Section 2.2

2. For $x \neq -1$, the differential equation may be written as $y\,dy = [x^2/(1+x^3)]dx$. Integrating both sides, with respect to the appropriate variables, we obtain the relation $y^2/2 = \frac{1}{3}ln|1+x^3| + c$. That is, $y(x) = \pm\sqrt{\frac{2}{3}ln|1+x^3| + c}$.

3. The differential equation may be written as $y^{-2}dy = -\sin x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-1} = \cos x + c$. That is, $(C - \cos x)y = 1$, in which C is an arbitrary constant. Solving for the dependent variable, explicitly, $y(x) = 1/(C - \cos x)$.

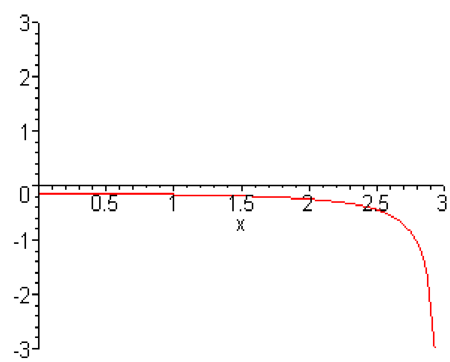
5. Write the differential equation as $\cos^{-2} 2y\,dy = \cos^2 x\,dx$, or $\sec^2 2y\,dy = \cos^2 x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $\tan 2y = \sin x \cos x + x + c$.

7. The differential equation may be written as $(y + e^y)dy = (x - e^{-x})dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation
$y^2 + 2\,e^y = x^2 + 2\,e^{-x} + c$.

8. Write the differential equation as $(1+y^2)dy = x^2\,dx$. Integrating both sides of the equation, we obtain the relation $y + y^3/3 = x^3/3 + c$, that is, $3y + y^3 = x^3 + C$.

9(a). The differential equation is separable, with $y^{-2}dy = (1-2x)dx$. Integration yields $-y^{-1} = x - x^2 + c$. Substituting $x = 0$ and $y = -1/6$, we find that $c = 6$. Hence the specific solution is $y^{-1} = x^2 - x - 6$. The *explicit form* is
$y(x) = 1/(x^2 - x - 6)$.

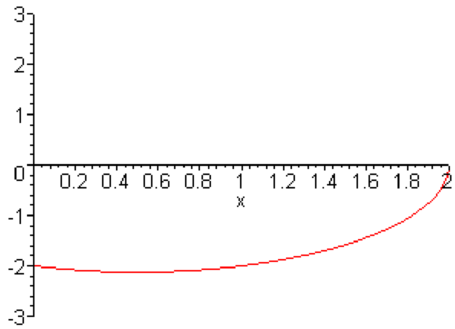
(b)



(c). Note that $x^2 - x - 6 = (x+2)(x-3)$. Hence the solution becomes *singular* at $x = -2$ and $x = 3$.

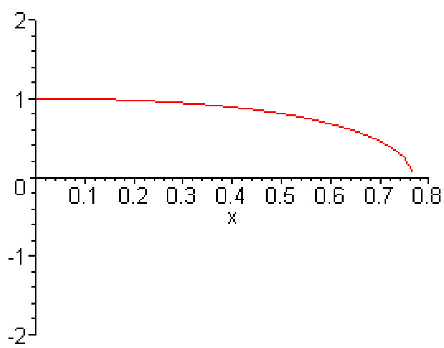
10(a). $y(x) = -\sqrt{2x - 2x^2 + 4}$.

$10(b)$.



$11(a)$. Rewrite the differential equation as $x\,e^x dx = -y\,dy$. Integrating both sides of the equation results in $x\,e^x - e^x = -y^2/2 + c$. Invoking the initial condition, we obtain $c = -1/2$. Hence $y^2 = 2e^x - 2x\,e^x - 1$. The *explicit form* of the solution is $y(x) = \sqrt{2e^x - 2x\,e^x - 1}$. The *positive* sign is chosen, since $y(0) = 1$.

