Chapter Eight

Section 8.1

2. The Euler formula for this problem is

$$y_{n+1} = y_n + h\big(5\,t_n - 3\sqrt{y_n}\,\big),$$

in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write

$$y_{n+1} = y_n + 5nh^2 - 3h\sqrt{y_n}\,,$$

with $y_0 = 2$.

(a). Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.59980	1.29288	1.07242	0.930175

(b). Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.61124	1.31361	1.10012	0.962552

The *backward* Euler formula is

$$y_{n+1} = y_n + h\big(5\,t_{n+1} - 3\sqrt{y_{n+1}}\,\big),$$

in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write

$$y_{n+1} = y_n + 5(n+1)h^2 - 3h\sqrt{y_{n+1}}\,,$$

with $y_0 = 2$. Solving for y_{n+1}, and choosing the *positive* root, we find that

$$y_{n+1} = \left[-\frac{3}{2}h + \frac{1}{2}\sqrt{(20n+29)h^2 + 4y_n}\right]^2.$$

(c). Backward Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.64337	1.37164	1.17763	1.05334

(d). Backward Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.63301	1.35295	1.15267	1.02407

3. The Euler formula for this problem is

$$y_{n+1} = y_n + h(2\,y_n - 3\,t_n\,),$$

in which $t_n = t_0 + nh$. Since $t_0 = 0$,

$$y_{n+1} = y_n + 2hy_n - 3nh^2\,,$$

with $y_0 = 1$.

(a). Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.2025	1.41603	1.64289	1.88590

(b). Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.20388	1.41936	1.64896	1.89572

The *backward* Euler formula is

$$y_{n+1} = y_n + h(2\,y_{n+1} - 3\,t_{n+1}\,),$$

in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write

$$y_{n+1} = y_n + 2h\,y_{n+1} - 3(n+1)h^2\,,$$

with $y_0 = 1$. Solving for y_{n+1}, we find that

$$y_{n+1} = \frac{y_n - 3(n+1)h^2}{1-2h}\,.$$

(c). Backward Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.20864	1.43104	1.67042	1.93076

(d). Backward Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.20693	1.42683	1.66265	1.91802

4. The Euler formula is

$$y_{n+1} = y_n + h[2\,t_n + exp(-t_n\,y_n)].$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + 2nh^2 + h\,exp(-nh\,y_n),$$

with $y_0 = 1$.

(a). Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.10244	1.21426	1.33484	1.46399

(b). Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.10365	1.21656	1.33817	1.46832

The *backward* Euler formula is

$$y_{n+1} = y_n + h[2\,t_{n+1} + exp(-t_{n+1}\,y_{n+1})].$$

Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1} = y_n + 2h^2(n+1) + h\,exp[-(n+1)h\,y_{n+1}],$$

with $y_0 = 1$. This equation cannot be solved *explicitly* for y_{n+1}. At each step, given the current value of y_n, the equation must be solved *numerically* for y_{n+1}.

(c). Backward Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.10720	1.22333	1.34797	1.48110

(d). Backward Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.10603	1.22110	1.34473	1.47688

6. The Euler formula for this problem is

$$y_{n+1} = y_n + h\left(t_n^2 - y_n^2\right) sin\, y_n \,.$$

Here $t_0 = 0$ and $t_n = nh$. So that

$$y_{n+1} = y_n + h\left(n^2h^2 - y_n^2\right) sin\, y_n \,,$$

with $y_0 = -1$.

(a). Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	-0.920498	-0.857538	-0.808030	-0.770038

(b). Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	-0.922575	-0.860923	-0.82300	-0.774965

The *backward* Euler formula is

$$y_{n+1} = y_n + h\left(t_{n+1}^2 - y_{n+1}^2\right) sin\, y_{n+1}.$$

Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1} = y_n + h\left[(n+1)^2 h^2 - y_{n+1}^2\right] sin\, y_{n+1}\,,$$

with $y_0 = -1$. Note that this equation cannot be solved *explicitly* for y_{n+1}. Given y_n, the transcendental equation

$$y_{n+1} + h\, y_{n+1}^2 sin\, y_{n+1} = y_n + h(n+1)^2 h^2$$

must be solved *numerically* for y_{n+1}.

(c). Backward Euler method with $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	-0.928059	-0.870054	-0.824021	-0.788686

(d). Backward Euler method with $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	-0.926341	-0.867163	-0.820279	-0.784275

8. The Euler formula

$$y_{n+1} = y_n + h\left(5\, t_n - 3\sqrt{y_n}\,\right),$$

in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write

$$y_{n+1} = y_n + 5nh^2 - 3h\sqrt{y_n}\,,$$

with $y_0 = 2$.

