Section 9.7

3. The equilibrium solutions of the ODE

$$\frac{dr}{dt} = r(r-1)(r-3)$$

are given by $r_1 = 0$, $r_2 = 1$ and $r_3 = 3$. Note that

$$\frac{dr}{dt} > 0 \text{ for } 0 < r < 1 \text{ and } r > 3 \ ; \quad \frac{dr}{dt} < 0 \text{ for } 1 < r < 3 \,.$$

$r = 0$ corresponds to an *unstable* critical point. The equilibrium solution $r_2 = 1$ is *asymptotically stable*, whereas the equilibrium solution $r_3 = 3$ is *unstable*. Since the critical values are *isolated*, a limit cycle is given by

$$r = 1 \,, \theta = t + t_0$$

which is *asymptotically stable.* Another periodic solution is found to be

$$r = 3 \,, \theta = t + t_0$$

which is *unstable.*

5. The equilibrium solutions of the ODE

$$\frac{dr}{dt} = sin\,\pi r$$

are given by $r = n$, $n = 0\,,1\,,2\,,\cdots$. Based on the *sign* of r' in the neighborhood of each critical value, the equilibrium solutions $r = 2k$, $k = 1\,,2\,,\cdots$ correspond to *unstable* periodic solutions, with $\theta = t + t_0$. The equilibrium solutions $r = 2k+1$, $k = 0\,,1\,,2\,,\cdots$ correspond to *stable* limit cycles, with $\theta = t + t_0$. The solution $r = 0$ represents an *unstable* critical point.

10. Given $F(x\,,y) = a_{11}\,x + a_{12}\,y$ and $G(x\,,y) = a_{21}\,x + a_{22}\,y$, it follows that

$$F_x + G_y = a_{11} + a_{22} \,.$$

Based on the hypothesis, $F_x + G_y$ is either *positive* or *negative* on the entire plane. By Theorem 9.7.2 , the system cannot have a nontrivial periodic solution.

12. Given that $F(x\,,y) = \,-\,2x - 3y - xy^2$ and $G(x\,,y) = y + x^3 - x^2y$,

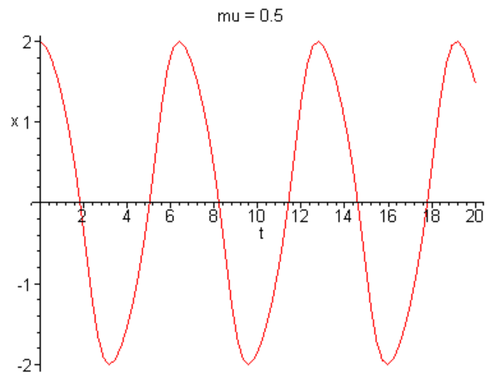
$$F_x + G_y = \,-\,1 - x^2 - y^2.$$

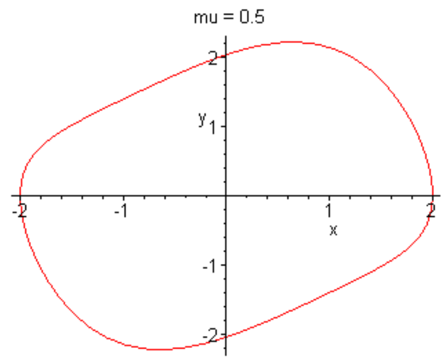
Since $F_x + G_y < 0$ on the entire plane, Theorem 9.7.2 asserts that the system cannot have a nontrivial periodic solution.

$14(a)$. Based on the given graphs, the following table shows the estimated values:

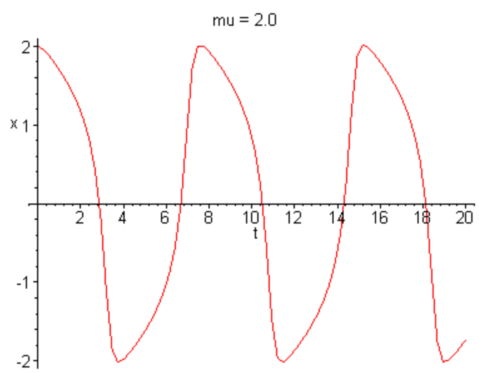
$\mu = 0.2$	$T \approx 6.29$
$\mu = 1.0$	$T \approx 6.66$
$\mu = 5.0$	$T \approx 11.60$

