)$. The *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\,.$$

Using the starting values generated by the Runge-Kutta method :

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.1	0.2	0.3
y_n	0.5	0.51016950	0.52413795	0.54210529

	$n=4(pre)$	$n=4(cor)$	$n=5(pre)$	$n=5(cor)$
t_n	0.4	0.4	0.5	0.5
y_n	0.56428532	0.56428577	0.59090816	0.59090918

(b). With $f_{n+1} = f(t_{n+1}, y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2}) \,.$$

In this problem,

$$f_{n+1} = \frac{y_{n+1}^2 + 2\, t_{n+1}\, y_{n+1}}{3 + t_{n+1}^2} \,.$$

Since the ODE is *nonlinear*, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[9\, \frac{y_{n+1}^2 + 2\, t_{n+1}\, y_{n+1}}{3 + t_{n+1}^2} + 19\, f_n - 5\, f_{n-1} + f_{n-2}\right]$$

at each time step.

	$n=4$	$n=5$
t_n	0.4	0.5
y_n	0.56428578	0.59090920

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h f_{n+1}].$$

Since the ODE is *nonlinear*, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h \frac{y_{n+1}^2 + 2\, t_{n+1}\, y_{n+1}}{3 + t_{n+1}^2}\right]$$

at each time step. We obtain the approximate values:

	$n=4$	$n=5$
t_n	0.4	0.5
y_n	0.56428588	0.59090952

The exact solution of the IVP is $y(t) = (3+t^2)/(6-t)$.

$6(a)$. The *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\, f_n - 59\, f_{n-1} + 37\, f_{n-2} - 9\, f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\ .$$

We use the starting values generated by the Runge-Kutta method :

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.1	0.2	0.3
y_n	-1.0	-0.924517	-0.864125	-0.816377

	$n=4(pre)$	$n=4(cor)$	$n=5(pre)$	$n=5(cor)$
t_n	0.4	0.4	0.5	0.5
y_n	-0.779832	-0.779693	-0.753311	-0.753135

(b). With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\ .$$

In this problem, $f_{n+1} = (t_{n+1}^2 - y_{n+1}^2)sin\, y_{n+1}$. Since the ODE is *nonlinear*, we obtain the *implicit* equation

$$y_{n+1} = y_n + \frac{h}{24}\left[9\left(t_{n+1}^2 - y_{n+1}^2\right)sin\, y_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2}\right].$$

	$n=4$	$n=5$
t_n	0.4	0.5
y_n	-0.779700	-0.753144

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h f_{n+1}].$$

Since the ODE is $nonlinear$, we obtain the *implicit* equation

$$y_{n+1} = \frac{1}{25}\left[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h\left(t_{n+1}^2 - y_{n+1}^2\right) sin\, y_{n+1}\right].$$

	$n = 4$	$n = 5$
t_n	0.4	0.5
y_n	$-\,0.779680$	$-\,0.753089$

$8(a)$. The *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\, f_n - 59\, f_{n-1} + 37\, f_{n-2} - 9\, f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\ .$$

We use the starting values generated by the Runge-Kutta method :

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.05	0.1	0.15
y_n	2.0	1.7996296	1.6223042	1.4672503

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.9257133	1.285148	2.408595	4.103495

(b). Since the ODE is $nonlinear$, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[45t_{n+1} - 27\sqrt{y_{n+1}}\, + 19\, f_n - 5\, f_{n-1} + f_{n-2}\right]$$

at each time step. We obtain the approximate values:

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.9257125	1.285148	2.408595	4.103495

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h f_{n+1}].$$

Since the ODE is *nonlinear*, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h\left(5t_{n+1} - 3\sqrt{y_{n+1}}\,\right)\right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.9257248	1.285158	2.408594	4.103493

The exact solution of the IVP is given *implicitly* by

$$\frac{1}{\left(2\sqrt{y} + 5t\right)^5 \left(t - \sqrt{y}\right)^2} = \frac{\sqrt{2}}{512}\,.$$

$9(a)$. The *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\, f_n - 59\, f_{n-1} + 37\, f_{n-2} - 9\, f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\,.$$

Using the starting values generated by the Runge-Kutta method :