(a). Euler method with $h = 0.025$:

	$n=20$	$n=40$	$n=60$	$n=80$
t_n	0.5	1.0	1.5	2.0
y_n	0.891830	1.25225	2.37818	4.07257

(b). Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.908902	1.26872	2.39336	4.08799

The *backward* Euler formula is

$$y_{n+1} = y_n + h\big(5\,t_{n+1} - 3\sqrt{y_{n+1}}\,\big),$$

in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write

$$y_{n+1} = y_n + 5(n+1)h^2 - 3h\sqrt{y_{n+1}}\,,$$

with $y_0 = 2$. Solving for y_{n+1}, and choosing the *positive* root, we find that

$$y_{n+1} = \left[-\frac{3}{2}h + \frac{1}{2}\sqrt{(20n+29)h^2 + 4y_n}\right]^2 .$$

(c). Backward Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.958565	1.31786	2.43924	4.13474

(d). Backward Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.942261	1.30153	2.24389	4.11908

9. The Euler formula for this problem is

$$y_{n+1} = y_n + h\sqrt{t_n + y_n}\ .$$

Here $t_0 = 0$ and $t_n = nh$. So that

$$y_{n+1} = y_n + h\sqrt{nh + y_n}\ ,$$

with $y_0 = 3$.

10. The Euler formula is

$$y_{n+1} = y_n + h[2\,t_n + exp(-\,t_n\,y_n)].$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + 2nh^2 + h\,exp(-\,nh\,y_n),$$

with $y_0 = 1$.

(a). Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	1.60729	2.46830	3.72167	5.45963

(b). Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	1.60996	2.47460	3.73356	5.47774

The *backward* Euler formula is

$$y_{n+1} = y_n + h[2\,t_{n+1} + exp(-\,t_{n+1}\,y_{n+1})].$$

Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1} = y_n + 2h^2(n+1) + h\,exp[-\,(n+1)h\,y_{n+1}],$$

with $y_0 = 1$. This equation cannot be solved *explicitly* for y_{n+1}. At each step, given the current value of y_n, the equation must be solved *numerically* for y_{n+1}.

(c). Backward Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	1.61792	2.49356	3.76940	5.53223

(d). Backward Euler method with $h = 0.0125$:

	$n=40$	$n=80$	$n=120$	$n=160$
t_n	0.5	1.0	1.5	2.0
y_n	1.61528	2.48723	3.75742	5.51404

11. The Euler formula is

$$y_{n+1} = y_n + h(4 - t_n\, y_n)/\big(1 + y_n^2\big).$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \big(4h - nh^2\, y_n\big)/\big(1 + y_n^2\big),$$

with $y_0 = -2$.

(a). Euler method with $h = 0.025$:

	$n=20$	$n=40$	$n=60$	$n=80$
t_n	0.5	1.0	1.5	2.0
y_n	-1.45865	-0.217545	1.05715	1.41487

(b). Euler method with $h = 0.0125$:

	$n=40$	$n=80$	$n=120$	$n=160$
t_n	0.5	1.0	1.5	2.0
y_n	-1.45322	-0.180813	1.05903	1.41244

The *backward* Euler formula is

$$y_{n+1} = y_n + h(4 - t_{n+1}\, y_{n+1})/\big(1 + y_{n+1}^2\big).$$

Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1}\big(1 + y_{n+1}^2\big) = y_n\big(1 + y_{n+1}^2\big) + \big[4h - (n+1)h^2\, y_{n+1}\big],$$

with $y_0 = -2$. This equation cannot be solved *explicitly* for y_{n+1}. At each step, given the current value of y_n, the equation must be solved *numerically* for y_{n+1}.

(c). Backward Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	-1.43600	-0.0681657	1.06489	1.40575

(d). Backward Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	-1.44190	-0.105737	1.06290	1.40789

12. The Euler formula is

$$y_{n+1} = y_n + h\left(y_n^2 + 2\,t_n\,y_n\right)/\left(3 + t_n^2\right).$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \left(h\,y_n^2 + 2\,nh^2\,y_n\right)/\left(3 + n^2h^2\right),$$

with $y_0 = 0.5$.

