Section 8.6

1. In vector notation, the initial value problem can be written as

$$\frac{d}{dt}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x+y+t\\ 4x-2y\end{pmatrix},\ \mathbf{x}(0) = \begin{pmatrix}1\\ 0\end{pmatrix}.$$

(a). The Euler formula is

$$\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix} = \begin{pmatrix}x_n\\ y_n\end{pmatrix} + h\begin{pmatrix}x_n+y_n+t_n\\ 4x_n-2y_n\end{pmatrix}.$$

That is,

$$\begin{aligned}x_{n+1} &= x_n + h(x_n+y_n+t_n)\\ y_{n+1} &= y_n + h(4x_n-2y_n).\end{aligned}$$

With $h = 0.1$, we obtain the values

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.26	1.7714	2.58991	3.82374	5.64246
y_n	0.76	1.4824	2.3703	3.60413	5.38885

(b). The Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = (x_n+y_n+t_n\,,4x_n-2y_n)^T$$

$$\mathbf{k}_{n2} = \left[x_n+\frac{h}{2}k_{n1}^1+y_n+\frac{h}{2}k_{n1}^2+t_n+\frac{h}{2}\,,4\left(x_n+\frac{h}{2}k_{n1}^1\right)-2\left(y_n+\frac{h}{2}k_{n1}^2\right)\right]^T$$

$$\mathbf{k}_{n3} = \left[x_n+\frac{h}{2}k_{n2}^1+y_n+\frac{h}{2}k_{n2}^2+t_n+\frac{h}{2}\,,4\left(x_n+\frac{h}{2}k_{n2}^1\right)-2\left(y_n+\frac{h}{2}k_{n2}^2\right)\right]^T$$

$$\mathbf{k}_{n4} = [x_n+hk_{n3}^1+y_n+hk_{n3}^2+t_n+h\,,4(x_n+hk_{n3}^1)-2(y_n+hk_{n3}^2)]^T.$$

With $h = 0.2$, we obtain the values:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.32493	1.93679	2.93414	4.48318	6.84236
y_n	0.758933	1.57919	2.66099	4.22639	6.56452

(c). With $h = 0.1$, we obtain

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.32489	1.9369	2.93459	4.48422	6.8444
y_n	0.759516	1.57999	2.66201	4.22784	6.56684

The exact solution of the IVP is

$$x(t) = e^{2t} + \frac{2}{9}e^{-3t} - \frac{1}{3}t - \frac{2}{9}$$
$$y(t) = e^{2t} - \frac{8}{9}e^{-3t} - \frac{2}{3}t - \frac{1}{9}\,.$$

$3(a)$. The Euler formula is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h\begin{pmatrix} -\,t_n\,x_n - y_n - 1 \\ x_n \end{pmatrix}.$$

That is,

$$x_{n+1} = x_n + h(\,-\,t_n\,x_n - y_n - 1)$$
$$y_{n+1} = y_n + h(x_n).$$

With $h = 0.1$, we obtain the values

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	0.582	0.117969	$-\,0.336912$	$-\,0.730007$	$-\,1.02134$
y_n	1.18	1.27344	1.27382	1.18572	1.02371

(b). The Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = (\,-\,t_n\,x_n - y_n - 1, x_n)^T$$
$$\mathbf{k}_{n2} = \left[\,-\left(t_n + \frac{h}{2}\right)\left(x_n + \frac{h}{2}k^1_{n1}\right) - \left(y_n + \frac{h}{2}k^2_{n1}\right) - 1\,, x_n + \frac{h}{2}k^1_{n1}\right]^T$$
$$\mathbf{k}_{n3} = \left[\,-\left(t_n + \frac{h}{2}\right)\left(x_n + \frac{h}{2}k^1_{n2}\right) - \left(y_n + \frac{h}{2}k^2_{n2}\right) - 1\,, x_n + \frac{h}{2}k^1_{n2}\right]^T$$
$$\mathbf{k}_{n4} = [\,-\,(t_n + h)(x_n + hk^1_{n3}) - (y_n + hk^2_{n3}\,) - 1\,, x_n + hk^1_{n3}]^T.$$

With $h = 0.2$, we obtain the values:

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	0.568451	0.109776	− 0.32208	− 0.681296	− 0.937852
y_n	1.15775	1.22556	1.20347	1.10162	0.937852

(c). With $h = 0.1$, we obtain

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	0.56845	0.109773	− 0.322081	− 0.681291	− 0.937841
y_n	1.15775	1.22557	1.20347	1.10161	0.93784