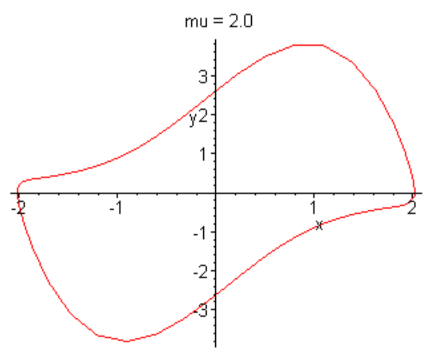
(b). The initial conditions were chosen as $x(0) = 2\,,\, y(0) = 0\,.$



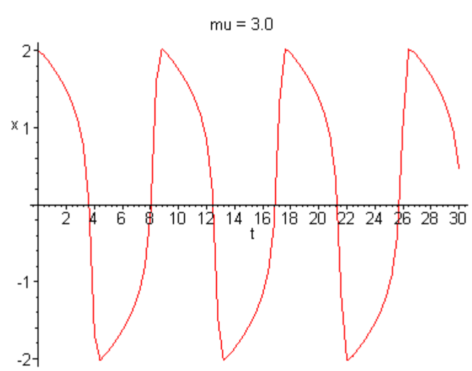


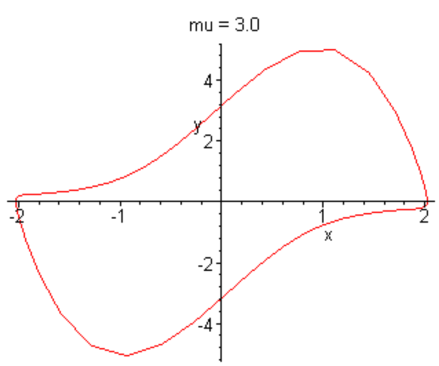
$T \approx 6.38\,.$



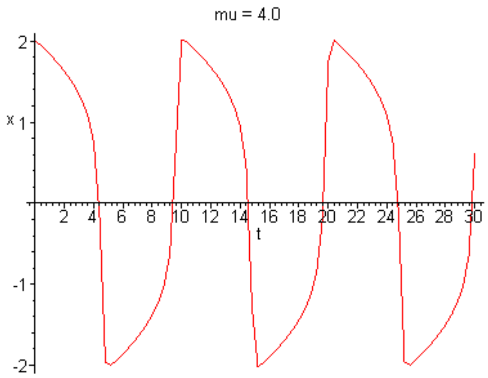


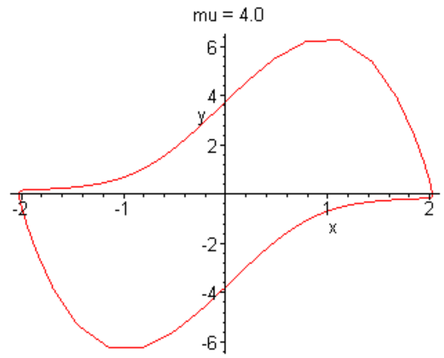
$T \approx 7.65\,.$





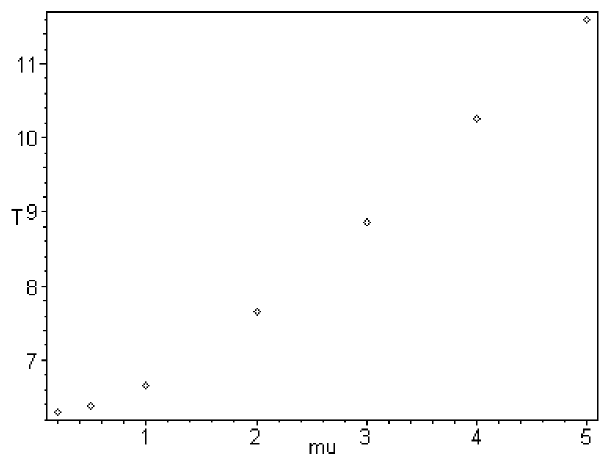
$T \approx 8.86$.