(a). Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.587987	0.791589	1.14743	1.70973

(b). Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.589440	0.795758	1.15693	1.72955

The *backward* Euler formula is

$$y_{n+1} = y_n + h\left(y_{n+1}^2 + 2\,t_{n+1}\,y_{n+1}\right)/\left(3 + t_{n+1}^2\right).$$

Since $t_0 = 0$ and $t_{n+1} = (n+1)h$, we can also write

$$y_{n+1}\left[3 + (n+1)^2h^2\right] - h\,y_{n+1}^2 = y_n\left[3 + (n+1)^2h^2\right] + 2(n+1)h^2\,y_{n+1},$$

with $y_0 = 0.5$. Note that although this equation can be solved *explicitly* for y_{n+1}, it is also possible to use a numerical equation solver. At each time step, given the current value of y_n, the equation may be solved *numerically* for y_{n+1}.

(c). Backward Euler method with $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.593901	0.808716	1.18687	1.79291

(d). Backward Euler method with $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.592396	0.804319	1.17664	1.77111

13. The Euler formula for this problem is

$$y_{n+1} = y_n + h(1 - t_n + 4\,y_n\,),$$

in which $t_n = t_0 + nh$. Since $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h - nh^2 + 4h\,y_n\,,$$

with $y_0 = 1$. With $h = 0.01$, a total number of 200 iterations is needed to reach $\bar{t} = 2$. With $h = 0.001$, a total of 2000 iterations are necessary.

14. The *backward* Euler formula is

$$y_{n+1} = y_n + h(1 - t_{n+1} + 4\,y_{n+1}\,).$$

Since the equation is linear, we can solve for y_{n+1} in terms of y_n :

$$y_{n+1} = \frac{y_n + h - h\,t_{n+1}}{1 - 4h}\,.$$

Here $t_0 = 0$ and $y_0 = 1$. With $h = 0.01$, a total number of 200 iterations is needed to reach $\bar{t} = 2$. With $h = 0.001$, a total of 2000 iterations are necessary.

18. Let $\phi(t)$ be a solution of the initial value problem. The *local* truncation error for the Euler method, on the interval $t_n \leq t \leq t_{n+1}$, is given by

$$e_{n+1} = \frac{1}{2}\phi''(\,\bar{t}_n)h^2,$$

where $t_n < \bar{t}_n < t_{n+1}$. Since $\phi'(t) = t^2 + [\phi(t)]^2$, it follows that

$$\begin{aligned}\phi''(t)\; &= 2t + 2\,\phi(t)\phi'(t) \\ &= 2t + 2t^2\phi(t) + 2[\phi(t)]^3.\end{aligned}$$

Hence

$$|e_{n+1}| \leq \left[t_{n+1} + t_{n+1}^2 M_{n+1} + M_{n+1}^3\right] h^2,$$

in which $M_{n+1} = max\{\phi(t) \,|\, t_n \leq t \leq t_{n+1}\}$.

20. Given that $\phi(t)$ is a solution of the initial value problem, the *local* truncation error for the Euler method, on the interval $t_n \leq t \leq t_{n+1}$, is

$$e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2,$$

where $t_n < \bar{t}_n < t_{n+1}$. Based on the ODE, $\phi'(t) = \sqrt{t + \phi(t)}$, and hence

$$\begin{aligned} \phi''(t) &= \frac{1 + \phi'(t)}{2\sqrt{t + \phi(t)}} \\ &= \frac{1}{2\sqrt{t + \phi(t)}} + \frac{1}{2}\,. \end{aligned}$$

Therefore

$$|e_{n+1}| \leq \frac{1}{4}\left[1 + \frac{1}{\sqrt{\bar{t}_n + \phi(\bar{t}_n)}}\right]h^2.$$

21. Let $\phi(t)$ be a solution of the initial value problem. The *local* truncation error for the Euler method, on the interval $t_n \leq t \leq t_{n+1}$, is given by

$$e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2,$$

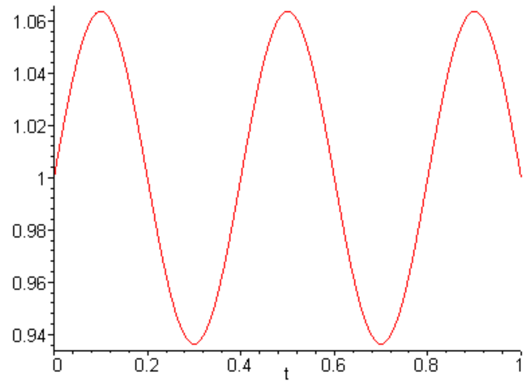
where $t_n < \bar{t}_n < t_{n+1}$. Since $\phi'(t) = 2t + exp[-t\,\phi(t)]$, it follows that

$$\begin{aligned} \phi''(t) &= 2 - 2[\phi(t) + t\,\phi'(t)] \cdot exp[-t\,\phi(t)] \\ &= 2 - \left\{\phi(t) + 2t^2 + t\,exp[-t\,\phi(t)]\right\} \cdot exp[-t\,\phi(t)]. \end{aligned}$$