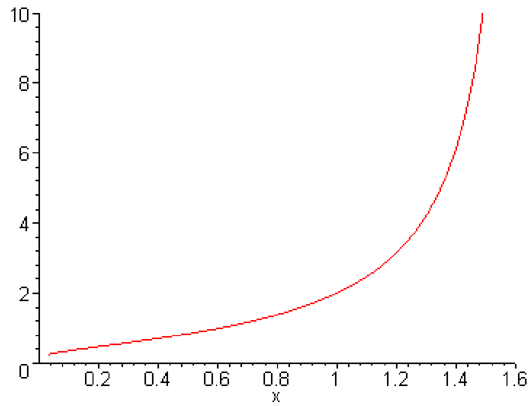
(b).



(c). The function under the radical becomes *negative* near $x = -1.7$ and $x = 0.76$.

$11(a)$. Write the differential equation as $r^{-2}dr = \theta^{-1}\,d\theta$. Integrating both sides of the equation results in the relation $-r^{-1} = ln\,\theta + c$. Imposing the condition $r(1) = 2$, we obtain $c = -1/2$. The *explicit form* of the solution is $r(\theta) = 2/(1 - 2\,ln\,\theta)$.

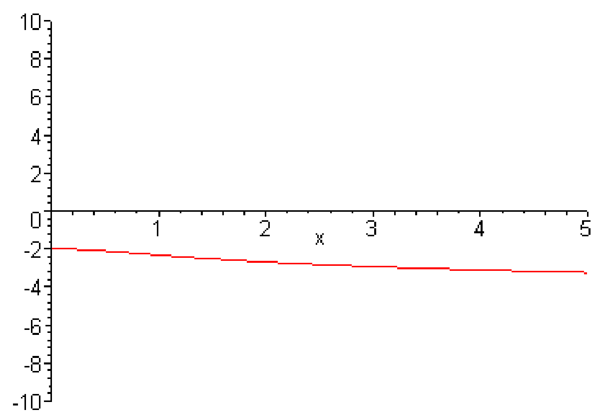
(b).



(c). Clearly, the solution makes sense only if $\theta > 0$. Furthermore, the solution becomes singular when $ln\,\theta = 1/2$, that is, $\theta = \sqrt{e}$.

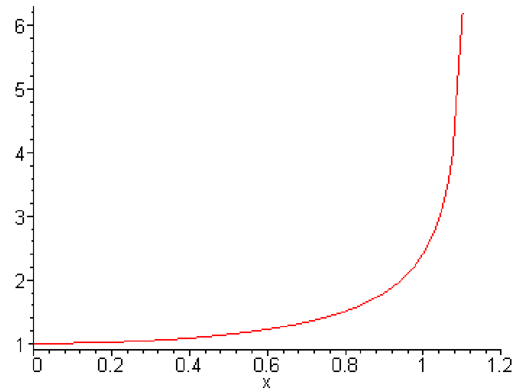
13(a). $y(x) = -\sqrt{2\,ln(1+x^2)+4}$.

(b).



14(a). Write the differential equation as $y^{-3}dy = x(1+x^2)^{-1/2}\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-2}/2 = \sqrt{1+x^2} + c$. Imposing the initial condition, we obtain $c = -3/2$. Hence the specific solution can be expressed as $y^{-2} = 3 - 2\sqrt{1+x^2}$. The *explicit form* of the solution is $y(x) = 1/\sqrt{3-2\sqrt{1+x^2}}$. The *positive* sign is chosen to satisfy the initial condition.

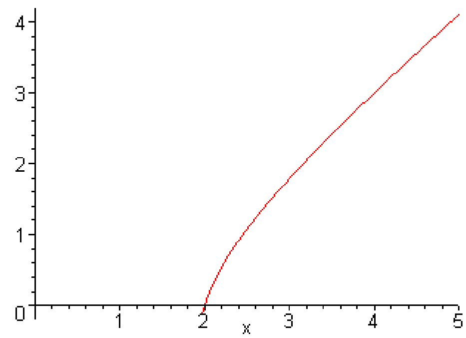
(b).



(c). The solution becomes singular when $2\sqrt{1+x^2} = 3$. That is, at $x = \pm\sqrt{5}/2$.

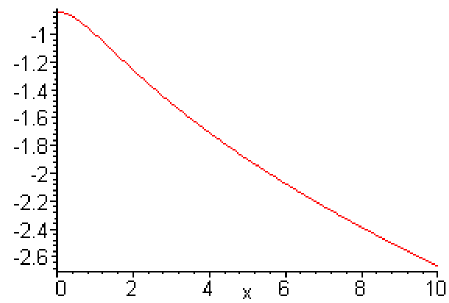
$15(a)$. $y(x) = -1/2 + \sqrt{x^2 - 15/4}$.

