Section 10.6

1. The steady-state solution, $v(x)$, satisfies the boundary value problem

$$v''(x) = 0\,,\ 0 < x < 50\,,\ v(0) = 10\ , v(50) = 40\,.$$

The general solution of the ODE is $v(x) = Ax + B$. Imposing the boundary conditions, we have

$$v(x) = \frac{40 - 10}{50}x + 10 = \frac{3x}{5} + 10\,.$$

2. The steady-state solution, $v(x)$, satisfies the boundary value problem

$$v''(x) = 0\,,\ 0 < x < 40\,,\ v(0) = 30\ , v(40) = -20\,.$$

The solution of the ODE is *linear*. Imposing the boundary conditions, we have

$$v(x) = \frac{-20 - 30}{40}x + 30 = -\frac{5x}{4} + 30\,.$$

4. The steady-state solution is also a solution of the boundary value problem given by $v''(x) = 0\,,\ 0 < x < L$, and the conditions $v'(0) = 0\,, v(L) = T$. The solution of the ODE is $v(x) = Ax + B$. The boundary condition $v'(0) = 0$ requires that $A = 0$. The other condition requires that $B = T$. Hence $v(x) = T$.

5. As in Prob. 4, the steady-state solution has the form $v(x) = Ax + B$. The boundary condition $v(0) = 0$ requires that $B = 0$. The boundary condition $v'(L) = 0$ requires that $A = 0$. Hence $v(x) = 0$.

6. The steady-state solution has the form $v(x) = Ax + B$. The first boundary condition, $v(0) = T$, requires that $B = T$. The other boundary condition, $v'(L) = 0$, requires that $A = 0$. Hence $v(x) = T$.

8. The steady-state solution, $v(x)$, satisfies the differential equation $v''(x) = 0$, along with the boundary conditions

$$v(0) = T\ ,\ v'(L) + v(L) = 0\,.$$

The general solution of the ODE is $v(x) = Ax + B$. The boundary condition $v'(0) = 0$ requires that $B = T$. It follows that $v(x) = Ax + T$, and

$$v'(L) + v(L) = A + AL + T\,.$$

The second boundary condition requires that $A = -T/(1 + L)$. Therefore

$$v(x) = -\frac{Tx}{1 + L} + T\,.$$

$10(a)$. Based on the *symmetry* of the problem, consider only *left* half of the bar. The steady-state solution satisfies the ODE $v''(x) = 0$, along with the boundary conditions $v(0) = 0$ and $v(50) = 100$. The solution of this boundary value problem is $v(x) = 2x$. It follows that the steady-state temperature is the *entire* rod is given by

$$f(x) = \begin{cases} 2x\,, & 0 \le x \le 50 \\ 200 - 2x\,, & 50 \le x \le 100\,. \end{cases}$$

(b). The heat conduction problem is formulated as

$$\begin{aligned} \alpha^2 u_{xx} &= u_t\,, & & 0 < x < 100\,,\ t > 0\,; \\ u(0\,,t) &= 20\,, & & u(100\,,t) = 0\,,\ t > 0\,; \\ u(x\,,0) &= f(x)\,, & & 0 < x < 100\,. \end{aligned}$$

First express the solution as $u(x\,,t) = g(x) + w(x\,,t)$, where $g(x) = -x/5 + 20$ and w satisfies the heat conduction problem

$$\begin{aligned} \alpha^2 w_{xx} &= w_t\,, & & 0 < x < 100\,,\ t > 0\,; \\ w(0\,,t) &= 0\,, & & w(100\,,t) = 0\,,\ t > 0\,; \\ w(x\,,0) &= f(x) - g(x)\,, & & 0 < x < 100\,. \end{aligned}$$

Based on the results in Section 10.5,

$$w(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2\alpha^2 t/10000} sin\frac{n\pi x}{100}\,,$$

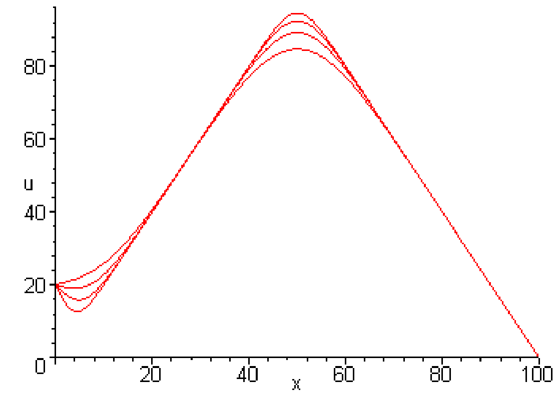
in which the coefficients c_n are the Fourier sine coefficients of $f(x) - g(x)$. That is,

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L [f(x) - g(x)] sin\frac{n\pi x}{L} dx \\ &= \frac{1}{50}\int_0^{100} [f(x) - g(x)] sin\frac{n\pi x}{100} dx \\ &= 40\,\frac{20\,sin\frac{n\pi}{2} - n\pi}{n^2\pi^2}\,. \end{aligned}$$