Hence

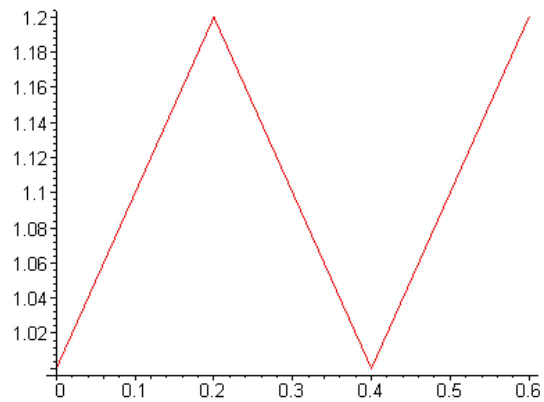
$$e_{n+1} = h^2 - \frac{h^2}{2}\left\{\phi(\bar{t}_n) + 2\,\bar{t}_n^{\,2} + \bar{t}_n exp[-\bar{t}_n\phi(\bar{t}_n)]\right\} \cdot exp[-\bar{t}_n\phi(\bar{t}_n)].$$

22(a). Direct integration yields $\phi(t) = \frac{1}{5\pi} sin\, 5\pi t + 1$.



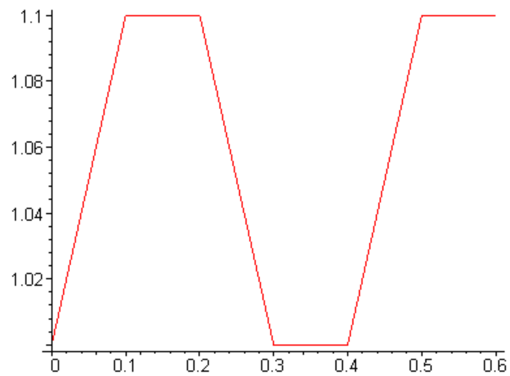
(b).

	$n=0$	$n=1$	$n=2$	$n=3$
t_n	0.0	0.2	0.4	0.6
y_n	1.0	1.2	1.0	1.2



(c).

	$n=0$	$n=2$	$n=4$	$n=6$
t_n	0.0	0.2	0.4	0.6
y_n	1.0	1.1	1.0	1.1



(d). Since $\phi''(t) = -5\pi \, sin \, 5\pi t$, the *local* truncation error for the Euler method, on the interval $t_n \leq t \leq t_{n+1}$, is given by

$$e_{n+1} = -\frac{5\pi h^2}{2} sin \, 5\pi \, \bar{t}_n \, .$$

In order to satisfy

$$|e_{n+1}| \leq \frac{5\pi h^2}{2} < 0.05 \, ,$$

it is necessary that

$$h < \frac{1}{\sqrt{50\pi}} \approx 0.08 \, .$$

$25(a)$. The Euler formula is

$$y_{n+1} = y_n + h(1 - t_n + 4\, y_n \,).$$

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.55	2.34	3.46	5.07

(b). The Euler formula for this problem is

$$y_{n+1} = y_n + h(3 + t_n - \, y_n \,).$$

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.20	1.39	1.57	1.74

(c). The Euler formula is

$$y_{n+1} = y_n + h(2\,y_n - 3\,t_n).$$

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.20	1.42	1.65	1.90

26(a).

$$1000 \cdot \begin{vmatrix} 6.0 & 18 \\ 2.0 & 6.0 \end{vmatrix} = 1000 \cdot (0) = 0\,.$$

(b).

$$1000 \cdot \begin{vmatrix} 6.01 & 18.0 \\ 2.00 & 6.00 \end{vmatrix} = 1000(0.06) = 60\,.$$

(c).

$$1000 \cdot \begin{vmatrix} 6.010 & 18.04 \\ 2.004 & 6.000 \end{vmatrix} = 1000(\,-0.09216) = \,-92.16\,.$$

27. Rounding to *three* digits, $a(b-c) \approx 0.224$. Likewise, to *three* digits, $ab \approx 0.702$ and $ac \approx 0.477$. It follows that $ab - ac \approx 0.225$.