Section 4.3

2. The general solution of the homogeneous equation is $y_c = c_1 e^t + c_2 e^{-t} + c_3 cos\, t +$ $+ c_4 sin\, t$. Let $g_1(t) = 3t$ and $g_2(t) = cos\, t$. By inspection, we find that $Y_1(t) = -3t$. Since $g_2(t)$ is a solution of the homogeneous equation, set $Y_2(t) = t(Acos\, t + Bsin\, t)$. Substitution into the given ODE and comparing the coefficients of similar term results in $A = 0$ and $B = -1/4$. Hence the general solution of the nonhomogeneous problem is

$$y(t) = y_c(t) - 3t - \frac{t}{4} sin\, t\,.$$

3. The characteristic equation corresponding to the homogeneous problem can be written as $(r+1)(r^2+1) = 0$. The solution of the homogeneous equation is $y_c = c_1 e^{-t} +$ $+ c_2 cos\, t + c_3 sin\, t$. Let $g_1(t) = e^{-t}$ and $g_2(t) = 4t$. Since $g_1(t)$ is a solution of the homogeneous equation, set $Y_1(t) = Ate^{-t}$. Substitution into the ODE results in $A = 1/2$. Now let $Y_2(t) = Bt + C$. We find that $B = -C = 4$. Hence the general solution of the nonhomogeneous problem is $y(t) = y_c(t) + te^{-t}/2 + 4(t-1)$.

4. The characteristic equation corresponding to the homogeneous problem can be written as $r(r+1)(r-1) = 0$. The solution of the homogeneous equation is $y_c = c_1 + c_2 e^t +$ $+ c_3 e^{-t}$. Since $g(t) = 2\, sin\, t$ is not a solution of the homogeneous problem, we can set $Y(t) = A\, cos\, t + B\, sin\, t$. Substitution into the ODE results in $A = 1$ and $B = 0$. Thus
the general solution is $y(t) = c_1 + c_2 e^t + c_3 e^{-t} + cos\, t$.

6. The characteristic equation corresponding to the homogeneous problem can be written as $(r^2+1)^2 = 0$. It follows that $y_c = c_1 cos\, t + c_2 sin\, t + t(c_3 cos\, t + c_4 sin\, t)$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A + Bcos\, 2t + Csin\, 2t$. Substitution into the ODE results in $A = 3$, $B = 1/9$, $C = 0$. Thus the general solution is $y(t) = y_c(t) + 3 + \frac{1}{9} cos\, 2t$.

7. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1) = 0$. Thus the homogeneous solution is

$$y_c = c_1 + c_2\, t + c_3 t^2 + c_4 e^{-t} + e^{t/2}\left[c_5 cos\left(\sqrt{3}\, t/2\right) + c_5 sin\left(\sqrt{3}\, t/2\right)\right].$$

Note the $g(t) = t$ is a solution of the homogenous problem. Consider a particular solution
of the form $Y(t) = t^3(At + B)$. Substitution into the ODE results in $A = 1/24$ and $B = 0$. Thus the general solution is $y(t) = y_c(t) + t^4/24$.

8. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r+1) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2\, t + c_3 t^2 + c_4 e^{-t}$. Since $g(t)$ is *not* a solution of the homogeneous problem, set $Y(t) = Acos\, 2t + Bsin\, 2t$. Substitution into the ODE results in $A = 1/40$ and $B = 1/20$. Thus the general solution

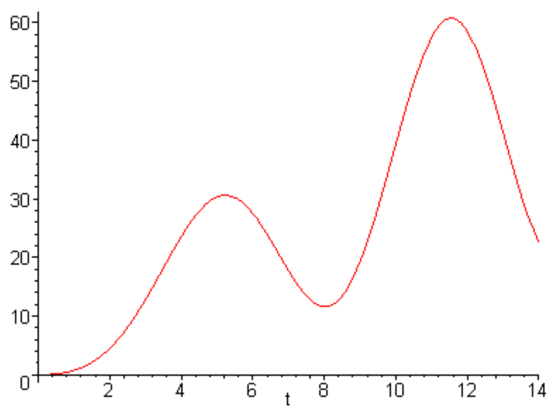
is $y(t) = y_c(t) + (cos\,2t + 2sin\,2t)/40$.