$4(a)$. The Euler formula gives

$$
\begin{aligned}
x_{n+1} &= x_n + h(x_n - y_n + x_n\, y_n) \\
y_{n+1} &= y_n + h(3x_n - 2y_n - x_n\, y_n).
\end{aligned}
$$

With $h = 0.1$, we obtain the values

	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	− 0.198	− 0.378796	− 0.51932	− 0.594324	− 0.588278
y_n	0.618	0.28329	− 0.0321025	− 0.326801	− 0.57545

(b). Given

$$
\begin{aligned}
f(t, x, y) &= x - y + x\, y \\
g(t, x, y) &= 3x - 2y - x\, y\,,
\end{aligned}
$$

the Runge-Kutta method uses the following intermediate calculations:

$$
\begin{aligned}
\mathbf{k}_{n1} &= [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T \\
\mathbf{k}_{n2} &= \left[f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right)\right]^T \\
\mathbf{k}_{n3} &= \left[f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right)\right]^T \\
\mathbf{k}_{n4} &= \left[f\left(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2\right), g\left(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2\right)\right]^T.
\end{aligned}
$$

With $h = 0.2$, we obtain the values:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	-0.196904	-0.372643	-0.501302	-0.561270	-0.547053
y_n	0.630936	0.298888	-0.0111429	-0.288943	-0.508303

(c). With $h = 0.1$, we obtain

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	-0.196935	-0.372687	-0.501345	-0.561292	-0.547031
y_n	0.630939	0.298866	-0.0112184	-0.28907	-0.508427

$5(a)$. The Euler formula gives

$$x_{n+1} = x_n + h[x_n(1 - 0.5\,x_n - 0.5\,y_n)]$$
$$y_{n+1} = y_n + h[y_n(-0.25 + 0.5\,x_n)].$$

With $h = 0.1$, we obtain the values

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	2.96225	2.34119	1.90236	1.56602	1.29768
y_n	1.34538	1.67121	1.97158	2.23895	2.46732

(b). Given

$$f(t,x,y) = x(1 - 0.5\,x - 0.5\,y)$$
$$g(t,x,y) = y(-0.25 + 0.5\,x),$$

the Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T$$
$$\mathbf{k}_{n2} = \left[f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n1}, y_n + \frac{h}{2}k^2_{n1}\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n1}, y_n + \frac{h}{2}k^2_{n1}\right)\right]^T$$
$$\mathbf{k}_{n3} = \left[f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n2}, y_n + \frac{h}{2}k^2_{n2}\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k^1_{n2}, y_n + \frac{h}{2}k^2_{n2}\right)\right]^T$$
$$\mathbf{k}_{n4} = \left[f\left(t_n + h, x_n + hk^1_{n3}, y_n + hk^2_{n3}\right), g\left(t_n + h, x_n + hk^1_{n3}, y_n + hk^2_{n3}\right)\right]^T.$$

With $h = 0.2$, we obtain the values:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	3.06339	2.44497	1.9911	1.63818	1.3555
y_n	1.34858	1.68638	2.00036	2.27981	2.5175

(c). With $h = 0.1$, we obtain

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	3.06314	2.44465	1.99075	1.63781	1.35514
y_n	1.34899	1.68699	2.00107	2.28057	2.51827

$6(a)$. The Euler formula gives

$$x_{n+1} = x_n + h[exp(-x_n + y_n) - cos\, x_n]$$
$$y_{n+1} = y_n + h[sin(x_n - 3\, y_n)].$$

With $h = 0.1$, we obtain the values

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.42386	1.82234	2.21728	2.61118	2.9955
y_n	2.18957	2.36791	2.53329	2.68763	2.83354

(b). The Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = [f(t_n, x_n, y_n), g(t_n, x_n, y_n)]^T$$
$$\mathbf{k}_{n2} = \left[f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right)\right]^T$$
$$\mathbf{k}_{n3} = \left[f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right), g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right)\right]^T$$
$$\mathbf{k}_{n4} = \left[f\left(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2\right), g\left(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2\right)\right]^T.$$

With $h = 0.2$, we obtain the values:

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.41513	1.81208	2.20635	2.59826	2.97806
y_n	2.18699	2.36233	2.5258	2.6794	2.82487

(c). With $h = 0.1$, we obtain

	$n=2$	$n=4$	$n=6$	$n=8$	$n=10$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.41513	1.81209	2.20635	2.59826	2.97806
y_n	2.18699	2.36233	2.52581	2.67941	2.82488

