Section 3.2

1.

$$W\left(e^{2t}, e^{-3t/2}\right) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}.$$

3.

$$W\left(e^{-2t}, t\,e^{-2t}\right) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}\,.$$

5.

$$W\left(e^{t}sin\,t\,, e^{t}cos\,t\right) = \begin{vmatrix} e^{t}sin\,t & e^{t}cos\,t \\ e^{t}(sin\,t + cos\,t) & e^{t}(cos\,t - sin\,t) \end{vmatrix} = -e^{2t}\,.$$

6.

$$W\left(cos^{2}\theta\,, 1 + cos\,2\theta\right) = \begin{vmatrix} cos^{2}\theta & 1 + cos\,2\theta \\ -2sin\,\theta\,cos\,\theta & -2\,sin\,2\theta \end{vmatrix} = 0\,.$$

7. Write the equation as $y'' + (3/t)y' = 1$. $p(t) = 3/t$ is continuous for all $t > 0$. Since $t_0 > 0$, the IVP has a unique solution for all $t > 0$.

9. Write the equation as $y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$. The coefficients are not continuous at $t = 0$ and $t = 4$. Since $t_0 \in (0\,, 4)$, the largest interval is $0 < t < 4$.

10. The coefficient $3ln|t|$ is discontinuous at $t = 0$. Since $t_0 > 0$, the largest interval of existence is $0 < t < \infty$.

11. Write the equation as $y'' + \frac{x}{x-3}y' + \frac{ln|x|}{x-3}y = 0$. The coefficients are discontinuous at $x = 0$ and $x = 3$. Since $x_0 \in (0\,, 3)$, the largest interval is $0 < x < 3$.

13. $y_1'' = 2$. We see that $t^2(2) - 2(t^2) = 0$. $y_2'' = 2\,t^{-3}$, with $t^2(y_2'') - 2(y_2) = 0$. Let $y_3 = c_1t^2 + c_2t^{-1}$, then $y_3'' = 2c_1 + 2c_2t^{-3}$. It is evident that y_3 is also a solution.

16. No. Substituting $y = sin(t^2)$ into the differential equation,

$$-4t^2sin\left(t^2\right) + 2cos\left(t^2\right) + 2t\,cos\left(t^2\right)p(t) + sin\left(t^2\right)q(t) = 0\,.$$

For the equation to be valid, we must have $p(t) = -1/t$, which is *not* continuous, or even defined, at $t = 0$.

17. $W(e^{2t}, g(t)) = e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$. Dividing both sides by e^{2t}, we find that g must satisfy the ODE $g' - 2g = 3e^{2t}$. Hence $g(t) = 3t\, e^{2t} + c\, e^{2t}$.

19. $W(f\,,g) = fg' - f'g$. Also, $W(u\,,v) = W(2f - g\,,f + 2g)$. Upon evaluation, $W(u\,,v) = 5fg' - 5f'g = 5W(f\,,g)$.

20. $W(f\,,g) = fg' - f'g = t\,cos\,t - sin\,t$, and $W(u\,,v) = -4fg' + 4f'g$. Hence $W(u\,,v) = -4t\,cos\,t + 4sin\,t$.

22. The general solution is $y = c_1e^{-3t} + c_2e^{-t}$. $W(e^{-3t}, e^{-t}) = 2e^{-4t}$, and hence the exponentials form a *fundamental set* of solutions. On the other hand, the *fundamental solutions* must also satisfy the conditions $y_1(1) = 1\,, y_1'(1) = 0\,; y_2(1) = 0\,, y_2'(1) = 1$. For y_1, the initial conditions require $c_1 + c_2 = e\,,\ -3c_1 - c_2 = 0$. The coefficients are $c_1 = -e^3/2\,, c_2 = 3e/2$. For the solution, y_2, the initial conditions require $c_1 + c_2 = 0$, $-3c_1 - c_2 = e$. The coefficients are $c_1 = -e^3/2\,, c_2 = e/2$. Hence the fundamental solutions are $\left\{y_1 = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)},\ y_2 = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}\right\}$.

23. Yes. $y_1'' = -4\,cos\,2t\,;\ y_2'' = -4\,sin\,2t$. $W(cos\,2t\,, sin\,2t) = 2$.

24. Clearly, $y_1 = e^t$ is a solution. $y_2' = (1+t)e^t, y_2'' = (2+t)e^t$. Substitution into the ODE results in $(2+t)e^t - 2(1+t)e^t + t\,e^t = 0$. Furthermore, $W(e^t, te^t) = e^{2t}$. Hence the solutions form a fundamental set of solutions.

26. Clearly, $y_1 = x$ is a solution. $y_2' = cos\,x\,, y_2'' = -sin\,x$. Substitution into the ODE results in $(1 - x\,cot\,x)(-sin\,x) - x(cos\,x) + sin\,x = 0$. $W(y_1, y_2) = x\,cos\,x - sin\,x$, which is *nonzero* for $0 < x < \pi$. Hence $\{x\,, sin\,x\}$ is a fundamental set of solutions.

28. $P = 1\,, Q = x\,, R = 1$. We have $P'' - Q' + R = 0$. The equation is *exact*. Note that $(y')' + (xy)' = 0$. Hence $y' + xy = c_1$. This equation is *linear*, with integrating factor $\mu = e^{x^2/2}$. Therefore the general solution is

$$y(x) = c_1 exp\big(-x^2/2\big)\int_{x_0}^{x} exp\big(u^2/2\big)du + c_2 exp\big(-x^2/2\big).$$

29. $P = 1\,, Q = 3x^2\,, R = x$. Note that $P'' - Q' + R = -5x$, and therefore the differential equation is *not* exact.

31. $P = x^2\,, Q = x\,, R = -1$. We have $P'' - Q' + R = 0$. The equation is *exact*. Write the equation as $(x^2y')' - (xy)' = 0$. Integrating, we find that $x^2y' - x\,y = c$. Divide both sides of the ODE by x^2. The resulting equation is *linear*, with integrating factor $\mu = 1/x$. Hence $(y/x)' = c\,x^{-3}$. The solution is $y(t) = c_1x^{-1} + c_2x$.

33. $P = x^2\,, Q = x\,, R = x^2 - \nu^2$. Hence the coefficients are $2P' - Q = 3x$ and $P'' - Q' + R = x^2{+}1 - \nu^2$. The *adjoint* of the original differential equation is given by $x^2\mu'' + 3x\,\mu'{+}(x^2{+}1 - \nu^2)\mu = 0\,$.

35. $P = 1\,, Q = 0\,, R = \,-\,x\,$. Hence the coefficients are given by $2P' - Q = 0$ and $P'' - Q' + R = \,-\,x\,$. Therefore the *adjoint* of the original equation is $\mu'' - x\,\mu = 0\,$.