$T \approx 10.25$.

(c). The period, T , appears to be a *quadratic* function of μ .



$15(a)$. Setting $x = u$ and $y = u'$, we obtain the system of equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + \mu\left(1 - \frac{1}{3}y^2\right)y\,.$$

(b). Evidently, $y = 0$. It follows that $x = 0$. Hence the only critical point of the system is at $(0\,,0)$. The components of the vector field are infinitely differentiable everywhere. Therefore the system is *almost linear*.

The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & \mu - \mu y^2 \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

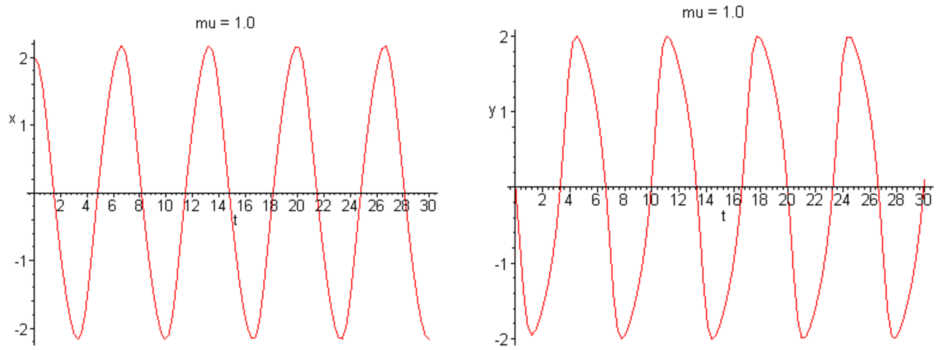
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix},$$

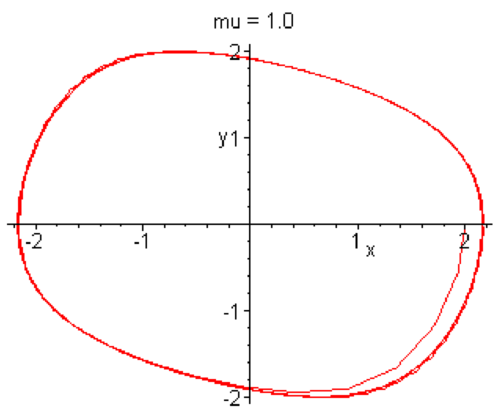
with eigenvalues

$$r_{1,2} = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}\ .$$

If $\mu = 0$, the equation reduces to the ODE for a simple harmonic oscillator. For the case $0 < \mu < 2$, the eigenvalues are *complex*, and the critical point is an *unstable spiral*. For $\mu \geq 2$, the eigenvalues are *real*, and the origin is an *unstable node*.

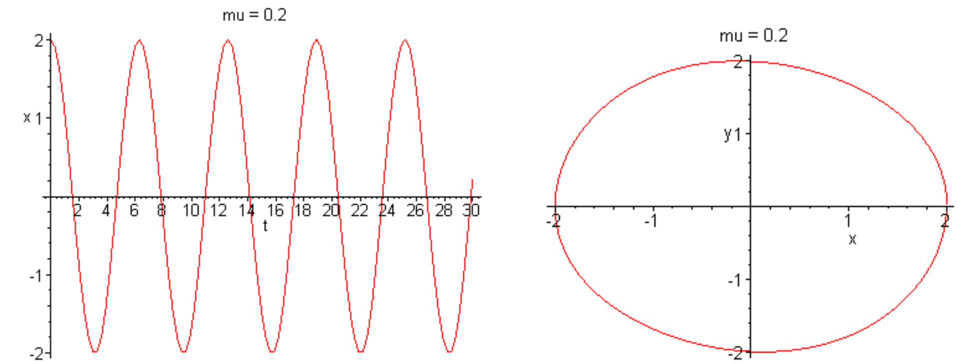
(c). The initial conditions were chosen as $x(0) = 2\,, y(0) = 0$.