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.05	0.1	0.15
y_n	3.0	3.087586	3.177127	3.268609

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	3.962186	5.108903	6.431390	7.923385

(b). With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\,.$$

In this problem, $f_{n+1} = \sqrt{t_{n+1} + y_{n+1}}$. Since the ODE is *nonlinear*, an equation solver must be implemented in order to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[9\sqrt{t_{n+1} + y_{n+1}} + 19\, f_n - 5\, f_{n-1} + f_{n-2}\right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	3.962186	5.108903	6.431390	7.923385

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h f_{n+1}].$$

Since the ODE is *nonlinear*, an equation solver is needed to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\, y_n - 36\, y_{n-1} + 16\, y_{n-2} - 3\, y_{n-3} + 12h\sqrt{t_{n+1} + y_{n+1}}\,\right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	3.962186	5.108903	6.431390	7.923385

The exact solution is given *implicitly* by

$$ln\left[\frac{2}{y + t - 1}\right] + 2\sqrt{t + y} - 2\, arctanh\, \sqrt{t + y} = t + 2\sqrt{3} - 2\, arctanh\, \sqrt{3}\ .$$

$10(a)$. The *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,,y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\;.$$

We use the starting values generated by the Runge-Kutta method :

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.05	0.1	0.15
y_n	1.0	1.051230	1.104843	1.160740

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	1.612622	2.480909	3.7451479	5.495872

(b). With $f_{n+1} = f(t_{n+1}\,,y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\;.$$

In this problem, $f_{n+1} = 2\,t_{n+1} + exp(-t_{n+1}\,y_{n+1})$. Since the ODE is $nonlinear$, an equation solver must be implemented in order to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\{9[2\,t_{n+1} + exp(-t_{n+1}\,y_{n+1})] + 19\,f_n - 5\,f_{n-1} + f_{n-2}\}$$

at each time step.

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	1.612622	2.480909	3.7451479	5.495872

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h f_{n+1}].$$

Since the ODE is $nonlinear$, we obtain the *implicit* equation

$$y_{n+1} = \frac{1}{25}\{48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h[2\,t_{n+1} + exp(-t_{n+1}\,y_{n+1})]\}.$$

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	1.612623	2.480905	3.7451473	5.495869

$11(a)$. The *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\, f_n - 59\, f_{n-1} + 37\, f_{n-2} - 9\, f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\ .$$

Using the starting values generated by the Runge-Kutta method :

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.05	0.1	0.15
y_n	$-\,2.0$	$-\,1.958833$	$-\,1.915221$	$-\,1.868975$

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	$-\,1.447639$	$-\,0.1436281$	1.060946	1.410122

(b). With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\, f_{n+1} + 19\, f_n - 5\, f_{n-1} + f_{n-2})\ .$$

In this problem,

$$f_{n+1} = \frac{4 - t_{n+1}\, y_{n+1}}{1 + y_{n+1}^2}\ .$$

Since the differential equation is *nonlinear*, an equation solver is used to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[9\, \frac{4 - t_{n+1}\, y_{n+1}}{1 + y_{n+1}^2} + 19\, f_n - 5\, f_{n-1} + f_{n-2}\right]$$

at each time step.

	$n=10$	$n=20$	$n=30$	$n=40$
t_n	0.5	1.0	1.5	2.0
y_n	$-\,1.447638$	$-\,0.1436767$	1.060913	1.410103

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h f_{n+1}].$$

Since the ODE is *nonlinear*, an equation solver must be implemented in order to approximate the solution of

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h\frac{4 - t_{n+1}\,y_{n+1}}{1 + y_{n+1}^2}\right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	-1.447621	-0.1447619	1.060717	1.410027

$12(a)$. The *predictor* formula is

$$y_{n+1} = y_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3}).$$

With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the *corrector* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\,.$$

We use the starting values generated by the Runge-Kutta method :

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0.0	0.05	0.1	0.15
y_n	0.5	0.5046218	0.5101695	0.5166666

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.5909091	0.8000000	1.166667	1.750000

