Section 7.6

2. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

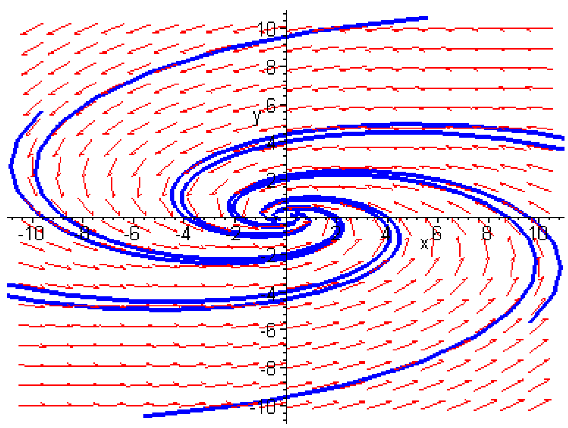
$$\begin{pmatrix} -1-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 2r + 5 = 0$. The roots of the characteristic equation are $r = -1 \pm 2i$. Substituting $r = -1 - 2i$, the two equations reduce to $\xi_1 + 2i\,\xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (-2i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2i\,,1)^T$. Hence one of the *complex-valued* solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-(1+2i)t} \\ &= \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-t}(cos\,2t - i\,sin\,2t) \\ &= e^{-t}\begin{pmatrix} -2\,sin\,2t \\ cos\,2t \end{pmatrix} + i\,e^{-t}\begin{pmatrix} -2\,cos\,2t \\ -\,sin\,2t \end{pmatrix}. \end{aligned}$$

Based on the real and imaginary parts of this solution, the general solution is

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} -2\,sin\,2t \\ cos\,2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\,cos\,2t \\ sin\,2t \end{pmatrix}.$$



3. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$. Hence one of the *complex-valued* solutions is given by

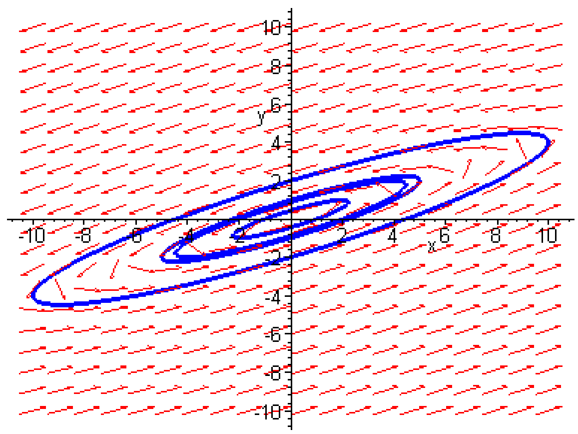
$$\begin{aligned}\mathbf{x}^{(1)} &= \begin{pmatrix}2+i\\1\end{pmatrix}e^{it}\\ &= \begin{pmatrix}2+i\\1\end{pmatrix}(cos\,t + i\,sin\,t)\\ &= \begin{pmatrix}2\,cos\,t - sin\,t\\ cos\,t\end{pmatrix} + i\begin{pmatrix}cos\,t + 2\,sin\,t\\ sin\,t\end{pmatrix}.\end{aligned}$$

Therefore the general solution is

$$\mathbf{x} = c_1\begin{pmatrix}2\,cos\,t - sin\,t\\ cos\,t\end{pmatrix} + c_2\begin{pmatrix}cos\,t + 2\,sin\,t\\ sin\,t\end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1\begin{pmatrix}5\,cos\,t\\ 2\,cos\,t + sin\,t\end{pmatrix} + c_2\begin{pmatrix}5\,sin\,t\\ -\,cos\,t + 2\,sin\,t\end{pmatrix}.$$



4. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix}2-r & -\,5/2\\ 9/5 & -\,1-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r + \frac{5}{2} = 0$. The roots of the characteristic equation are $r = (1 \pm 3i)/2$. With $r = (1 + 3\,i)/2$, the equations reduce to the single equation $(3-3i)\xi_1 - 5\,\xi_2 = 0$. The corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (5\,, 3 - 3\,i)^T$. Hence one of the *complex-valued* solutions is

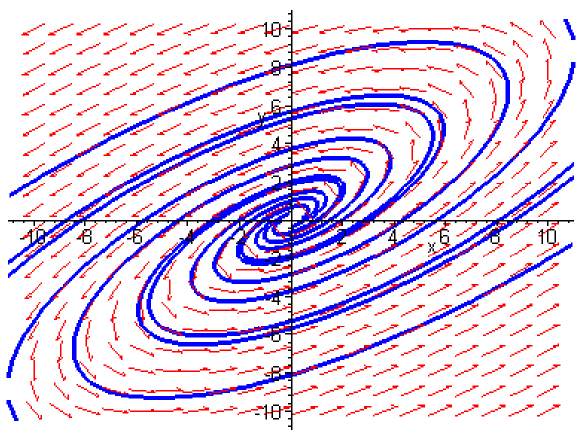
$$\begin{aligned}\mathbf{x}^{(1)} &= \begin{pmatrix} 5 \\ 3-3i \end{pmatrix} e^{(1+3i)t/2} \\ &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{t/2}\left(cos\,\frac{3}{2}t + i\,sin\,\frac{3}{2}t\right) \\ &= e^{t/2}\begin{pmatrix} 2\,cos\,\frac{3}{2}t - sin\,\frac{3}{2}t \\ cos\,\frac{3}{2}t \end{pmatrix} + ie^{t/2}\begin{pmatrix} cos\,\frac{3}{2}t + 2\,sin\,\frac{3}{2}t \\ sin\,\frac{3}{2}t \end{pmatrix}.\end{aligned}$$

