Section 11.6

1. The *sine expansion* of $f(x) = 1$, on $0 < x < 1$, is given by

$$f(x) = 2\sum_{m=1}^{\infty} \frac{1 - cos\, m\pi}{m\pi}\, sin\, m\pi x\,,$$

with partial sums

$$S_n(x) = 2\sum_{m=1}^{n} \frac{1 - cos\, m\pi}{m\pi}\, sin\, m\pi x\,.$$

The *mean square error* in this problem is

$$R_n = \int_0^1 |1 - S_n(x)|^2 dx\,.$$

Several values are shown in the Table :

n	5	10	15	20
R_n	0.067	0.04	0.026	0.02

Further numerical calculation shows that $R_n < 0.02$ for $n \geq 21$.

$3(a)$. The *sine expansion* of $f(x) = x(1-x)$, on $0 < x < 1$, is given by

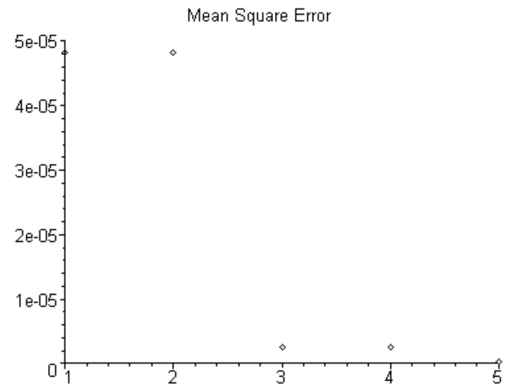
$$f(x) = 2\sum_{m=1}^{\infty} \frac{1 - cos\, m\pi}{m\pi}\, sin\, m\pi x\,,$$

with partial sums

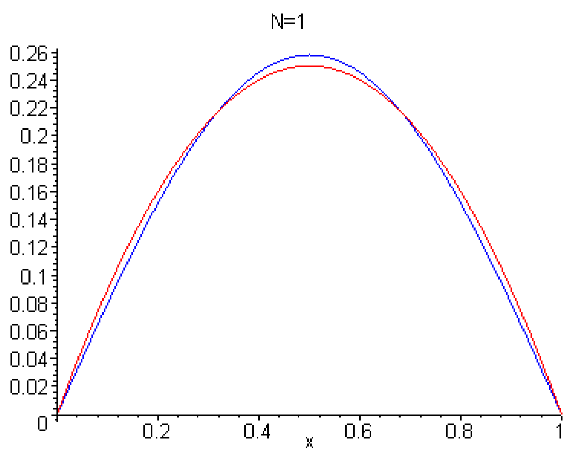
$$S_n(x) = 4\sum_{m=1}^{n} \frac{1 - cos\, m\pi}{m^3\pi^3}\, sin\, m\pi x\,.$$

(b, c). The *mean square error* in this problem is

$$R_n = \int_0^1 |\, x(1-x) - S_n(x)|^2 dx\,.$$



We find that $R_1 = 0.000048$. The graphs of $f(x)$ and $S_1(x)$ are plotted below:



$6(a)$. The function is bounded on intervals not containing $x = 0$, so for $\varepsilon > 0$,

$$\int_{\varepsilon}^{1} f(x)dx = \int_{\varepsilon}^{1} x^{-1/2}\,dx = 2 - 2\sqrt{\varepsilon}\,.$$

Hence the improper integral is evaluated as

$$\int_{0}^{1} f(x)dx = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} x^{-1/2}\,dx = 2\,.$$

On the other hand, $f^2(x) = 1/x$ for $x \neq 0$, and

$$\int_{\varepsilon}^{1} f^2(x)dx = \int_{\varepsilon}^{1} x^{-1}dx = -\,ln\sqrt{\varepsilon}\,.$$

Therefore the improper integral does not exist.

(b). Since $f^2(x) \equiv 1$, it is evident that the *Riemann integral* of $f^2(x)$ exists. Let

$$P_N = \{0 = x_1\,, x_2, \cdots, x_{N+1} = 1\}$$

be a *partition* of $[0\,,1]$ into equal subintervals. We can always choose a *rational* point, ξ_i , in each of the subintervals so that the Riemann sum

$$R(\xi_1, \xi_2, \cdots, \xi_N) = \sum_{n=1}^{N} f(\xi_n)\frac{1}{N} = 1\,.$$

Likewise, can always choose an *irrational* point, η_i , in each of the subintervals so that the Riemann sum

$$R(\eta_1, \eta_2, \cdots, \eta_N) = \sum_{n=1}^{N} f(\eta_n)\frac{1}{N} = -1\,.$$

It follows that $f(x)$ is *not* Riemann integrable.

8. With $P_0(x) = 1$ and $P_1(x) = x$, the normalization conditions are satisfied. Using the usual inner product on $[-1\,,1]$,

$$\int_{-1}^{1} P_0(x)P_1(x)dx = 0$$

and hence the polynomials are also orthogonal. Let $P_2(x) = a_2x^2 + a_1x + a_0$. The normalization condition requires that $a_2 + a_1 + a_0 = 1$. For orthogonality, we need

$$\int_{-1}^{1} \left(a_2x^2 + a_1x + a_0\right)dx = 0 \text{ and } \int_{-1}^{1} x\left(a_2x^2 + a_1x + a_0\right)dx = 0\,.$$

It follows that $a_2 = 3/2\,, a_1 = 0$ and $a_0 = -1/2$. Hence $P_2(x) = (3x^2 - 1)/2$. Now let $P_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. The coefficients must be chosen so that $a_3 + a_2 + a_1 + a_0 = 1$ and the orthogonality conditions

$$\int_{-1}^{1} P_i(x)P_j(x)dx = 0 \quad (i \neq j)$$

are satisfied. Solution of the resulting algebraic equations leads to $a_3 = 5/2\,, a_2 = 0$, $a_1 = -3/2$ and $a_0 = 0$. Therefore $P_3(x) = (5x^3 - 3x)/2$.

11. The implied sequence of coefficients is $a_n = 1\,, n \geq 1$. Since the limit of these coefficients is *not* zero, the series cannot be an eigenfunction expansion.

13. Consider the eigenfunction expansion

$$f(x) = \sum_{i=1}^{\infty} a_i\phi_i(x).$$

Formally,

$$f^2(x) = \sum_{i=1}^{\infty} a_i^2 \phi_i^2(x) + 2 \sum_{i \neq j} a_i a_j \, \phi_i(x) \phi_j(x) \, .$$

Integrating term-by-term,

$$\begin{aligned} \int_0^1 r(x) f^2(x) dx \; &= \; \sum_{i=1}^{\infty} \int_0^1 a_i^2 r(x) \phi_i^2(x) dx + 2 \sum_{i \neq j} \int_0^1 a_i a_j \, r(x) \phi_i(x) \phi_j(x) dx \\ &= \sum_{i=1}^{\infty} a_i^2 \int_0^1 \phi_i^2(x) dx \, , \end{aligned}$$

since the eigenfunctions are orthogonal. Assuming that they are also normalized,

$$\int_0^1 r(x) f^2(x) dx = \sum_{i=1}^{\infty} a_i^2 \, .$$