(b). With $f_{n+1} = f(t_{n+1}\,, y_{n+1})$, the fourth order *Adams-Moulton* formula is

$$y_{n+1} = y_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})\,.$$

In this problem,

$$f_{n+1} = \frac{y_{n+1}^2 + 2\,t_{n+1}\,y_{n+1}}{3 + t_{n+1}^2}\ .$$

Since the ODE is *nonlinear*, an equation solver is needed to approximate the solution of

$$y_{n+1} = y_n + \frac{h}{24}\left[9\,\frac{y_{n+1}^2 + 2\,t_{n+1}\,y_{n+1}}{3 + t_{n+1}^2} + 19\,f_n - 5\,f_{n-1} + f_{n-2}\right]$$

at each time step.

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.5909091	0.8000000	1.166667	1.750000

(c). The fourth order *backward differentiation* formula is

$$y_{n+1} = \frac{1}{25}[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12hf_{n+1}].$$

Since the ODE is *nonlinear*, we obtain the *implicit* equation

$$y_{n+1} = \frac{1}{25}\left[48\,y_n - 36\,y_{n-1} + 16\,y_{n-2} - 3\,y_{n-3} + 12h\,\frac{y_{n+1}^2 + 2\,t_{n+1}\,y_{n+1}}{3 + t_{n+1}^2}\right].$$

	$n = 10$	$n = 20$	$n = 30$	$n = 40$
t_n	0.5	1.0	1.5	2.0
y_n	0.5909092	0.8000002	1.166667	1.750001

The exact solution of the IVP is $y(t) = (3 + t^2)/(6 - t)$.

13. Both *Adams* methods entail the approximation of $f(t\,,y)$, on the interval $[t_n\,,t_{n+1}]$, by a polynomial. Approximating $\phi'(t) = P_1(t) \equiv A$, which is a *constant* polynomial, we have

$$\begin{aligned}\phi(t_{n+1}) - \phi(t_n) &= \int_{t_n}^{t_{n+1}} A\,dt \\ &= A(t_{n+1} - t_n) = Ah\,.\end{aligned}$$

Setting $A = \lambda\,f_n + (1-\lambda)f_{n-1}$, where $0 \le \lambda \le 1$, we obtain the approximation

$$y_{n+1} = y_n + h[\lambda\,f_n + (1-\lambda)f_{n-1}].$$

An appropriate choice of λ yields the familiar Euler formula. Similarly, setting

$$A = \lambda\,f_n + (1-\lambda)f_{n+1}\,,$$

where $0 \le \lambda \le 1$, we obtain the approximation

$$y_{n+1} = y_n + h[\lambda f_n + (1-\lambda) f_{n+1}].$$

14. For a *third order* Adams-Bashforth formula, we approximate $f(t, y)$, on the interval $[t_n, t_{n+1}]$, by a *quadratic* polynomial using the points (t_{n-2}, y_{n-2}), (t_{n-1}, y_{n-1}) and (t_n, y_n). Let $P_3(t) = At^2 + Bt + C$. We obtain the system of equations

$$\begin{aligned} At_{n-2}^2 + Bt_{n-2} + C &= f_{n-2} \\ At_{n-1}^2 + Bt_{n-1} + C &= f_{n-1} \\ At_n^2 + Bt_n + C &= f_n . \end{aligned}$$

For computational purposes, assume that $t_0 = 0$, and $t_n = nh$. It follows that

$$\begin{aligned} A &= \frac{f_n - 2f_{n-1} + f_{n-2}}{2\,h^2} \\ B &= \frac{(3-2n)f_n + (4n-4)f_{n-1} + (1-2n)f_{n-2}}{2\,h} \\ C &= \frac{n^2 - 3n + 2}{2} f_n + \left(2n - n^2\right) f_{n-1} + \frac{n^2 - n}{2} f_{n-2} . \end{aligned}$$