The general solution is

$$\mathbf{x} = c_1 e^{t/2}\begin{pmatrix} 2\,cos\,\frac{3}{2}t - sin\,\frac{3}{2}t \\ cos\,\frac{3}{2}t \end{pmatrix} + c_2 e^{t/2}\begin{pmatrix} cos\,\frac{3}{2}t + 2\,sin\,\frac{3}{2}t \\ sin\,\frac{3}{2}t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 e^{t/2}\begin{pmatrix} 5\,cos\,\frac{3}{2}t \\ 3\,cos\,\frac{3}{2}t + 3sin\,\frac{3}{2}t \end{pmatrix} + c_2 e^{t/2}\begin{pmatrix} 5\,sin\,\frac{3}{2}t \\ -3\,cos\,\frac{3}{2}t + 3sin\,\frac{3}{2}t \end{pmatrix}.$$



5. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

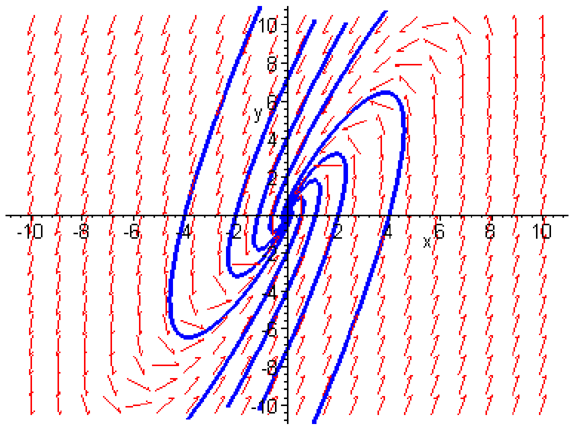
$$\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1 - i$ reduces the system of equations to $(2+i)\xi_1 - \xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (1\,, 2+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1\,, 2-i)^T$. Hence one of the *complex-valued* solutions is given by

$$
\begin{aligned}
\mathbf{x}^{(1)} &= \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-(1+i)t} \\
&= \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t}(\cos t - i \sin t) \\
&= e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + ie^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}.
\end{aligned}
$$

The general solution is

$$
\mathbf{x} = c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ -\cos t + 2\sin t \end{pmatrix}.
$$



6. Solution of the ODEs is based on the analysis of the algebraic equations

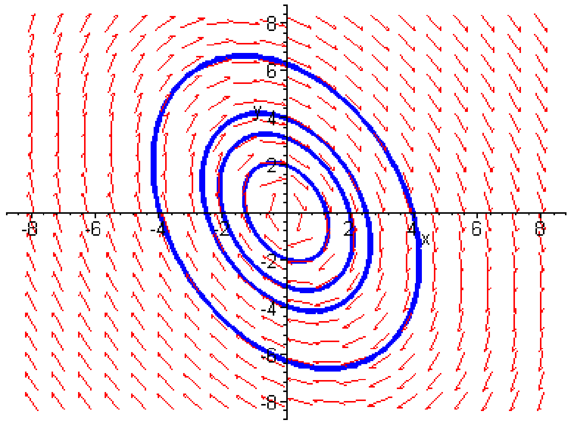
$$
\begin{pmatrix} 1-r & 2 \\ -5 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 9 = 0$. The roots of the characteristic equation are $r = \pm 3\,i$. Setting $r = 3\,i$, the two equations reduce to $(1 - 3\,i)\xi_1 + 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (-2\,, 1 - 3i)^T$. Hence one of the *complex-valued* solutions is given by

$$
\begin{aligned}
\mathbf{x}^{(1)} &= \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} e^{3it} \\
&= \begin{pmatrix} -2 \\ 1-3i \end{pmatrix}(\cos 3t + i \sin 3t) \\
&= \begin{pmatrix} -2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} + i\begin{pmatrix} -2\sin 3t \\ -3\cos 3t + \sin 3t \end{pmatrix}.
\end{aligned}
$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin 3t \\ 3\cos 3t - \sin 3t \end{pmatrix}.$$



8. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -3-r & 0 & 2 \\ 1 & -1-r & 0 \\ -2 & -1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 + 4r^2 + 7r + 6 = 0$, with roots $r_1 = -2$, $r_2 = -1 - \sqrt{2}\, i$ and $r_3 = -1 + \sqrt{2}\, i$. Setting $r = -2$, the equations reduce to

$$\begin{aligned} -\xi_1 + 2\xi_3 &= 0 \\ \xi_1 + \xi_2 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2\,,\,-2\,,1)^T$. With $r = -1 - \sqrt{2}\, i$, the system of equations is equivalent to

$$\begin{aligned} \left(2 - i\sqrt{2}\right)\xi_1 - 2\xi_3 &= 0 \\ \xi_1 + i\sqrt{2}\,\xi_2 &= 0\,. \end{aligned}$$

An eigenvector is given by $\boldsymbol{\xi}^{(2)} = \left(-i\sqrt{2}\,,1\,,\,-1 - i\sqrt{2}\right)^T$. Hence one of the *complex-valued* solutions is given by

$$\begin{aligned}\mathbf{x}^{(2)} &= \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1-i\sqrt{2} \end{pmatrix} e^{-\left(1+i\sqrt{2}\right)it} \\ &= \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1-i\sqrt{2} \end{pmatrix} e^{-t}\left(\cos\sqrt{2}\,t - i\sin\sqrt{2}\,t\right) \\ &= e^{-t}\begin{pmatrix} -\sqrt{2}\sin\sqrt{2}\,t \\ \cos\sqrt{2}\,t \\ -\cos\sqrt{2}\,t-\sqrt{2}\sin\sqrt{2}\,t \end{pmatrix} + ie^{-t}\begin{pmatrix} -\sqrt{2}\cos\sqrt{2}\,t \\ -\sin\sqrt{2}\,t \\ -\sqrt{2}\cos\sqrt{2}\,t-\sin\sqrt{2}\,t \end{pmatrix}.\end{aligned}$$

The other complex-valued solution is $\mathbf{x}^{(3)} = \overline{\boldsymbol{\xi}^{(2)}}\, e^{r_3 t}$. The general solution is

$$\begin{aligned}\mathbf{x} &= c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + \\ &+ c_2\, e^{-t}\begin{pmatrix} \sqrt{2}\sin\sqrt{2}\,t \\ -\cos\sqrt{2}\,t \\ \cos\sqrt{2}\,t+\sqrt{2}\sin\sqrt{2}\,t \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} \sqrt{2}\cos\sqrt{2}\,t \\ \sin\sqrt{2}\,t \\ \sqrt{2}\cos\sqrt{2}\,t+\sin\sqrt{2}\,t \end{pmatrix}.\end{aligned}$$

It is easy to see that all solutions converge to the equilibrium point $(0\,,0\,,0)$.