10. From Prob. 22 in Section 4.2 , the homogeneous solution is

$$y_c = c_1 cos\,t + c_2 sin\,t \; + \; + t[c_3 cos\,t + c_4 sin\,t].$$

Since $g(t)$ is *not* a solution of the homogeneous problem, substitute $Y(t) = At + B$ into the ODE to obtain $A = 3$ and $B = 4$. Thus the general solution is $y(t) = y_c(t) + 3t + 4$. Invoking the initial conditions, we find that $c_1 = -4$, $c_2 = -4$, $c_3 = 1$, $c_4 = -3/2$. Therefore the solution of the initial value problem is

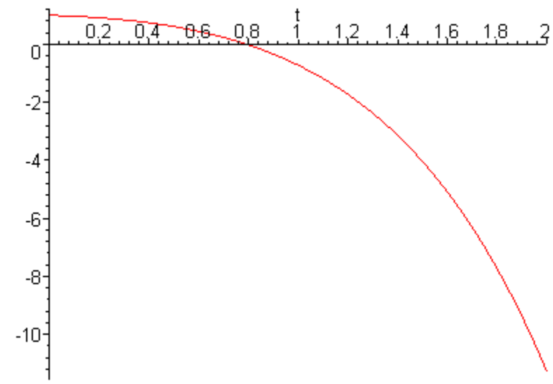
$$y(t) = (t-4)cos\,t - (3t/2 + 4)sin\,t + 3t + 4\,.$$



11. The characteristic equation can be written as $r(r^2 - 3r + 2) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{2t}$. Let $g_1(t) = e^t$ and $g_2(t) = t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1(t) = Ate^t$. Substitution into the ODE results in $A = -1$. Now let $Y_2(t) = Bt^2 + Ct$. Substitution into the ODE results in $B = 1/4$ and $C = 3/4$. Therefore the general solution is

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} - te^t + \left(t^2 + 3t\right)/4\,.$$

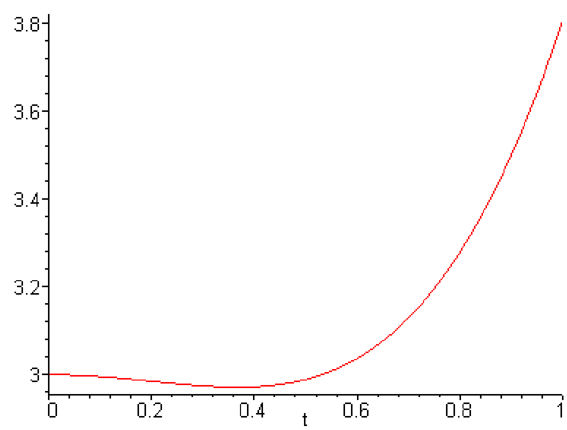
Invoking the initial conditions, we find that $c_1 = 1$, $c_2 = c_3 = 0$. The solution of the initial value problem is $y(t) = 1 - te^t + (t^2 + 3t)/4$.



12. The characteristic equation can be written as $(r-1)(r+3)(r^2+4)=0$. Hence the homogeneous solution is $y_c = c_1e^t + c_2e^{-3t} + c_3cos\,2t + c_4sin\,2t$. None of the terms in $g(t)$ is a solution of the homogeneous problem. Therefore we can assume a form $Y(t) = Ae^{-t} + Bcos\,t + Csin\,t$. Substitution into the ODE results in $A = 1/20$, $B = -2/5$, $C = -4/5$. Hence the general solution is

$$y(t) = c_1e^t + c_2e^{-3t} + c_3cos\,2t + c_4sin\,2t + e^{-t}/20 - (2cos\,t + 4sin\,t)/5\,.$$

Invoking the initial conditions, we find that $c_1 = 81/40$, $c_2 = 73/520$, $c_3 = 77/65$, $c_4 = -49/130$.



14. From Prob. 4, the homogeneous solution is $y_c = c_1 + c_2e^t + c_3e^{-t}$. Consider the terms $g_1(t) = te^{-t}$ and $g_2(t) = 2cos\,t$. Note that since $r = -1$ is a *simple* root of the characteristic equation, Table 4.3.1 suggests that we set $Y_1(t) = t(At+B)e^{-t}$. The function $2cos\,t$ is *not* a solution of the homogeneous equation. We can simply choose $Y_2(t) = Ccos\,t + Dsin\,t$. Hence the particular solution has the form