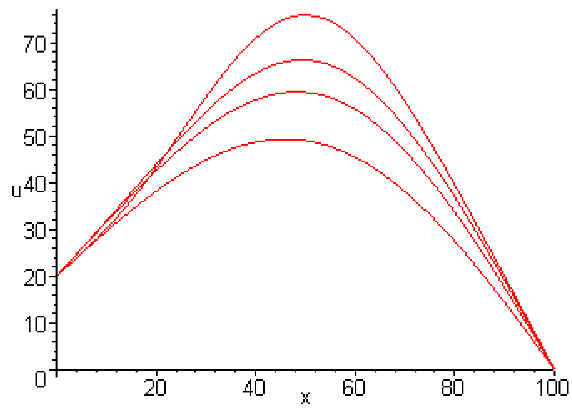
Finally, the *thermal diffusivity* of copper is $1.14\,cm^2/sec$. Therefore the temperature distribution in the rod is

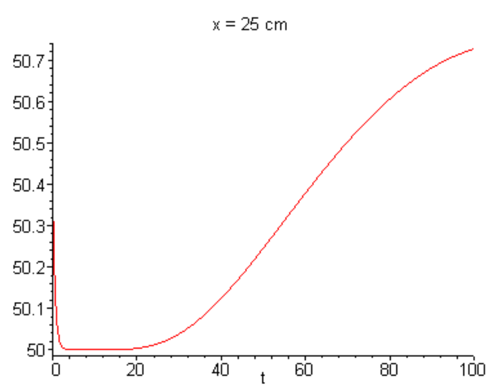
$$u(x\,,t) = 20 - \frac{x}{5} + \frac{40}{\pi^2}\sum_{n=1}^{\infty} \frac{20\,sin\frac{n\pi}{2} - n\pi}{n^2}\, e^{-1.14\,n^2\pi^2 t/10000} sin\frac{n\pi x}{100}\,.$$

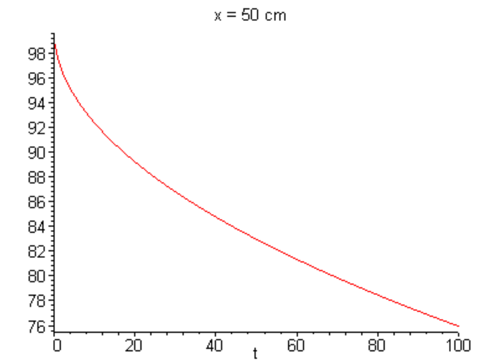
(c). $t = 5\,, 10, 20, 40\,sec$:

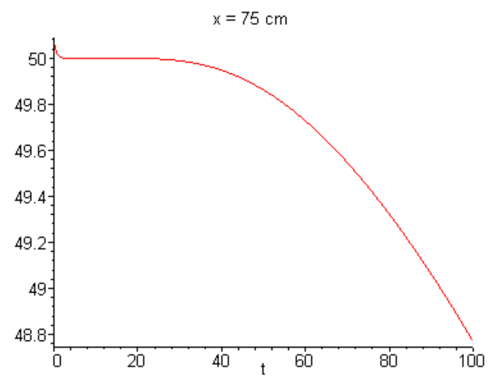


$t = 100, 200, 300, 500\,sec$:

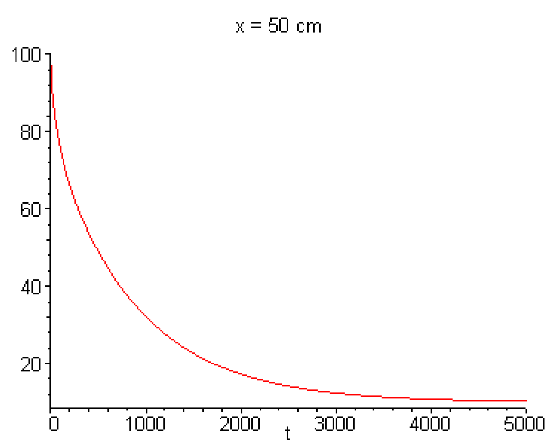








(d). The steady-state temperature of the center of the rod will be $g(50) = 10°C$.



Using a one-term approximation,

$$u(x\,,t) \approx 10 + \frac{800 - 40\,\pi}{\pi^2}\;e^{-1.14\,\pi^2 t/10000}.$$

Numerical investigation shows that $10 < u(50\,,t) < 11$ for $t \geq 3755\,sec$.

$11(a)$. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx} &= u_t\,, & 0 < x < 30\,,\ t > 0\,; \\ u(0\,,t) &= 30\,, & u(30\,,t) = 0\,,\ t > 0\,; \\ u(x\,,0) &= f(x)\,, & 0 < x < 30\,, \end{aligned}$$

in which the initial condition is given by $f(x) = x(60 - x)/30$. Express the solution as $u(x\,,t) = v(x) + w(x\,,t)$, where $v(x) = 30 - x$ and w satisfies the heat conduction problem

$$\begin{aligned} w_{xx} &= w_t\,, & 0 < x < 30\,,\ t > 0\,; \\ w(0\,,t) &= 0\,, & w(30\,,t) = 0\,,\ t > 0\,; \\ w(x\,,0) &= f(x) - v(x)\,, & 0 < x < 30\,. \end{aligned}$$

As shown in Section 10.5,

$$w(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2 t/900} sin\frac{n\pi x}{30}\,,$$

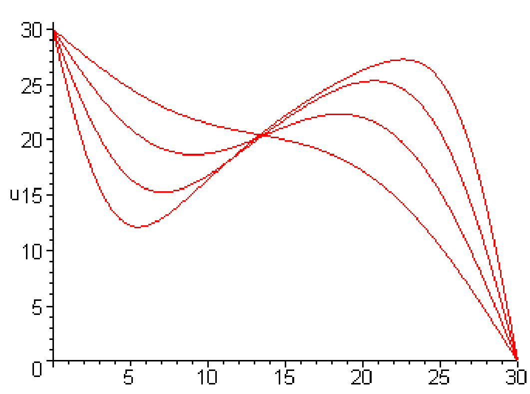
in which the coefficients c_n are the Fourier sine coefficients of $f(x) - v(x)$. That is,