10. Solution of the system of ODEs requires that

$$\begin{pmatrix} -3-r & 2 \\ -1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2+4\,r+5=0$, with roots $r=-2\pm i$. Substituting $r=-2+i$, the equations are equivalent to $\xi_1-(1-i)\xi_2=0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)}=(1-i\,,1)^T$. One of the *complex-valued* solutions is given by

$$\begin{aligned}\mathbf{x}^{(1)} &= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(-2+i)t} \\ &= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{-2t}(\cos t + i\sin t) \\ &= e^{-2t}\begin{pmatrix} \cos t+\sin t \\ \cos t \end{pmatrix} + ie^{-2t}\begin{pmatrix} -\cos t+\sin t \\ \sin t \end{pmatrix}.\end{aligned}$$

Hence the general solution is

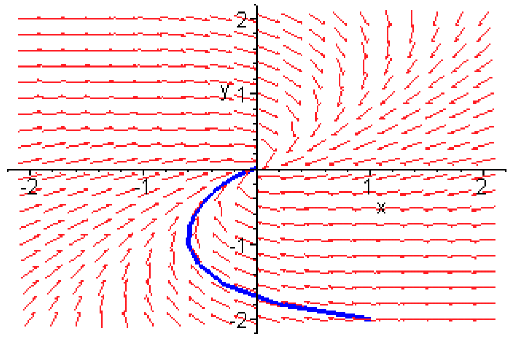
$$\mathbf{x} = c_1\, e^{-2t}\begin{pmatrix} \cos t+\sin t \\ \cos t \end{pmatrix} + c_2\, e^{-2t}\begin{pmatrix} -\cos t+\sin t \\ \sin t \end{pmatrix}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 - c_2 &= 1 \\ c_1 &= -2\,. \end{aligned}$$

Solving for the coefficients, the solution of the initial value problem is

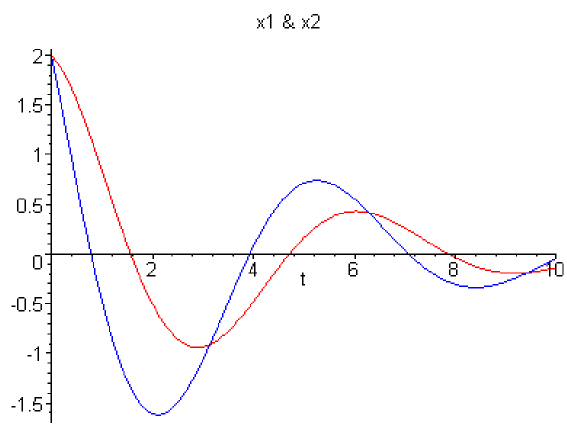
$$\mathbf{x} = -2e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} - 3e^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}$$
$$= e^{-2t}\begin{pmatrix} \cos t - 5\sin t \\ -2\cos t - 3\sin t \end{pmatrix}.$$



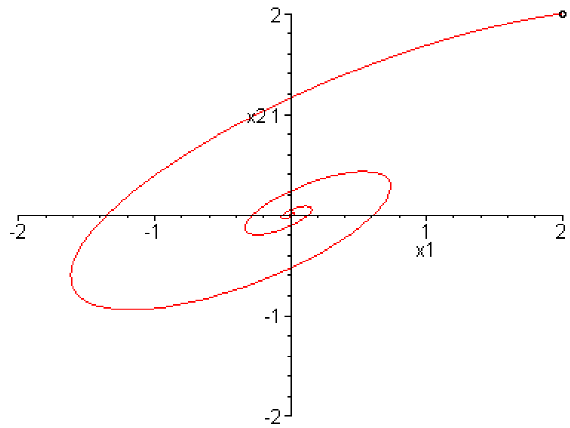
11(a). With $\mathbf{x}(0) = (2\ ,2)^T$, the solution is

$$\mathbf{x} = e^{-t/4}\begin{pmatrix} 2\cos t - 2\sin t \\ 2\cos t \end{pmatrix}.$$

11(b).



$11(c)$.



12. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{4}{5}-r & 2 \\ -1 & \frac{6}{5}-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $25\,r^2 - 10\,r + 26 = 0$, with roots $r = \frac{1}{5} \pm i$. Setting $r = 1/5 + i$, the two equations reduce to $\xi_1 - (1-i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1-i\,,1)^T$. One of the *complex-valued* solutions is given by

$$\begin{aligned}
\mathbf{x}^{(1)} &= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(\frac{1}{5}+i)t} \\
&= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{t/5}(\cos t + i\sin t) \\
&= e^{t/5}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + ie^{t/5}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.
\end{aligned}$$

Hence the general solution is

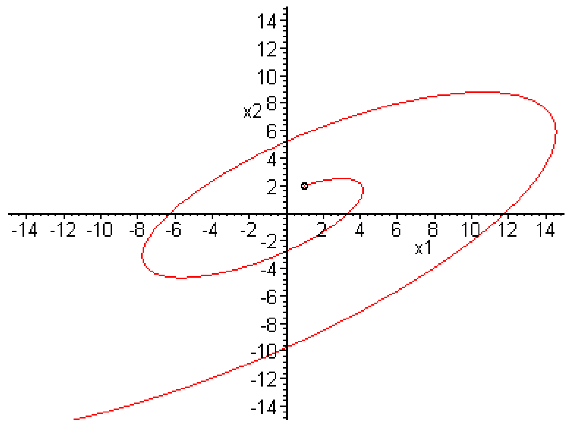
$$\mathbf{x} = c_1 e^{t/5}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{t/5}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