(b).



$16(a)$. Rewrite the differential equation as $4y^3 dy = x(x^2+1)dx$. Integrating both sides
of the equation results in $y^4 = (x^2+1)^2/4 + c$. Imposing the initial condition, we obtain $c = 0$. Hence the solution may be expressed as $(x^2+1)^2 - 4y^4 = 0$. The *explicit* form of the solution is $y(x) = -\sqrt{(x^2+1)/2}$. The *sign* is chosen based on $y(0) = -1/\sqrt{2}$.

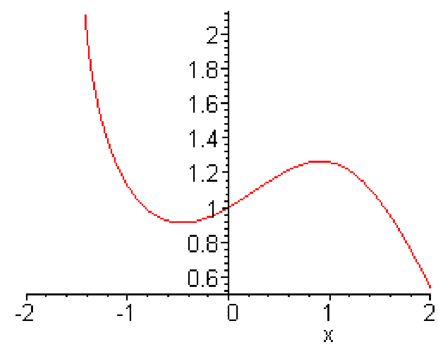
(b).



(c). The solution is valid for all $x \in \mathbb{R}$.

$17(a)$. $y(x) = -5/2 - \sqrt{x^3 - e^x + 13/4}$.

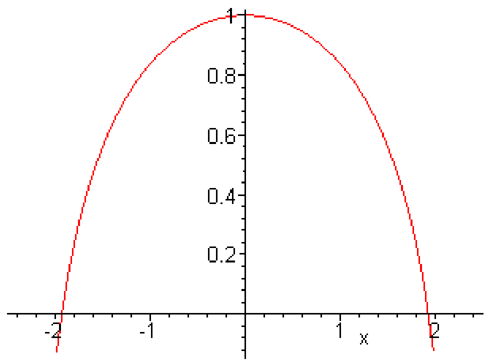
(b).



(c). The solution is valid for $x > -1.45$. This value is found by estimating the root of $4x^3 - 4e^x + 13 = 0$.

$18(a)$. Write the differential equation as $(3 + 4y)dy = (e^{-x} - e^x)dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $3y + 2y^2 = -(e^x + e^{-x}) + c$. Imposing the initial condition, $y(0) = 1$, we obtain $c = 7$. Thus, the solution can be expressed as $3y + 2y^2 = -(e^x + e^{-x}) + 7$. Now by *completing the square* on the left hand side, $2(y + 3/4)^2 = -(e^x + e^{-x}) + 65/8$. Hence the *explicit* form of the solution is $y(x) = -3/4 + \sqrt{65/16 - cosh\, x}$.

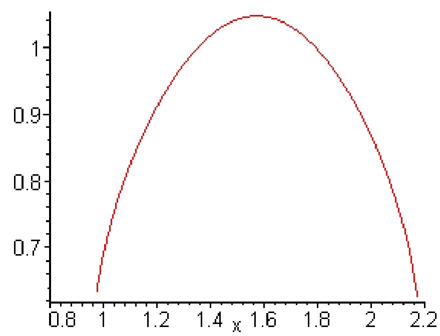
(b).



(c). Note the $65 - 16\,cosh\,x \geq 0$, as long as $|x| > 2.1$. Hence the solution is valid on the interval $-2.1 < x < 2.1$.

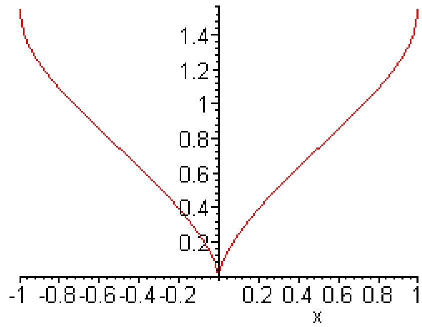
$19(a)$. $y(x) = -\pi/3 + \frac{1}{3} sin^{-1}(3\,cos^2 x)$.

(b).