$$Y(t) = t(At+B)e^{-t} + Ccos\,t + Dsin\,t\,.$$

15. The characteristic equation can be written as $(r^2-1)^2 = 0$. The roots are given

as $r = \pm 1$, each with *multiplicity two*. Hence the solution of the homogeneous problem is $y_c = c_1 e^t + c_2 t e^t + c_3 e^{-t} + c_4 t e^{-t}$. Let $g_1(t) = e^t$ and $g_2(t) = sin\, t$. The function e^t is a solution of the homogeneous problem. Since $r = 1$ has multiplicity two, we set $Y_1(t) = At^2 e^t$. The function $sin\, t$ is *not* a solution of the homogeneous equation. We can set $Y_2(t) = Bcos\, t + Csin\, t$. Hence the particular solution has the form

$$Y(t) = At^2 e^t + Bcos\, t + Csin\, t\,.$$

16. The characteristic equation can be written as $r^2(r^2+4) = 0$, with roots $r = 0, \pm 2i$. The root $r = 0$ has multiplicity two, hence the homogeneous solution is $y_c = c_1 + c_2 t +$ $+ c_3 cos\, 2t + c_4 sin\, 2t$. The functions $g_1(t) = sin\, 2t$ and $g_2(t) = 4$ are solutions of the homogenous equation. The complex roots have multiplicity *one*, therefore we need to set $Y_1(t) = At\, cos\, 2t + Bt\, sin\, 2t$. Now $g_2(t) = 4$ is associated with the *double* root $r = 0$. Based on Table 4.3.1, set $Y_2(t) = Ct^2$. Finally, $g_3(t) = te^t$ (and its derivatives) is independent of the homogeneous solution. Therefore set $Y_3(t) = (Dt + E)e^t$. Conclude that the particular solution has the form

$$Y(t) = At\, cos\, 2t + Bt\, sin\, 2t + Ct^2 + (Dt + E)e^t.$$

18. The characteristic equation can be written as $r^2(r^2 + 2r + 2) = 0$, with roots $r = 0$, with multiplicity *two*, and $r = -1 \pm i$. The homogeneous solution is $y_c = c_1 + c_2 t +$ $+ c_3 e^{-t} cos\, t + c_4 e^{-t} sin\, t$. The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set $Y_1(t) = Ae^t + (Bt + C)e^{-t}$. Now $g_2(t) = e^{-t} sin\, t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set $Y_2(t) = t(D\, e^{-t} cos\, t + E\, e^{-t} sin\, t)$. It follows that the particular solution has the form

$$Y(t) = Ae^t + (Bt + C)e^{-t} + t\big(D\, e^{-t} cos\, t + E\, e^{-t} sin\, t\big).$$

19. Differentiating $y = u(t)v(t)$, successively, we have

$$\begin{aligned} y' &= u'v + uv' \\ y'' &= u''v + 2u'v' + uv'' \\ &\vdots \\ y^{(n)} &= \sum_{j=0}^{n} \binom{n}{j} u^{(n-j)} v^{(j)} \end{aligned}$$

Setting $v(t) = e^{\alpha t}$, $v^{(j)} = \alpha^j e^{\alpha t}$. So for any $p = 1, 2, \cdots, n$,

$$y^{(p)} = e^{\alpha t} \sum_{j=0}^{p} \binom{p}{j} \alpha^j\, u^{(p-j)}.$$

It follows that

$$L\big[e^{\alpha t}u\big] = e^{\alpha t}\sum_{p=0}^{n}\left[a_{\,n-p}\sum_{j=0}^{p}\binom{p}{j}\alpha^j\,u^{(p-j)}\right] \qquad (*).$$

It is evident that the right hand side of Eq. $(*)$ is of the form

$$e^{\alpha t}\big[k_0\,u^{(n)} + k_1\,u^{(n-1)} + \cdots + k_{\,n-1}u' + k_n\,u\big]\,.$$

Hence operator equation $L[e^{\alpha t}u] = e^{\alpha t}(b_0\,t^m + b_1\,t^{m-1} + \cdots + b_{\,m-1}t + b_m\,)$ can be written as

$$k_0\,u^{(n)} + k_1\,u^{(n-1)} + \cdots + k_{\,n-1}u' + k_n\,u =$$

$$= b_0\,t^m + b_1\,t^{m-1} + \cdots + b_{\,m-1}t + b_m\,.$$

The coefficients $k_{\,i}\,,\, i = 0, 1, \cdots, n$ can be determined by collecting the like terms in the double summation in Eq. $(*)$. For example, k_0 is the coefficient of $u^{(n)}$. The *only* term that contains $u^{(n)}$ is when $p = n$ and $j = 0$. Hence $k_0 = a_0$. On the other hand, k_n is the coefficient of $u(t)$. The inner summation in $(*)$ contains terms with u, given by $\alpha^p u$ (when $j = p$), for each $p = 0, 1, \cdots, n$. Hence