(b). Let $\mathbf{x}(0) = (x_1^0\,, x_2^0)^T$. The solution of the initial value problem is

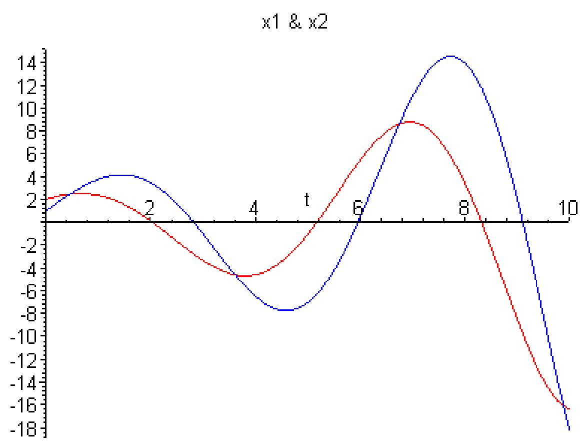
$$\begin{aligned}
\mathbf{x} &= x_2^0\, e^{t/5}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + (x_2^0 - x_1^0)e^{t/5}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}. \\
&= e^{t/5}\begin{pmatrix} x_1^0 \cos t + (2\,x_2^0 - x_1^0)\sin t \\ x_2^0 \cos t + (x_2^0 - x_1^0)\sin t \end{pmatrix}.
\end{aligned}$$

With $\mathbf{x}(0) = (1\ ,2)^T$, the solution is

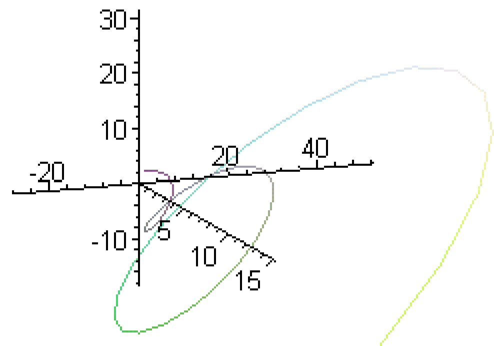
$$\mathbf{x} = e^{t/5}\begin{pmatrix} cos\,t + 3\,sin\,t \\ 2\,cos\,t + sin\,t \end{pmatrix}.$$



(c).



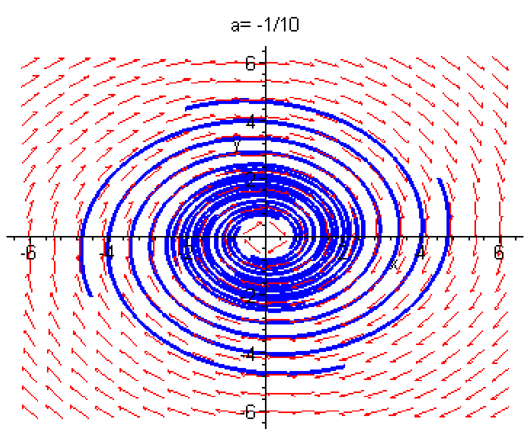
(d).

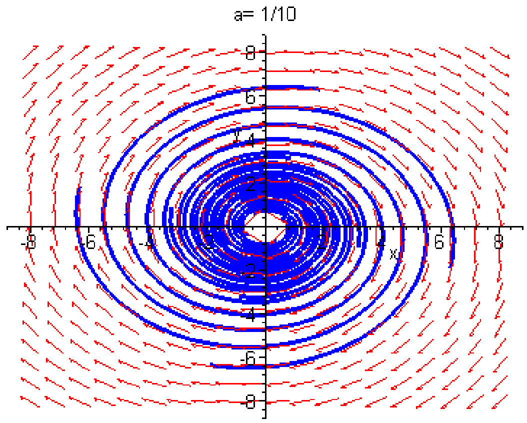


$13(a)$. The characteristic equation of the coefficient matrix is $r^2 - 2\alpha r + 1 + \alpha^2$, with roots $r = \alpha \pm i$.

(b). When $\alpha < 0$ and $\alpha > 0$, the equilibrium point $(0, 0)$ is a *stable* spiral and an *unstable* spiral, respectively. The equilibrium point is a *center* when $\alpha = 0$.

(c).



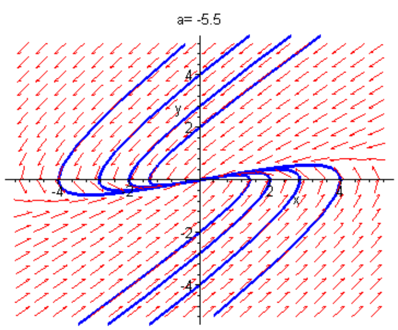


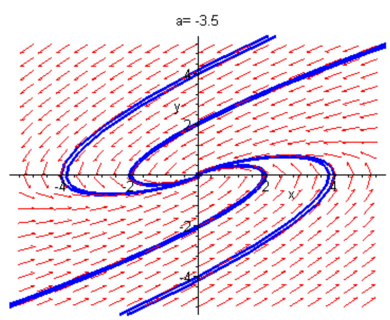
14(a). The roots of the characteristic equation, $r^2 - \alpha\, r + 5 = 0$, are

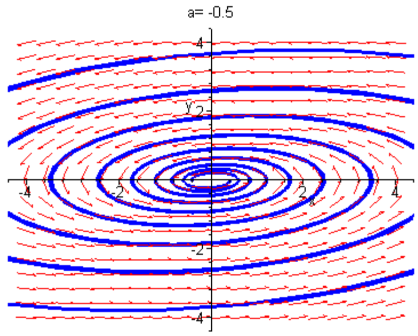
$$r_{1,2} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 20}\,.$$

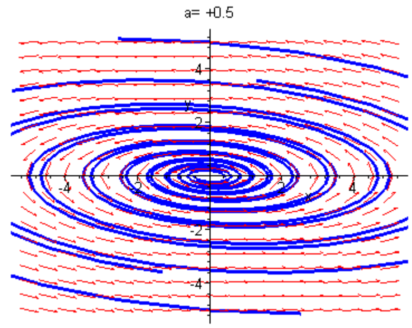
(b). Note that the roots are *complex* when $-\sqrt{20} < \alpha < \sqrt{20}$. For the case when $\alpha \in \left(-\sqrt{20}\,, 0\right)$, the equilibrium point $(0\,,0)$ is a *stable* spiral. On the other hand, when $\alpha \in \left(0\,,\sqrt{20}\right)$, the equilibrium point is an *unstable* spiral. For the case $\alpha = 0$, the roots are purely imaginary, so the equilibrium point is a *center*. When $\alpha^2 > 20$, the roots are *real* and *distinct*. The equilibrium point becomes a *node*, with its stability dependent on the sign of α. Finally, the case $\alpha^2 = 20$ marks the transition from spirals to nodes.