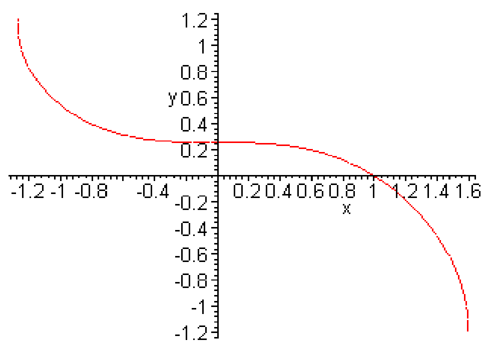
$20(a)$. Rewrite the differential equation as $y^2 dy = arcsin\,x/\sqrt{1-x^2}\,dx$. Integrating both sides of the equation results in $y^3/3 = (arcsin\,x)^2/2 + c$. Imposing the condition $y(0) = 0$, we obtain $c = 0$. The *explicit* form of the solution is $y(x) = \sqrt[3]{\frac{3}{2}}(arcsin\,x)^{2/3}$.

(b).



(c). Evidently, the solution is defined for $-1 \leq x \leq 1$.

22. The differential equation can be written as $(3y^2 - 4)dy = 3x^2 dx$. Integrating both sides, we obtain $y^3 - 4y = x^3 + c$. Imposing the initial condition, the specific solution is $y^3 - 4y = x^3 - 1$. Referring back to the differential equation, we find that $y' \to \infty$ as $y \to \pm 2/\sqrt{3}$. The respective values of the abscissas are $x = -1.276\,, 1.598$.



Hence the solution is valid for $-1.276 < x < 1.598$.

24. Write the differential equation as $(3 + 2y)dy = (2 - e^x)dx$. Integrating both sides, we obtain $3y + y^2 = 2x - e^x + c$. Based on the specified initial condition, the solution can be written as $3y + y^2 = 2x - e^x + 1$. *Completing the square*, it follows that $y(x) = -3/2 + \sqrt{2x - e^x + 13/4}$. The solution is defined if $2x - e^x + 13/4 \geq 0$, that is, $-1.5 \leq x \leq 2$ (*approximately*). In that interval, $y' = 0$, for $x = ln\,2$. It can be verified that $y''(ln\,2) < 0$. In fact, $y''(x) < 0$ on the interval of definition. Hence the solution attains a global maximum at $x = ln\,2$.

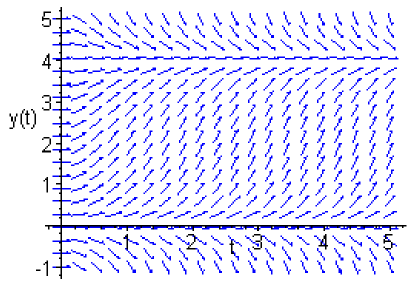
26. The differential equation can be written as $(1 + y^2)^{-1}dy = 2(1 + x)dx$. Integrating both sides of the equation, we obtain $arctan y = 2x + x^2 + c$. Imposing the given initial
condition, the specific solution is $arctan y = 2x + x^2$. Therefore, $y(x) = tan(2x + x^2)$. Observe that the solution is defined as long as $-\pi/2 < 2x + x^2 < \pi/2$. It is easy to see that $2x + x^2 \geq -1$. Furthermore, $2x + x^2 = \pi/2$ for $x = -2.6$ and 0.6 . Hence the solution is valid on the interval $-2.6 < x < 0.6$. Referring back to the differential

equation, the solution is *stationary* at $x = -1$. Since $y''(x) > 0$ on the entire interval of
definition, the solution attains a global minimum at $x = -1$.

28(a). Write the differential equation as $y^{-1}(4-y)^{-1}dy = t(1+t)^{-1}dt$. Integrating both sides of the equation, we obtain $ln\,|y| - ln|y-4| = 4t - 4ln|1+t| + c$. Taking the *exponential* of both sides, it follows that $|y/(y-4)| = C\,e^{4t}/(1+t)^4$. It follows that as $t \to \infty$, $|y/(y-4)| = |1 + 4/(y-4)| \to \infty$. That is, $y(t) \to 4$.