$$k_n = \sum_{p=0}^{n} a_{\,n-p}\,\alpha^p\,.$$

$21(a)$. Clearly, e^{2t} is a solution of $y' - 2y = 0$, and te^{-t} is a solution of the differential equation $y'' + 2y' + y = 0$. The latter ODE has characteristic equation $(r+1)^2 = 0$. Hence $(D-2)[3e^{2t}] = 3(D-2)[e^{2t}] = 0$ and $(D+1)^2[te^{-t}] = 0$. Furthermore, we have $(D-2)(D+1)^2[te^{-t}] = (D-2)[0] = 0$, and $(D-2)(D+1)^2[3e^{2t}] =$
$= (D+1)^2(D-2)[3e^{2t}] = (D+1)^2[0] = 0$.

(b). Based on Part (a),

$$\begin{aligned}(D-2)(D+1)^2\big[(D-2)^3(D+1)Y\big] &= (D-2)(D+1)^2\big[3e^{2t} - te^{-t}\big]\\ &= 0\,,\end{aligned}$$

since the operators are linear. The implied operations are associative and commutative. Hence

$$(D-2)^4(D+1)^3Y = 0\,.$$

The operator equation corresponds to the solution of a linear homogeneous ODE with characteristic equation $(r-2)^4(r+1)^3 = 0$. The roots are $r = 2$, with multiplicity 4 and $r = -1$, with multiplicity 3. It follows that the given homogeneous solution is

$$Y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4t^3e^{2t} + c_5e^{-t} + c_6te^{-t} + c_7t^2e^{-t},$$

which is a linear combination of seven independent solutions.

22(15). Observe that $(D-1)[e^t]=0$ and $(D^2+1)[sin\,t]=0$. Hence the operator $H(D)=(D-1)(D^2+1)$ is an annihilator of $e^t+sin\,t$. The operator corresponding to the left hand side of the given ODE is $(D^2-1)^2$. It follows that

$$(D+1)^2(D-1)^3\big(D^2+1\big)Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$Y(t)=c_1e^{-t}+c_2te^{-t}+c_3e^t+c_4te^t+c_5t^3e^t+c_6cos\,t+c_7sin\,t\,.$$

After examining the homogeneous solution of Prob. 15 , and eliminating duplicate terms, we have

$$Y(t)=c_5t^3e^t+c_6cos\,t+c_7sin\,t\,.$$

22(16). We find that $D[4]=0$, $(D-1)^2[te^t]=0$, and $(D^2+4)[sin\,2t]=0$.
The operator $H(D)=D(D-1)^2(D^2+4)$is an annihilator of $t^2+te^t+sin\,2t$. The operator corresponding to the left hand side of the ODE is $D^2(D^2+4)$. It follows that

$$D^3(D-1)^2\big(D^2+4\big)^2Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$Y(t)=c_1+c_2t+c_3t^2+c_4e^t+c_5te^t+c_6cos\,2t+c_7sin\,2t+c_8tcos\,2t+c_9tsin\,2t\,.$$

After examining the homogeneous solution of Prob. 16 , and eliminating duplicate terms, we have

$$Y(t)=c_3t^2+c_4e^t+c_5te^t+c_8tcos\,2t+c_9tsin\,2t\,.$$

22(18). Observe that $(D-1)[e^t]=0$, $(D+1)^2[te^{-t}]=0$. The function $e^{-t}sin\,t$ is a solution of a second order ODE with characteristic roots $r=\,-1\pm i$. It follows that $(D^2+2D+2)[e^{-t}sin\,t]=0$. Therefore the operator

$$H(D)=(D-1)(D+1)^2\big(D^2+2D+2\big)$$

is an annihilator of $3e^t+2te^{-t}+e^{-t}sin\,t$. The operator corresponding to the left hand side of the given ODE is $D^2(D^2+2D+2)$. It follows that

$$D^2(D-1)(D+1)^2\big(D^2+2D+2\big)^2Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$\begin{aligned}Y(t)=\;&c_1+c_2t+c_3e^t+c_4e^{-t}+c_5te^{-t}+\\&+e^{-t}(c_6cos\,t+c_7sin\,t)+te^{-t}(c_8cos\,t+c_9sin\,t\,)\,.\end{aligned}$$

After examining the homogeneous solution of Prob. 18 , and eliminating duplicate terms, we have

$$Y(t) = c_3e^t + c_4e^{-t} + c_5te^{-t} + te^{-t}(c_8cos\, t + c_9sin\, t\,).$$