(c).

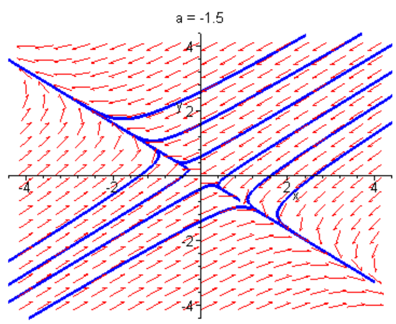


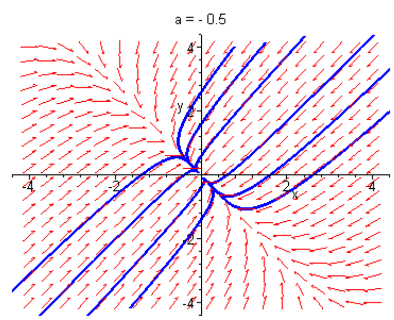


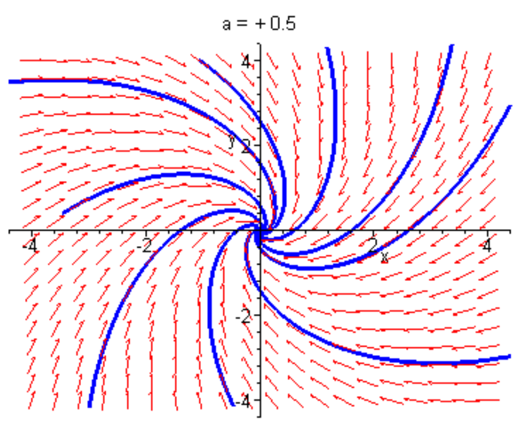




17. The characteristic equation of the coefficient matrix is $r^2 + 2r + 1 + \alpha = 0$, with roots given formally as $r_{1,2} = -1 \pm \sqrt{-\alpha}$. The roots are *real* provided that $\alpha \le 0$. First note that the *sum* of the roots is -2 and the *product* of the roots is $1 + \alpha$. For *negative* values of α , the roots are distinct, with one always negative. When $\alpha < -1$, the roots have *opposite* signs. Hence the equilibrium point is a *saddle*. For the case $-1 < \alpha < 0$, the roots are both *negative*, and the equilibrium point is a *stable node*. $\alpha = -1$ represents a transition from saddle to node. When $\alpha = 0$, both roots are equal. For the case $\alpha > 0$, the roots are complex conjugates, with negative real part. Hence the equilibrium point is a *stable spiral.*







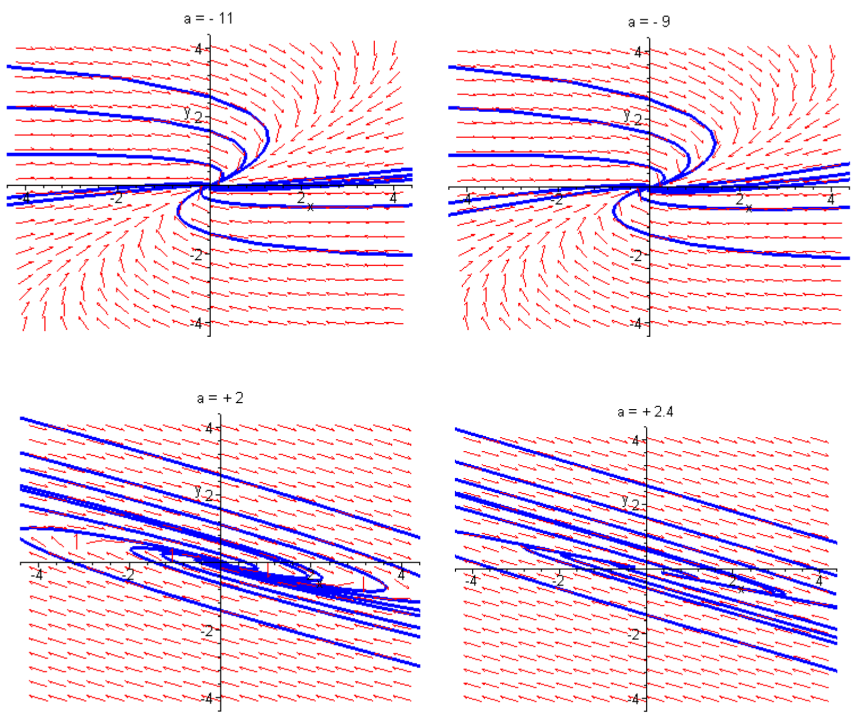
19. The characteristic equation for the system is given by

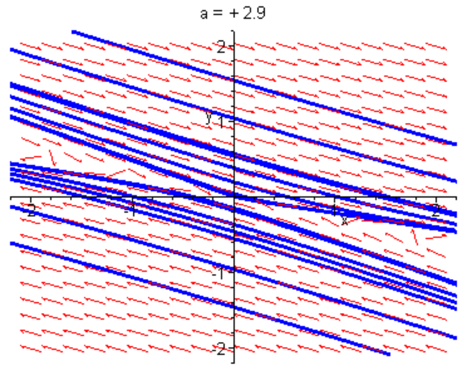
$$r^2 + (4-\alpha)r + 10 - 4\alpha = 0.$$