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L [f(x) - g(x)] sin\frac{n\pi x}{L} dx \\ &= \frac{1}{15}\int_0^{30} [f(x) - g(x)] sin\frac{n\pi x}{30} dx \\ &= 60\,\frac{2(1 - cos\,n\pi) - n^2\pi^2(1 + cos\,n\pi)}{n^3\pi^3}\,. \end{aligned}$$

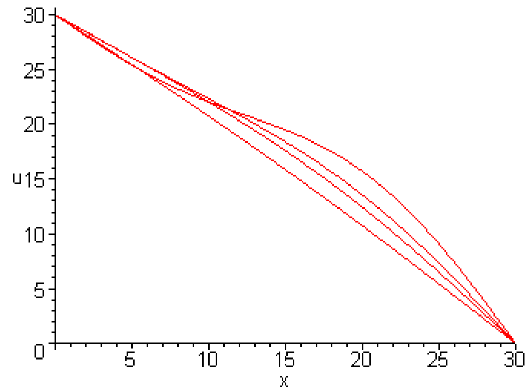
Therefore the temperature distribution in the rod is

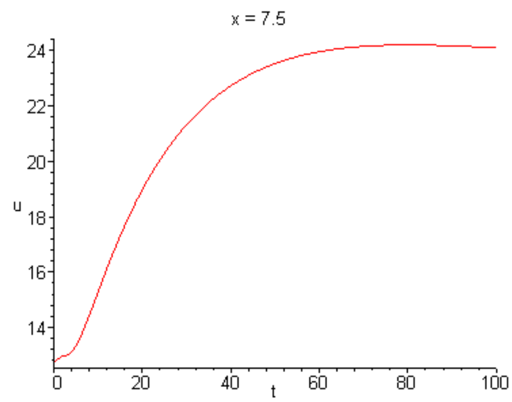
$$u(x\,,t) = 30 - x + \frac{60}{\pi^3}\sum_{n=1}^{\infty}\frac{2(1 - cos\,n\pi) - n^2\pi^2(1 + cos\,n\pi)}{n^3}\, e^{-n^2\pi^2 t/900} sin\frac{n\pi x}{30}\,.$$

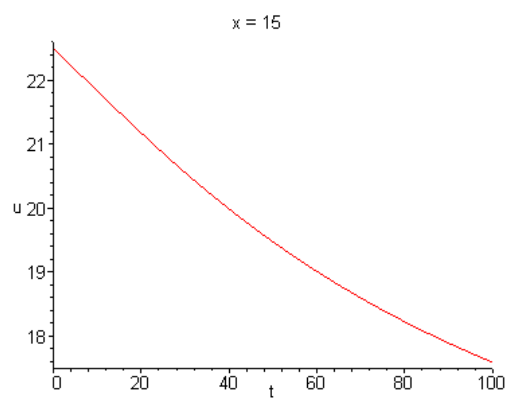
(b). $t = 5\,, 10, 20, 40\,sec$:

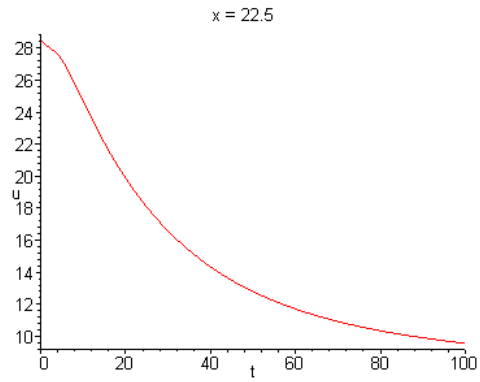


$t = 50, 75, 100, 200\, sec$:

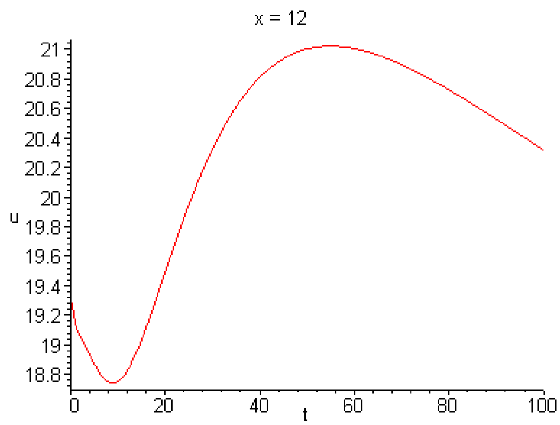








(c).



Based on the *heat conduction equation*, the rate of change of the temperature at any given point is proportional to the *concavity* of the graph of u versus x, that is, u_{xx}. Evidently, near $t = 60$, the concavity of $u(x,t)$ changes.

$13(a)$. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx} &= 4\,u_t\,, & 0 < x < 40\,,\ t > 0\,; \\ u_x(0,t) &= 0\,, & u_x(40,t) = 0\,,\ t > 0\,; \\ u(x,0) &= f(x)\,, & 0 < x < 40\,, \end{aligned}$$

in which the initial condition is given by $f(x) = x(60 - x)/30$.