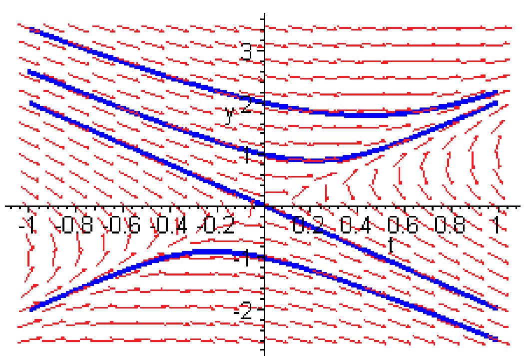
(b). Setting $y(0) = 2$, we obtain that $C = 1$. Based on the initial condition, the solution may be expressed as $y/(y-4) = -e^{4t}/(1+t)^4$. Note that $y/(y-4) < 0$, for all $t \geq 0$. Hence $y < 4$ for all $t \geq 0$. Referring back to the differential equation, it follows that y' is always *positive*. This means that the solution is *monotone increasing*. We find that the root of the equation $e^{4t}/(1+t)^4 = 399$ is near $t = 2.844$.

(c). Note the $y(t) = 4$ is an equilibrium solution. Examining the local direction field,

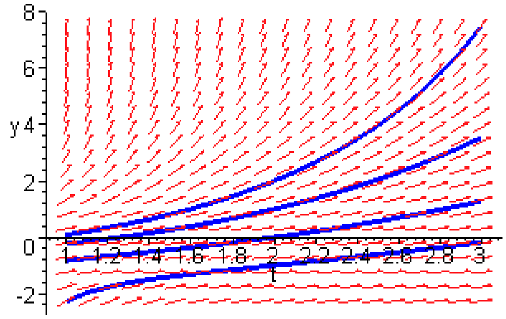


we see that if $y(0) > 0$, then the corresponding solutions converge to $y = 4$. Referring back to part (a), we have $y/(y-4) = [y_0/(y_0-4)]e^{4t}/(1+t)^4$, for $y_0 \neq 4$. Setting $t = 2$, we obtain $y_0/(y_0-4) = (3/e^2)^4 y(2)/(y(2)-4)$. Now since the function $f(y) = y/(y-4)$ is *monotone* for $y < 4$ and $y > 4$, we need only solve the equations $y_0/(y_0-4) = -399(3/e^2)^4$and $y_0/(y_0-4) = 401(3/e^2)^4$. The respective solutions are $y_0 = 3.6622$ and $y_0 = 4.4042$.

30(f).



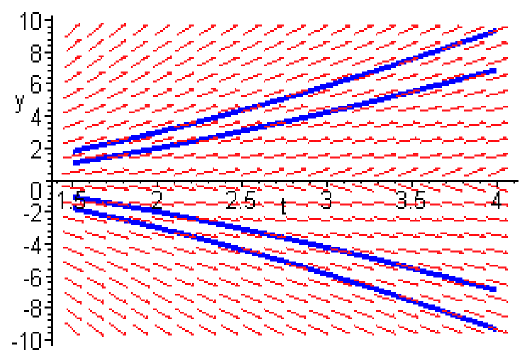
31(c)



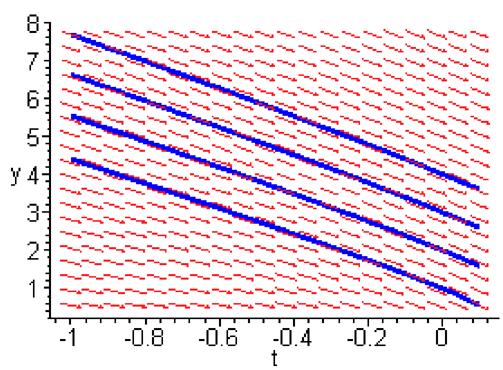
32(a). Observe that $(x^2+3y^2)/2xy = \frac{1}{2}\left(\frac{y}{x}\right)^{-1} + \frac{3}{2}\frac{y}{x}$. Hence the differential equation is *homogeneous*.

(b). The substitution $y = x\,v$ results in $v + x\,v' = (x^2+3x^2v^2)/2x^2v$. The transformed equation is $v' = (1+v^2)/2xv$. This equation is *separable*, with general solution $v^2+1 = c\,x$. In terms of the original dependent variable, the solution is $x^2+y^2 = c\,x^3$.

(c).