The roots are

$$r_{1,2} = -2 + \frac{\alpha}{2} \pm \sqrt{\alpha^2 + 8\alpha - 24}\,.$$

First note that the roots are *complex* when $-4-2\sqrt{10} < \alpha < -4+2\sqrt{10}$. We also find that when $-4-2\sqrt{10} < \alpha < 2$, the equilibrium point is a stable *spiral*. For the case $\alpha = 2$, the equilibrium point is a *center*. When $2 < \alpha < -4+2\sqrt{10}$, the equilibrium point is an unstable *spiral*. For all other cases, the roots are *real*. When $\alpha > 2.5$, the roots have *opposite* signs, with the equilibrium point being a *saddle*. For the case $-4+2\sqrt{10} < \alpha < 2.5$, the roots are both *positive*, and the equilibrium point is an unstable *node*. Finally, when $\alpha < -4-2\sqrt{10}$, both roots are negative, with the equilibrium point being a stable *node*.

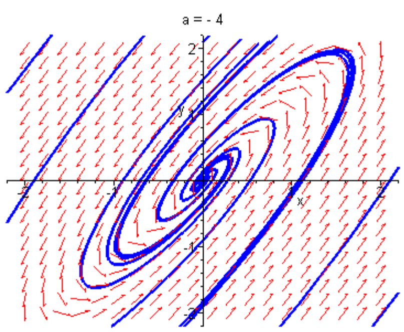


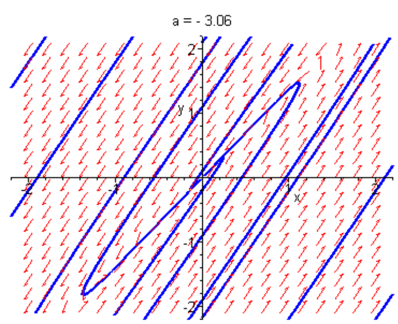


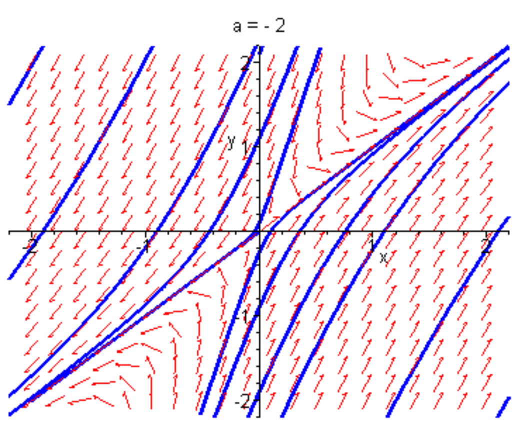
20. The characteristic equation is $r^2 + 2\,r - (24 + 8\alpha) = 0$, with roots

$$r_{1,2} = \; -1 \pm \sqrt{25 + 8\alpha}\;.$$

The roots are *complex* when $\alpha < \; -25/8$. Since the real part is negative, the origin is a stable *spiral*. Otherwise the roots are real. When $-25 < \alpha < \; -3$, both roots are negative, and hence the equilibrium point is a stable *node*. For $\alpha > \; -3$, the roots are of opposite sign and the origin is a *saddle*.







22. Based on the method in Prob. 19 of Section 7.5, setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the

algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation for the system is $r^2 + 1 = 0$, with roots $r_{1,2} = \pm i$. With $r = i$, the equations reduce to the single equation $\xi_1 - (2+i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i, 1)^T$. One *complex-valued* solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} t^i .$$

We can write $t^i = e^{i\,ln\,t}$. Hence

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{i\,ln\,t} \\ &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} [cos(ln\,t) + i\,sin(ln\,t)] \\ &= \begin{pmatrix} 2\,cos(ln\,t) - sin(ln\,t) \\ cos(ln\,t) \end{pmatrix} + i \begin{pmatrix} cos(ln\,t) + 2\,sin(ln\,t) \\ sin(ln\,t) \end{pmatrix}. \end{aligned}$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2\,cos(ln\,t) - sin(ln\,t) \\ cos(ln\,t) \end{pmatrix} + c_2 \begin{pmatrix} cos(ln\,t) + 2\,sin(ln\,t) \\ sin(ln\,t) \end{pmatrix}.$$

Other combinations are also possible.

24(a). The characteristic equation of the system is

$$r^3 + \frac{2}{5}r^2 + \frac{81}{80}r - \frac{17}{160} = 0,$$

with eigenvalues $r_1 = 1/10$, and $r_{2,3} = -1/4 \pm i$. For $r = 1/10$, simple calculations reveal that a corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (0, 0, 1)^T$. Setting $r = -1/4 - i$, we obtain the system of equations

$$\begin{aligned} \xi_1 - i\,\xi_2 &= 0 \\ \xi_3 &= 0\,. \end{aligned}$$

A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (i, 1, 0)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10}.$$

Another solution, which is *complex-valued*, is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-\left(\frac{1}{4}+i\right)t}$$
$$= \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-t/4}(cos\, t - i\, sint)$$
$$= e^{-t/4}\begin{pmatrix} sin\, t \\ cos\, t \\ 0 \end{pmatrix} + ie^{-t/4}\begin{pmatrix} cos\, t \\ -\, sin\, t \\ 0 \end{pmatrix}.$$

Using the real and imaginary parts of $\mathbf{x}^{(2)}$, the general solution is constructed as