As shown in the discussion on rods with *insulated ends*, the solution is given by

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2\alpha^2 t/1600} cos\frac{n\pi x}{40}\,,$$

where c_n are the Fourier cosine coefficients. In this problem,

$$\begin{aligned} c_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{1}{20}\int_0^{40}\frac{x(60-x)}{30}dx \\ &= 400/9\,, \end{aligned}$$

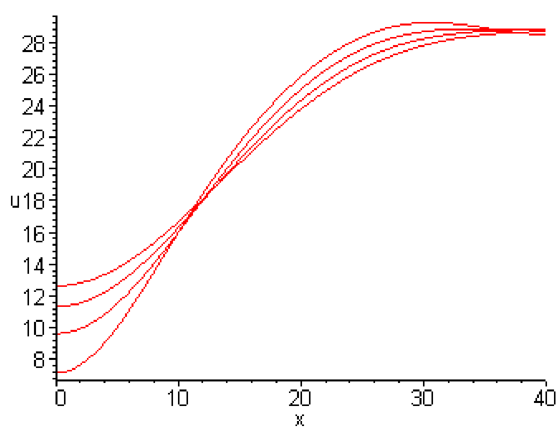
and for $n \geq 1$,

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{20}\int_0^{40}\frac{x(60-x)}{30}\,cos\frac{n\pi x}{40}dx \\ &= -\frac{160(3+\,cos\,n\pi)}{3n^2\pi^2}\,. \end{aligned}$$

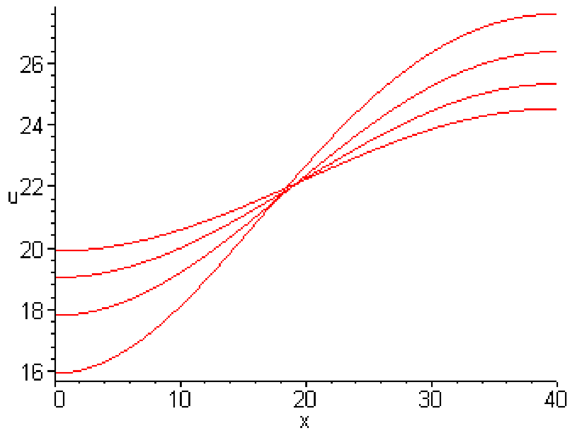
Therefore the temperature distribution in the rod is

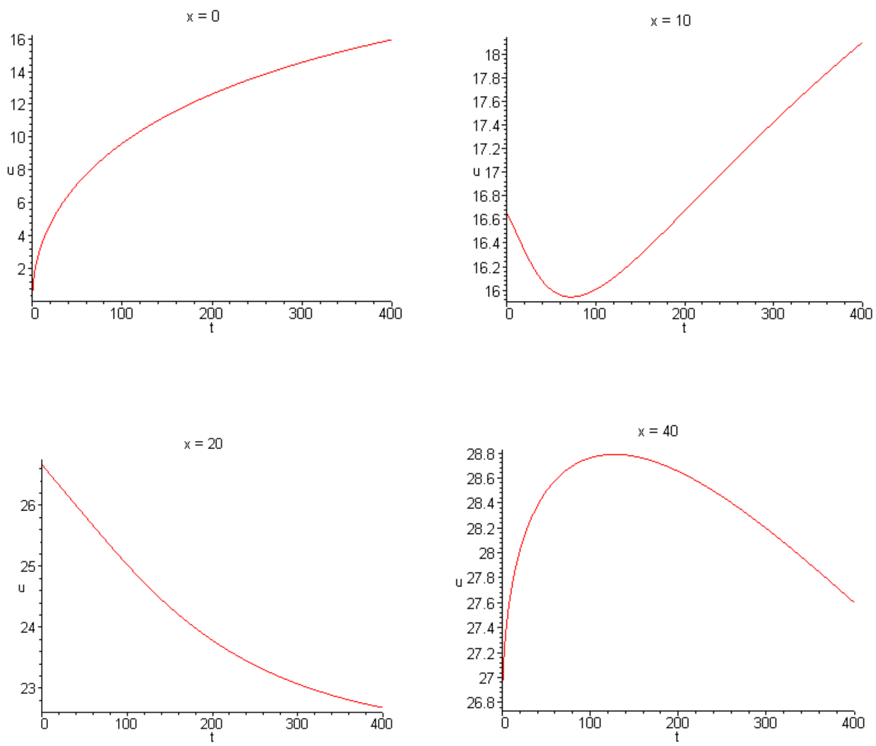
$$u(x\,,t) = \frac{200}{9} - \frac{160}{3\pi^2}\sum_{n=1}^{\infty}\frac{(3+\,cos\,n\pi)}{n^2}\,e^{-n^2\pi^2 t/6400}cos\frac{n\pi x}{40}\,.$$

(b). $t = 50, 100, 150, 200\,sec$:



$t = 40, 600, 800, 1000\, sec$:





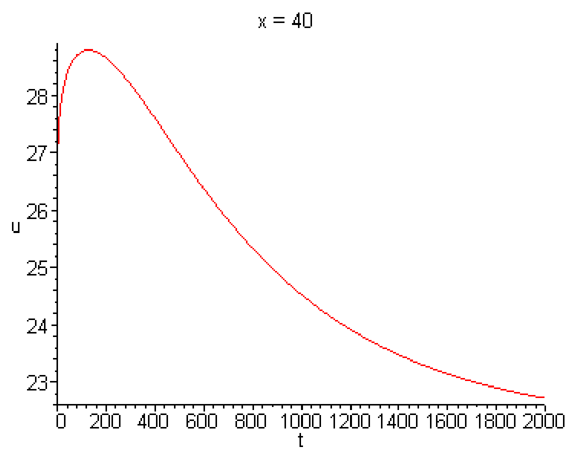
(c). Since

$$\lim_{t\to\infty} e^{-n^2\pi^2 t/6400} cos \frac{n\pi x}{40} = 0$$

for each x, it follows that the steady-state temperature is $u_\infty = 200/9$.