7. The Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = [x_n - 4\,y_n,\; -x_n + y_n]^T$$
$$\mathbf{k}_{n2} = \left[x_n + \frac{h}{2}k_{n1}^1 - 4\left(y_n + \frac{h}{2}k_{n1}^2\right),\; -\left(x_n + \frac{h}{2}k_{n1}^1\right) + y_n + \frac{h}{2}k_{n1}^2\right]^T$$
$$\mathbf{k}_{n3} = \left[x_n + \frac{h}{2}k_{n2}^1 - 4\left(y_n + \frac{h}{2}k_{n2}^2\right),\; -\left(x_n + \frac{h}{2}k_{n2}^1\right) + y_n + \frac{h}{2}k_{n2}^2\right]^T$$
$$\mathbf{k}_{n4} = \left[x_n + hk_{n3}^1 - 4(y_n + hk_{n3}^2),\; -(x_n + hk_{n3}^1) + y_n + hk_{n3}^2\right]^T.$$

Using $h = 0.04$, we obtain the following values:

	$n=5$	$n=10$	$n=15$	$n=20$	$n=25$
t_n	0.2	0.4	0.6	0.8	1.0
x_n	1.3204	1.9952	3.2992	5.7362	10.227
y_n	− 0.25085	− 0.66245	− 1.3752	− 2.6435	− 4.9294

The exact solution is given by

$$\phi(t) = \frac{e^{-t} + e^{3t}}{2}\,,\; \psi(t) = \frac{e^{-t} - e^{3t}}{4}\,,$$

and the associated tabulated values:

	$n=5$	$n=10$	$n=15$	$n=20$	$n=25$
t_n	0.2	0.4	0.6	0.8	1.0
$\phi(t_n)$	1.3204	1.9952	3.2992	5.7362	10.227
$\psi(t_n)$	− 0.25085	− 0.66245	− 1.3752	− 2.6435	− 4.9294

8. Let $y = x'$. The second order ODE can be transformed into the first order system

$$x' = y$$
$$y' = t - 3x - t^2 y\,,$$

with initial conditions $x(0) = 1$, $y(0) = 2$. Given

$$f(t, x, y) = y$$
$$g(t, x, y) = t - 3x - t^2 y\,,$$

the Runge-Kutta method uses the following intermediate calculations:

$$\mathbf{k}_{n1} = \left[y_n, t_n - 3\,x_n - t_n^2\,y_n\right]^T$$
$$\mathbf{k}_{n2} = \left[y_n + \frac{h}{2}k_{n1}^2, g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n1}^1, y_n + \frac{h}{2}k_{n1}^2\right)\right]^T$$
$$\mathbf{k}_{n3} = \left[y_n + \frac{h}{2}k_{n2}^2, g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{n2}^1, y_n + \frac{h}{2}k_{n2}^2\right)\right]^T$$
$$\mathbf{k}_{n4} = \left[y_n + hk_{n3}^2, g\left(t_n + h, x_n + hk_{n3}^1, y_n + hk_{n3}^2\right)\right]^T.$$

With $h = 0.1$, we obtain the following values:

	$n = 5$	$n = 10$
t_n	0.5	1.0
x_n	1.543	0.07075
y_n	1.14743	-1.3885

9. The *predictor* formulas are

$$x_{n+1} = x_n + \frac{h}{24}(55\,f_n - 59\,f_{n-1} + 37\,f_{n-2} - 9\,f_{n-3})$$
$$y_{n+1} = y_n + \frac{h}{24}(55\,g_n - 59\,g_{n-1} + 37\,g_{n-2} - 9\,g_{n-3}).$$

With $f_{n+1} = x_{n+1} - 4\,y_{n+1}$ and $g_{n+1} = -x_{n+1} + y_{n+1}$, the *corrector* formulas are

$$x_{n+1} = x_n + \frac{h}{24}(9\,f_{n+1} + 19\,f_n - 5\,f_{n-1} + f_{n-2})$$
$$y_{n+1} = y_n + \frac{h}{24}(9\,g_{n+1} + 19\,g_n - 5\,g_{n-1} + g_{n-2}).$$

We use the starting values from the *exact solution* :

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
t_n	0	0.1	0.2	0.3
x_n	1.0	1.12883	1.32042	1.60021
y_n	0.0	-0.11057	-0.250847	-0.429696

One time step using the *predictor-corrector* method results in the approximate values:

	$n = 4(pre)$	$n = 4(cor)$
t_n	0.4	0.4
x_n	1.99445	1.99521
y_n	-0.662064	-0.662442