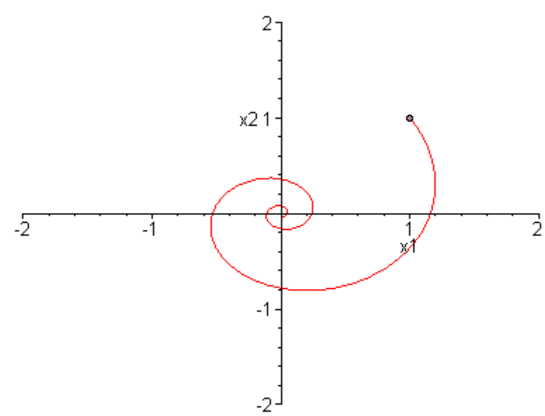
$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10} + c_2\, e^{-t/4}\begin{pmatrix} sin\, t \\ cos\, t \\ 0 \end{pmatrix} + c_3\, e^{-t/4}\begin{pmatrix} cos\, t \\ -\, sin\, t \\ 0 \end{pmatrix}.$$

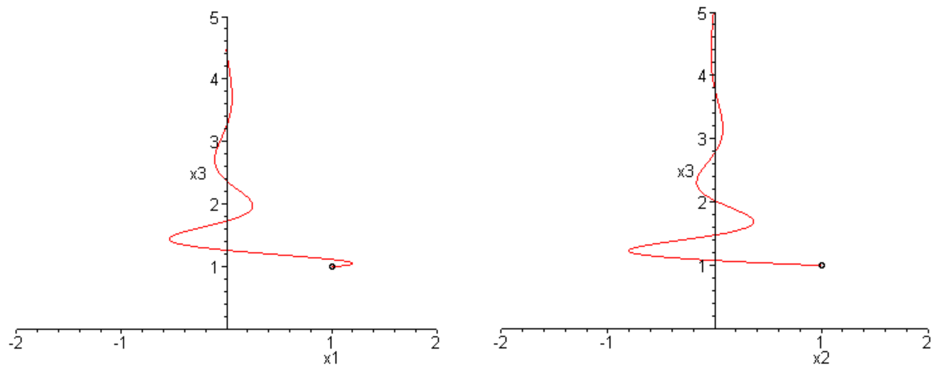
(b). Let $\mathbf{x}(0) = (x_1^0\,, x_2^0\,, x_3^0)$. The solution can be written as

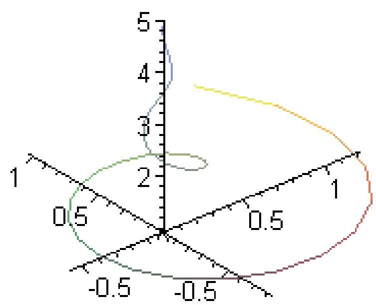
$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ x_3^0\, e^{t/10} \end{pmatrix} + e^{-t/4}\begin{pmatrix} x_2^0\, sin\, t + x_1^0\, cos\, t \\ x_2^0\, cos\, t - x_1^0\, sin\, t \\ 0 \end{pmatrix}.$$

With $\mathbf{x}(0) = (1\,, 1\,, 1)$, the solution of the initial value problem is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ e^{t/10} \end{pmatrix} + e^{-t/4}\begin{pmatrix} sin\, t + cos\, t \\ cos\, t - sin\, t \\ 0 \end{pmatrix}.$$







$25(a)$. Based on Probs. $18-20$ of Section 7.1, the system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

With $R_1 = R_2 = 4\,ohms$, $C = \frac{1}{2}\,farads$ and $L = 8\,henrys$, the eigenvalue problem is

$$\begin{pmatrix} -\frac{1}{2} - r & -\frac{1}{8} \\ 2 & -\frac{1}{2} - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b). The characteristic equation of the system is $r^2 + r + \frac{1}{2} = 0$, with eigenvalues

$$r_{1,2} = -\frac{1}{2} \pm \frac{1}{2}i\,.$$

Setting $r = -1/2 + i/2$, the algebraic equations reduce to $4i\xi_1 + \xi_2 = 0$. It follows that $\boldsymbol{\xi}^{(1)} = (1\,,\,-4i)^T$. Hence one *complex-valued* solution is

$$
\begin{aligned}
\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} &= \begin{pmatrix} 1 \\ -4i \end{pmatrix} e^{(-1+i)t/2} \\
&= \begin{pmatrix} 1 \\ -4i \end{pmatrix} e^{-t/2}[cos(t/2) + i\,sin(t/2)] \\
&= e^{-t/2}\begin{pmatrix} cos(t/2) \\ 4\,sin(t/2) \end{pmatrix} + ie^{-t/2}\begin{pmatrix} sin(t/2) \\ -4\,cos(t/2) \end{pmatrix}.
\end{aligned}
$$

Therefore the general solution is

$$
\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} cos(t/2) \\ 4\,sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} sin(t/2) \\ -4\,cos(t/2) \end{pmatrix}.
$$

(c). Imposing the initial conditions, we arrive at the equations $c_1 = 2$ and $c_2 = -\frac{3}{4}$, and

$$
\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t/2}\begin{pmatrix} 2\,cos(t/2) - \frac{3}{4}\,sin(t/2) \\ 8\,sin(t/2) + 3\,cos(t/2) \end{pmatrix}.
$$

(d). Since the eigenvalues have *negative* real parts, all solutions converge to the origin.

$26(a)$. The characteristic equation of the system is

$$
r^2 + \frac{1}{RC}r + \frac{1}{CL} = 0\,,
$$

with eigenvalues

$$
r_{1,2} = -\frac{1}{2RC} \pm \frac{1}{2RC}\sqrt{1 - \frac{4R^2C}{L}}\ .
$$

The eigenvalues are real and different provided that

$$
1 - \frac{4R^2C}{L} > 0\,.
$$

The eigenvalues are complex conjugates as long as

$$
1 - \frac{4R^2C}{L} < 0\,.
$$

(b). With the specified values, the eigenvalues are $r_{1,2} = -1 \pm i$. The eigenvector corresponding to $r = -1 + i$ is $\boldsymbol{\xi}^{(1)} = (1\,,\,-4i)^T$. Hence one *complex-valued* solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(-1+i)t}$$
$$= \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{-t}(\cos t + i \sin t)$$
$$= e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.$$

Therefore the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.$$

(c). Imposing the initial conditions, we arrive at the equations

$$c_1 = 2$$
$$-c_1 + c_2 = 1\,,$$

with $c_1 = 2$ and $c_2 = 3$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t}\begin{pmatrix} 2\cos t + 3\sin t \\ \cos t - 5\sin t \end{pmatrix}.$$

(d). Since $Re(r_{1,2}) = -1$, all solutions converge to the origin.