(d). We first note that

$$u(40\,,t) = \frac{200}{9} - \frac{160}{3\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n(3+\,cos\,n\pi)}{n^2}\;e^{-n^2\pi^2t/6400}.$$



For large values of t, an approximation is given by

$$u(40\,,t) \approx \frac{200}{9} + \frac{320}{3\pi^2}\;e^{-\pi^2t/6400}.$$

Numerical investigation shows that $22.22 < u(40\,,t) < 23.22$ for $t \geq 1550\,sec$.

$16(a)$. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx} &= u_t\,, & 0 < x < 30\,,\; t > 0\,; \\ u(0\,,t) &= 0\,, & u_x(30\,,t) = 0\,,\; t > 0\,; \\ u(x\,,0) &= f(x)\,, & 0 < x < 30\,, \end{aligned}$$

in which the initial condition is given by $f(x) = 30 - x$. Based on the results of Prob. 15,
the solution is given by

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\,e^{-(2n-1)^2\pi^2t/3600}sin\frac{n\pi x}{60}\,,$$

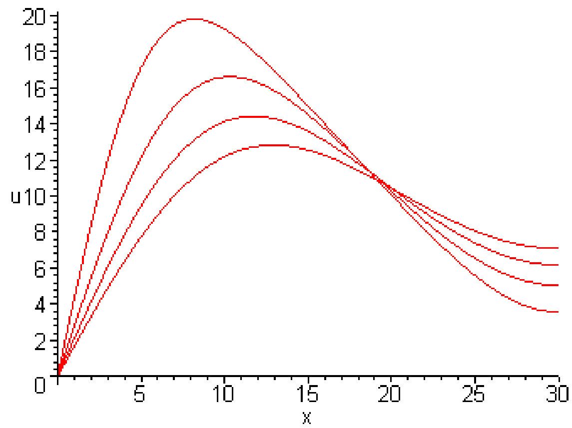
in which

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x)sin\frac{(2n-1)\pi x}{60}dx \\ &= \frac{1}{15}\int_0^{30}(30-x)sin\frac{(2n-1)\pi x}{60}dx \\ &= 120\frac{2\,cos\,n\pi + (2n-1)\pi}{(2n-1)^2\pi^2}\,. \end{aligned}$$

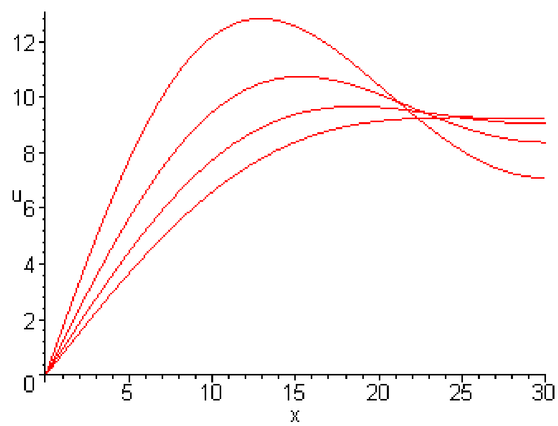
Therefore the solution of the heat conduction problem is

$$u(x\,,t) = 120\sum_{n=1}^{\infty}\frac{2\,cos\,n\pi + (2n-1)\pi}{(2n-1)^2\pi^2}\,e^{-(2n-1)^2\pi^2 t/3600}sin\frac{n\pi x}{60}\,.$$

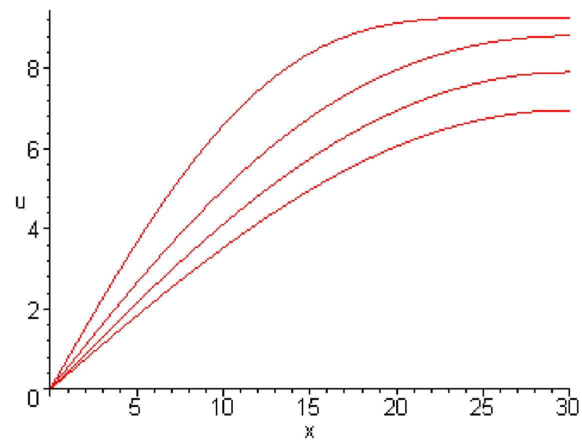
(b). $t = 10, 20, 30, 40\,sec$:

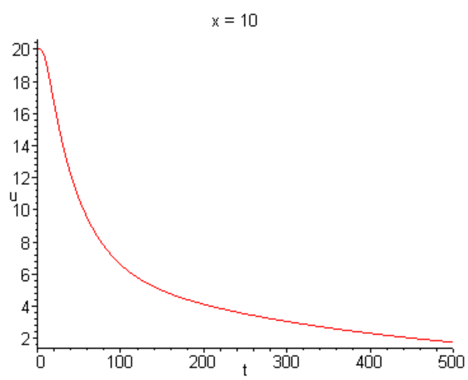


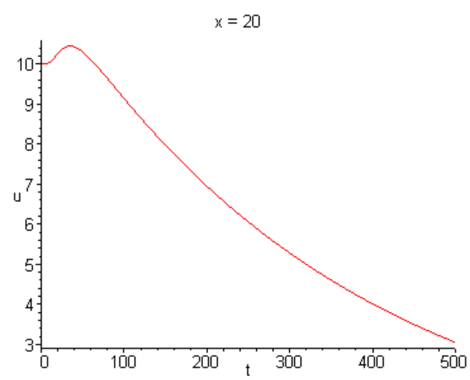
$t = 40, 60, 80, 100\,sec$:

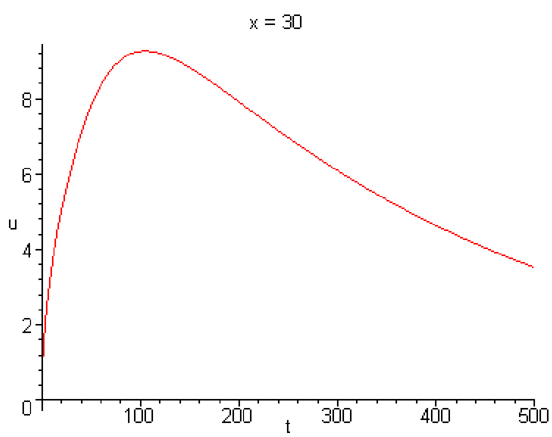


$t = 100, 150, 200, 250\, sec$:

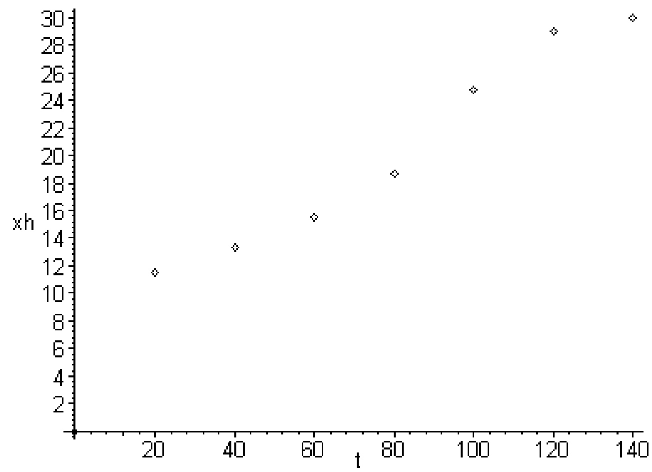






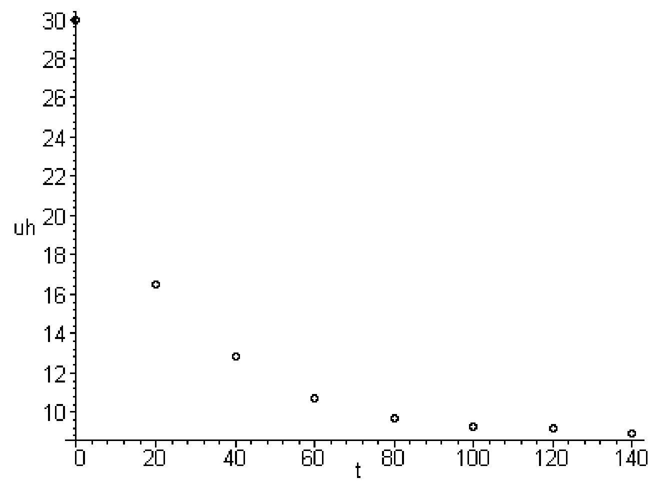


(c).



The location of x_h moves from $x = 0$ to $x = 30$.

(d).



$17(a)$. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx} &= u_t\,, & 0 < x < 30\,,\ t > 0\,; \\ u(0\,,t) &= 40\,, & u_x(30\,,t) = 0\,,\ t > 0\,; \\ u(x\,,0) &= 30 - x\,, & 0 < x < 30\,, \end{aligned}$$

The steady-state temperature satisfies the boundary value problem

$$v'' = 0\,, v(0) = 40 \text{ and } v'(30) = 0\,.$$

It easy to see we must have $v(x) = 40$. Express the solution as

$$u(x\,,t) = 40 + w(x\,,t)\,,$$

in which w satisfies the heat conduction problem

$$\begin{aligned} w_{xx} &= w_t\,, && 0 < x < 30\,,\ t > 0\,; \\ w(0\,,t) &= 0\,, && w_x(30\,,t) = 0\,,\ t > 0\,; \\ w(x\,,0) &= \,-10 - x\,, && 0 < x < 30\,. \end{aligned}$$

As shown in Prob. 15, the solution is given by

$$w(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-(2n-1)^2\pi^2 t/3600} sin\frac{n\pi x}{60}\,,$$

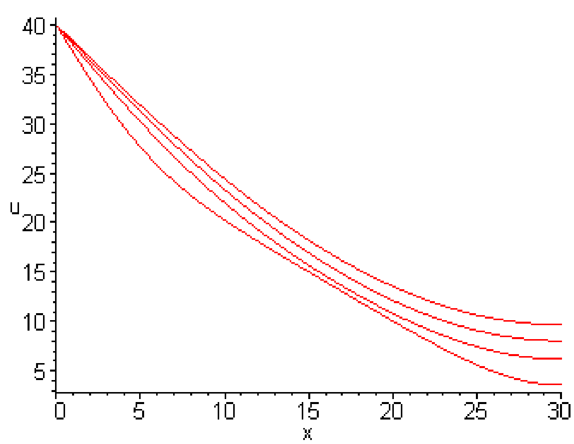
in which

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin\frac{(2n-1)\pi x}{60} dx \\ &= \frac{1}{15}\int_0^{30} (\,-10 - x) sin\frac{(2n-1)\pi x}{60} dx \\ &= 40\frac{6\,cos\,n\pi - (2n-1)\pi}{(2n-1)^2\pi^2}\,. \end{aligned}$$