We then have

$$\begin{aligned} y_{n+1} - y_n \; &= \int_{t_n}^{t_{n+1}} \left[At^2 + Bt + C\right] dt \\ &= Ah^3\left(n^2 + n + \frac{1}{3}\right) + Bh^2\left(n + \frac{1}{2}\right) + Ch , \end{aligned}$$

which yields

$$y_{n+1} - y_n = \frac{h}{12}(23\, f_n - 16\, f_{n-1} + 5\, f_{n-2}) .$$

15. For a *third order* Adams-Moulton formula, we approximate $f(t, y)$, on the interval $[t_n, t_{n+1}]$, by a *quadratic* polynomial using the points (t_{n-1}, y_{n-1}), (t_n, y_n) and (t_{n+1}, y_{n+1}). Let $P_3(t) = \alpha t^2 + \beta t + \gamma$. This time we obtain the system of algebraic equations

$$\begin{aligned} \alpha t_{n-1}^2 + \beta t_{n-1} + \gamma &= f_{n-1} \\ \alpha t_n^2 + \beta t_n + \gamma &= f_n \\ \alpha t_{n+1}^2 + \beta t_{n+1} + \gamma &= f_{n+1} . \end{aligned}$$

For computational purposes, again assume that $t_0 = 0$, and $t_n = nh$. It follows that

$$\alpha = \frac{f_{n-1} - 2f_n + f_{n+1}}{2\,h^2}$$
$$\beta = \frac{-\,(2n+1)f_{n-1} + 4nf_n + (1-2n)f_{n+1}}{2\,h}$$
$$\gamma = \frac{n^2+n}{2}f_{n-1} + \left(1-n^2\right)f_n + \frac{n^2-n}{2}f_{n+1}\,.$$

We then have

$$\begin{aligned} y_{n+1} - y_n &= \int_{t_n}^{t_{n+1}} \left[\alpha t^2 + \beta t + \gamma\right] dt \\ &= \alpha h^3\left(n^2 + n + \frac{1}{3}\right) + \beta h^2\left(n + \frac{1}{2}\right) + \gamma h\,, \end{aligned}$$

which results in

$$y_{n+1} - y_n = \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})\,.$$

Section 8.5

$1(a)$. The *general* solution of the ODE is $y(t) = c\,e^t + 2 - t$. Imposing the initial condition, $y(0) = 2$, the solution of the IVP is $\phi_1(t) = 2 - t$.

(b). If instead, the initial condition $y(0) = 2.001$ is given, the solution of the IVP is $\phi_2(t) = 0.001\,e^t + 2 - t$. We then have $\phi_2(t) - \phi_1(t) = 0.001\,e^t$.

3. The solution of the initial value problem is $\phi(t) = e^{-100\,t} + t$.

(a, b). Based on the exact solution, the *local truncation error* for both of the Euler methods is

$$|e_{loc}| \leq \frac{10^4}{2} e^{-100\,\bar{t}_n} h^2.$$

Hence $|e_n| \leq 5000\,h^2$, for all $0 < \bar{t}_n < 1$. Furthermore, the local truncation error is *greatest* near $t = 0$. Therefore $|e_1| \leq 5000\,h^2 < 0.0005$ for $h < 0.0003$. Now the truncation error accumulates at each time step. Therefore the *actual* time step should be much smaller than $h \approx 0.0003$. For example, with $h = 0.00025$, we obtain the data

	Euler	*B.Euler*	$\phi(t)$
$t = 0.05$	0.056323	0.057165	0.056738
$t = 0.1$	0.100040	0.100051	0.100045

Note that the total number of time steps needed to reach $t = 0.1$ is $N = 400$.

(c). Using the Runge-Kutta method, comparisons are made for several values of h :

$h = 0.1$:

	$\phi(t)$	y_n	$y_n - \phi(t_n)$
$t = 0.05$	0.056738	0.057416	0.000678
$t = 0.1$	0.100045	0.100055	0.000010

$h = 0.005$:

	$\phi(t)$	y_n	$y_n - \phi(t_n)$
$t = 0.05$	0.056738	0.056766	0.000027
$t = 0.1$	0.100045	0.100046	0.0000004

$6(a)$. Using the method of *undetermined coefficients*, it is easy to show that the general solution of the ODE is $y(t) = c\,e^{\lambda t} + t^2$. Imposing the initial condition, it follows that $c = 0$ and hence the solution of the IVP is $\phi(t) = t^2$.

(b). Using the Runge-Kutta method, with $h = 0.01$, numerical solutions are generated