$27(a)$. Suppose that $c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{0}$. Since **a** and **b** are the real and imaginary parts of the vector $\boldsymbol{\xi}^{(1)}$, respectively, $\mathbf{a} = \left(\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}}\right)/2$ and $\mathbf{b} = \left(\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}}\right)/2i$. Hence

$$c_1\left(\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}}\right) - ic_2\left(\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}}\right) = \mathbf{0}\,,$$

which leads to

$$(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\overline{\boldsymbol{\xi}^{(1)}} = \mathbf{0}\,.$$

Now since $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}^{(1)}}$ are *linearly independent*, we must have

$$c_1 - ic_2 = 0$$
$$c_1 + ic_2 = 0\,.$$

It follows that $c_1 = c_2 = 0$.

(c). Recall that

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t + \mathbf{b}\sin\mu t)\,.$$

Consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, for some t_0. We can then write

$$c_1 e^{\lambda t_0}(\mathbf{a}\, cos\, \mu t_0 - \mathbf{b}\, sin\, \mu t_0) + c_2 e^{\lambda t_0}(\mathbf{a}\, cos\, \mu t_0 + \mathbf{b}\, sin\, \mu t_0) = \mathbf{0}\,. \;\; (*)$$

Rearranging the terms, and dividing by the exponential,

$$(c_1 + c_2) cos\, \mu t_0\, \mathbf{a} \; + (c_2 - c_1) sin\, \mu t_0\, \mathbf{b} = \mathbf{0}\,.$$

From Part (b), since **a** and **b** are *linearly independent*, it follows that

$$(c_1 + c_2) cos\, \mu t_0 = (c_2 - c_1) sin\, \mu t_0 = 0\,.$$

Without loss of generality, assume that the trigonometric factors are *nonzero*. Otherwise proceed again from Equation $(*)$, above. We then conclude that

$$c_1 + c_2 = 0 \text{ and } c_2 - c_1 = 0\,,$$

which leads to $c_1 = c_2 = 0$. Thus $\mathbf{u}(t_0)$ and $\mathbf{v}(t_0)$ are linearly independent for some t_0, and hence the functions are linearly independent at every point.

28(a). Let $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$\begin{aligned} x_2' &= u'' \\ &= -\frac{k}{m}\, u\,. \end{aligned}$$

In terms of the new variables, we obtain the system of two first order ODEs

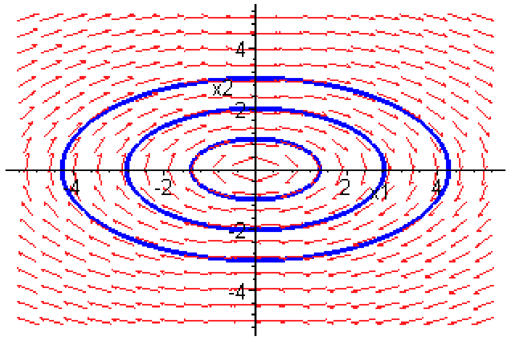
$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}\, x_1\,. \end{aligned}$$

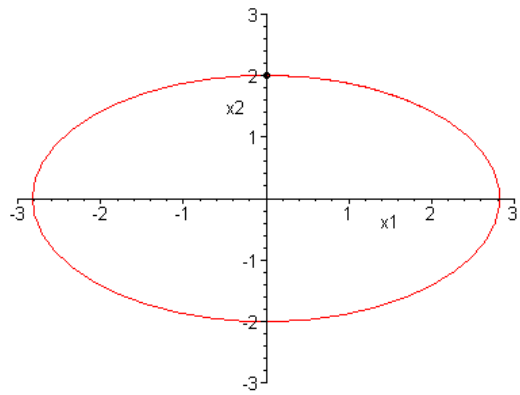
(b). The associated eigenvalue problem is

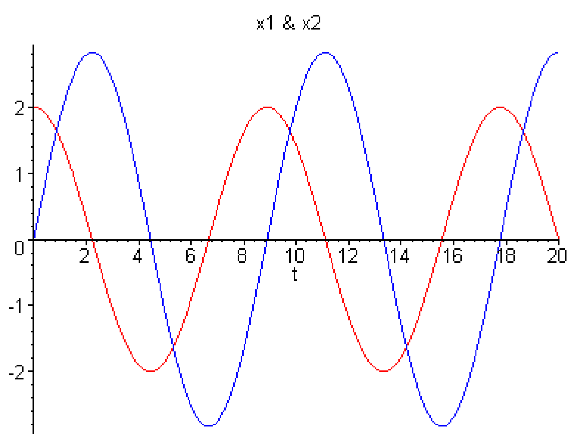
$$\begin{pmatrix} -r & 1 \\ -k/m & -r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + k/m = 0$, with roots $r_{1,2} = \pm\, i\sqrt{k/m}$.

(c). Since the eigenvalues are purely imaginary, the origin is a *center*. Hence the phase curves are *ellipses*, with a *clockwise* flow. For computational purposes, let $k = 1$ and $m = 2$.







(d). The general solution of the second order equation is

$$u(t) = c_1 cos\sqrt{\frac{k}{m}}\,t + c_2 sin\sqrt{\frac{k}{m}}\,t\,.$$

The general solution of the system of ODEs is given by

$$\mathbf{x} = c_1 \begin{pmatrix} \sqrt{\frac{m}{k}}\, sin\sqrt{\frac{k}{m}}\, t \\ cos\sqrt{\frac{k}{m}}\, t \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{\frac{m}{k}}\, cos\sqrt{\frac{k}{m}}\, t \\ -\, sin\sqrt{\frac{k}{m}}\, t \end{pmatrix}.$$

It is evident that the natural frequency of the system is equal to $Im(r_{1,2})$.