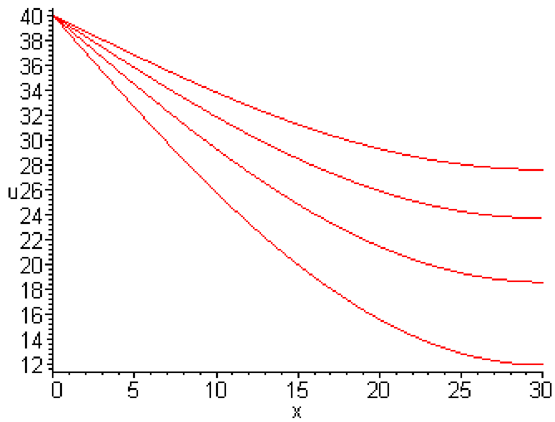
Therefore the solution of the *original* heat conduction problem is

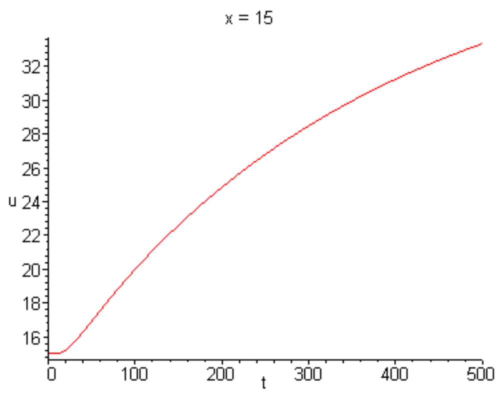
$$u(x\,,t) = 40 + 40\sum_{n=1}^{\infty}\frac{6\,cos\,n\pi - (2n-1)\pi}{(2n-1)^2\pi^2}\, e^{-(2n-1)^2\pi^2 t/3600} sin\frac{n\pi x}{60}\,.$$

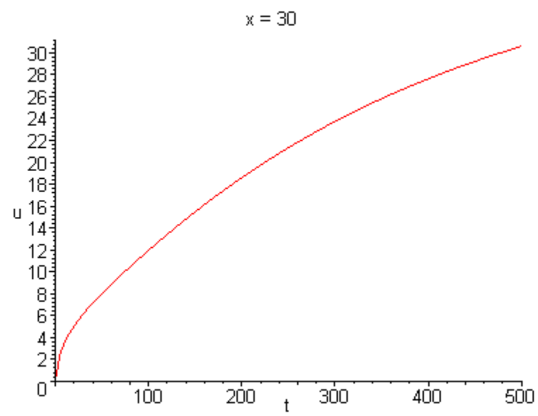
(b). $t = 10, 30, 50, 70\,sec$:



$t = 100, 200, 300, 400\, sec$:







(c). Observe the concavity of the curves. Note also that the temperature at the *insulated* end tends to the value of the fixed temperature at the boundary $x = 0$.

18. Setting $\lambda = \mu^2$, the general solution of the ODE $X'' + \mu^2 X = 0$ is

$$X(x) = k_1 e^{i\mu x} + k_2 e^{-i\mu x}.$$

The boundary conditions $y'(0) = y'(L) = 0$ lead to the system of equations

$$\begin{aligned} \mu k_1 - \mu k_2 &= 0 \\ \mu k_1 e^{i\mu L} - \mu k_2 e^{-i\mu L} &= 0\,. \end{aligned} \qquad (*)$$

If $\mu = 0$, then the solution of the ODE is $X = Ax + B$. The boundary conditions require that $X = B$.

If $\mu \neq 0$, then the system algebraic equations has a *nontrivial* solution if and only if the coefficient matrix is *singular*. Set the determinant equal to zero to obtain

$$e^{-i\mu L} - e^{i\mu L} = 0\,.$$

Let $\mu = \nu + i\sigma$. Then $i\mu L = i\nu L - \sigma L$, and the previous equation can be written as

$$e^{\sigma L} e^{-i\nu L} - e^{-\sigma L} e^{i\nu L} = 0\,.$$

Using Euler's relation, $e^{i\nu L} = cos\,\nu L + i\,sin\,\nu L$, we obtain

$$e^{\sigma L}(cos\,\nu - i\,sin\,\nu) - e^{-\sigma L}(cos\,\nu + i\,sin\,\nu) = 0\,.$$

Equating the real and imaginary parts of the equation,

$$\left(e^{\sigma L} - e^{-\sigma L}\right) cos\,\nu L = 0$$
$$\left(e^{\sigma L} + e^{-\sigma L}\right) sin\,\nu L = 0\,.$$

Based on the second equation, $\nu L = n\pi$, $n \in \mathbb{I}$. Since $cos\,nL \neq 0$, it follows that $e^{\sigma L} = e^{-\sigma L}$, or $e^{2\sigma L} = 1$. Hence $\sigma = 0$, and $\mu = n\pi/L$, $n \in \mathbb{I}$.

Note that if $\sigma \neq 0$, then the last two equations have no solution. It follows that the system
of equations $(*)$ has *no nontrivial solutions*.