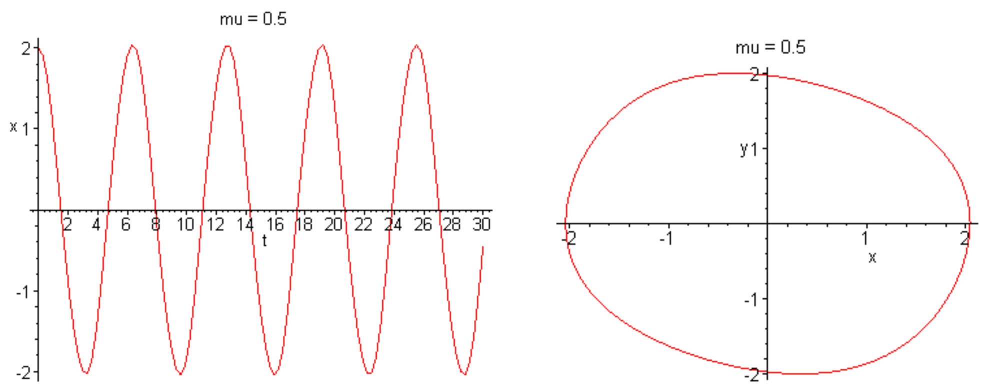


$A \approx 2.16$ and $T \approx 6.65$.

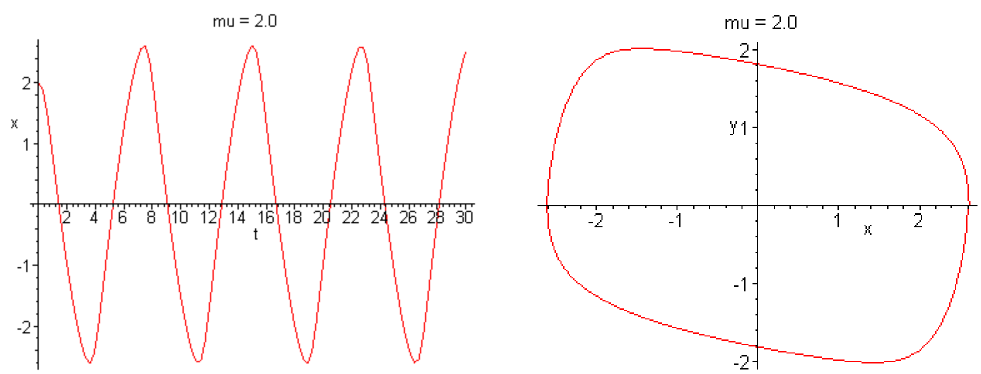
(d).



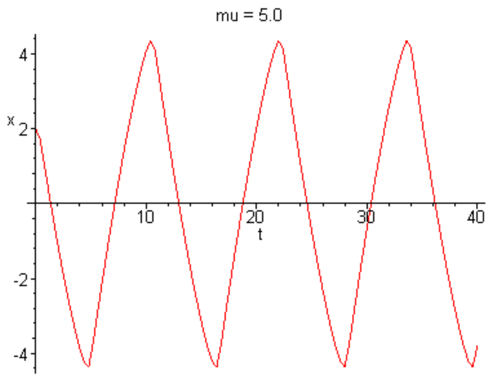
$A \approx 2.00$ and $T \approx 6.30$.

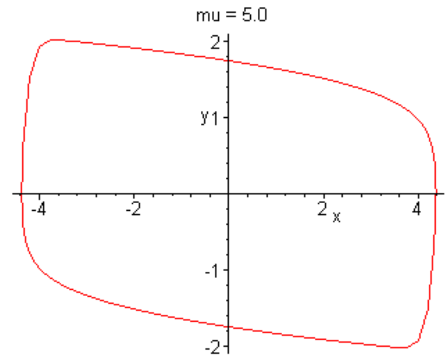


$A \approx 2.04$ and $T \approx 6.38$.



$A \approx 2.6$ and $T \approx 7.62$.





$A \approx 4.37$ and $T \approx 11.61$.

(e).

	A	T
$\mu = 0.2$	2.00	6.30
$\mu = 0.5$	2.04	6.38
$\mu = 1.0$	2.16	6.65
$\mu = 2.0$	2.6	7.62
$\mu = 5.0$	4.37	11.61