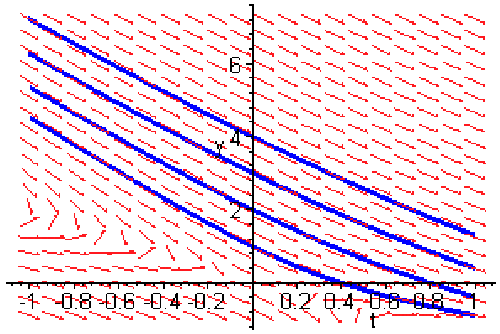
33(c).



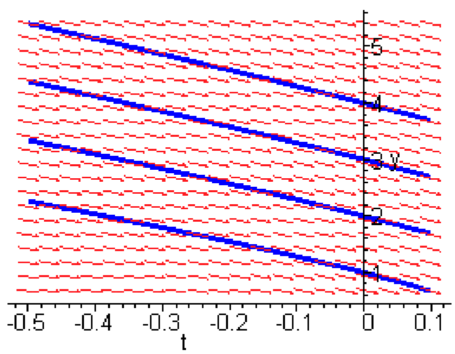
$34(a)$. Observe that $-(4x+3y)/(2x+y) = -2 - \frac{y}{x}\left[2+\frac{y}{x}\right]^{-1}$. Hence the differential equation is *homogeneous*.

(b). The substitution $y = x\,v$ results in $v + x\,v' = -2 - v/(2+v)$. The transformed equation is $v' = -(v^2+5v+4)/(2+v)x$. This equation is *separable*, with general solution $(v{+}4)^2|v{+}1| = C/x^3$. In terms of the original dependent variable, the solution is $(4x+y)^2|x{+}y| = C$.

(c).



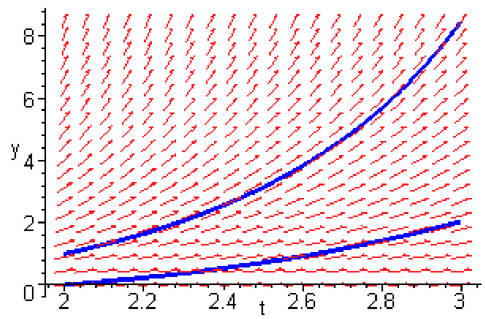
$35(c)$.



$36(a)$. Divide by x^2 to see that the equation is homogeneous. Substituting $y = x\,v$, we obtain $x\,v' = (1+v)^2$. The resulting differential equation is separable.

(b). Write the equation as $(1+v)^{-2}dv = x^{-1}dx$. Integrating both sides of the equation, we obtain the general solution $-1/(1+v) = ln|x| + c$. In terms of the original dependent variable, the solution is $y = x\,[C - ln|x|]^{-1} - x$.

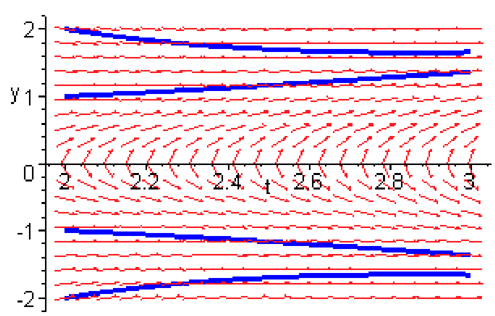
(c).



$37(a)$. The differential equation can be expressed as $y' = \frac{1}{2}\left(\frac{y}{x}\right)^{-1} - \frac{3}{2}\frac{y}{x}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (1-5v^2)/2v$. Separating variables, we have $\frac{2v}{1-5v^2}dv = \frac{1}{x}dx$.

(b). Integrating both sides of the transformed equation yields $-\frac{1}{5}$
$ln|1-5v^2| = ln|x| + c$,
that is, $1-5v^2 = C/|x|^5$. In terms of the original dependent variable, the general solution is $5y^2 = x^2 - C/|x|^3$.

(c).



$38(a)$. The differential equation can be expressed as $y' = \frac{3}{2}\frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^{-1}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (v^2-1)/2v$, that is, $\frac{2v}{v^2-1}dv = \frac{1}{x}dx$.

(b). Integrating both sides of the transformed equation yields $ln|v^2-1| = ln|x| + c$, that is, $v^2 - 1 = C|x|$. In terms of the original dependent variable, the general solution is $y^2 = C\,x^2|x| + x^2$.

(c).