20(a). Consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substitution into the partial differential equation results in

$$\alpha^2 X'' T = T'.$$

Divide both sides of the differential equation by the product XT to obtain

$$\frac{X''}{X} = \frac{T'}{\alpha^2\,T}\,.$$

Since both sides of the resulting equation are functions of different variables, each must be equal to a constant, say $-\lambda$. We obtain the ordinary differential equations

$$X'' + \lambda X = 0 \text{ and } T' + \lambda\alpha^2\,T = 0\,.$$

Invoking the first boundary condition,

$$u(0\,,t) = X(0)T(t) = 0\,.$$

At the other boundary,

$$u_x(L\,,t) + \gamma\,u(L\,,t) = [X'(L) + \gamma\,X(L)]T(t) = 0\,.$$

Since these conditions are valid for all $t > 0$, it follows that

$$X(0) = 0 \text{ and } X'(L) + \gamma\,X(L) = 0\,.$$

(b). We consider the boundary value problem

$$X'' + \lambda X = 0\,,\ 0 < x < L\,; \qquad (*)$$
$$X(0) = 0\ ,\ X'(L) + \gamma\, X(L) = 0\,.$$

Assume that λ is real, with $\lambda = -\mu^2$. The general solution of the ODE is

$$X(x) = c_1 cosh(\mu x) + c_2 sinh(\mu x)\,.$$

The first boundary condition requires that $c_1 = 0$. Imposing the second boundary condition,

$$c_2\, \mu cosh(\mu L) + \gamma\, c_2 sinh(\mu L) = 0\,.$$

If $c_2 \neq 0$, then $\mu cosh(\mu L) + \gamma\, sinh(\mu L) = 0$, which can also be written as

$$(\mu + \gamma)e^{\mu L} - (\mu + \gamma)e^{-\mu L} = 0\,.$$

If $\gamma = -\mu$, then it follows that $cosh(\mu L) = sinh(\mu L)$, and hence $\mu = 0$. If $\gamma \neq -\mu$, then $e^{\mu L} = e^{-\mu L}$ again implies that $\mu = 0$. For the case $\mu = 0$, the general solution is $X(x) = Ax + B$. Imposing the boundary conditions, we have $B = 0$ and

$$A + \gamma\, AL = 0\,.$$

If $\gamma = -1/L$, then $X(x) = Ax$ is a solution of $(*)$. Otherwise $A = 0$.

(c). Let $\lambda = \mu^2$, with $\mu > 0$. The general solution of $(*)$ is

$$X(x) = c_1 cos(\mu x) + c_2 sin(\mu x)\,.$$

The first boundary condition requires that $c_1 = 0$. From the second boundary condition,

$$c_2\, \mu cos(\mu L) + \gamma\, c_2 sin(\mu L) = 0\,.$$

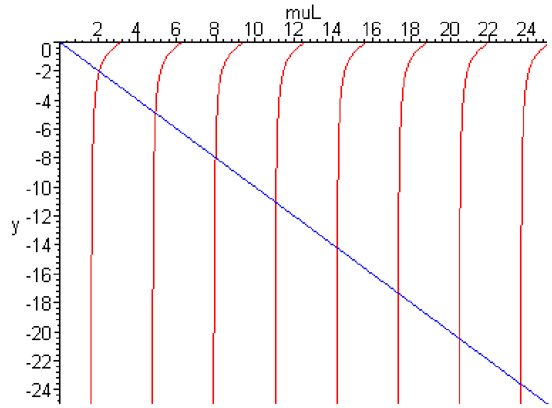
For a nontrivial solution, we must have

$$\mu cos(\mu L) + \gamma\, sin(\mu L) = 0\,.$$

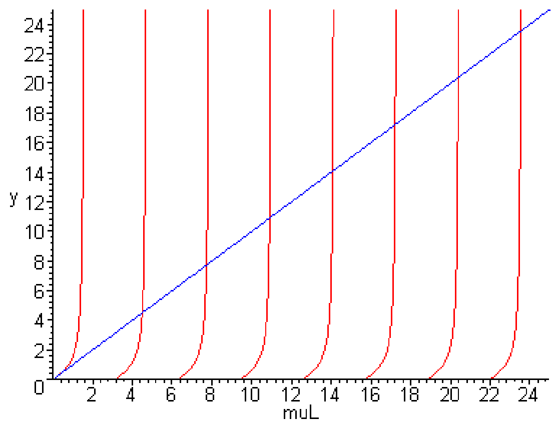
(d). The last equation can also be written as

$$tan\, \mu L = -\frac{\mu}{\gamma}\,. \qquad (**)$$

The eigenvalues λ obtained from the solutions of $(**)$, which are *infinite* in number. In the graph below, we assume $\gamma L = 1$.



For $\gamma L = -1$:



Denote the nonzero solutions of $(**)$ by $\mu_1, \mu_2, \mu_3, \cdots$.

(e). We can in principle calculate the eigenvalues $\lambda_n = \mu_n^2$. Hence the associated eigenfunctions are $X_n = sin\,\mu_n x$. Furthermore, the solutions of the temporal equations are $T_n = exp(-\alpha^2\mu_n^2\,t)$. The fundamental solutions of the heat conduction problem are given as

$$u_n(x\,,t) = e^{-\alpha^2\mu_n^2\,t} sin\,\mu_n x\,,$$

which lead to the general solution

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-\alpha^2\mu_n^2\,t} sin\,\mu_n x\,.$$