Section 8.5

$1(a)$. The *general* solution of the ODE is $y(t) = c\,e^t + 2 - t$. Imposing the initial condition, $y(0) = 2$, the solution of the IVP is $\phi_1(t) = 2 - t$.

(b). If instead, the initial condition $y(0) = 2.001$ is given, the solution of the IVP is $\phi_2(t) = 0.001\,e^t + 2 - t$. We then have $\phi_2(t) - \phi_1(t) = 0.001\,e^t$.

3. The solution of the initial value problem is $\phi(t) = e^{-100\,t} + t$.

(a, b). Based on the exact solution, the *local truncation error* for both of the Euler methods is

$$|e_{loc}| \leq \frac{10^4}{2} e^{-100\,\bar{t}_n} h^2.$$

Hence $|e_n| \leq 5000\,h^2$, for all $0 < \bar{t}_n < 1$. Furthermore, the local truncation error is *greatest* near $t = 0$. Therefore $|e_1| \leq 5000\,h^2 < 0.0005$ for $h < 0.0003$. Now the truncation error accumulates at each time step. Therefore the *actual* time step should be much smaller than $h \approx 0.0003$. For example, with $h = 0.00025$, we obtain the data

	Euler	*B.Euler*	$\phi(t)$
$t = 0.05$	0.056323	0.057165	0.056738
$t = 0.1$	0.100040	0.100051	0.100045

Note that the total number of time steps needed to reach $t = 0.1$ is $N = 400$.

(c). Using the Runge-Kutta method, comparisons are made for several values of h :

$h = 0.1$:

	$\phi(t)$	y_n	$y_n - \phi(t_n)$
$t = 0.05$	0.056738	0.057416	0.000678
$t = 0.1$	0.100045	0.100055	0.000010

$h = 0.005$:

	$\phi(t)$	y_n	$y_n - \phi(t_n)$
$t = 0.05$	0.056738	0.056766	0.000027
$t = 0.1$	0.100045	0.100046	0.0000004

$6(a)$. Using the method of *undetermined coefficients*, it is easy to show that the general solution of the ODE is $y(t) = c\,e^{\lambda t} + t^2$. Imposing the initial condition, it follows that $c = 0$ and hence the solution of the IVP is $\phi(t) = t^2$.

(b). Using the Runge-Kutta method, with $h = 0.01$, numerical solutions are generated for various values of λ :

$\lambda = 1$:

	$\phi(t)$	y_n	$\lvert y_n - \phi(t_n)\rvert$
$t = 0.25$	0.0625	$0.0624999..$	2×10^{-11}
$t = 0.5$	0.25	0.25	0
$t = 0.75$	0.5625	0.5625	0
$t = 1.0$	1.0	1.0	0

$\lambda = 10$:

	$\phi(t)$	y_n	$\lvert y_n - \phi(t_n)\rvert$
$t = 0.25$	0.0625	$0.0624998..$	2.215×10^{-7}
$t = 0.5$	0.25	0.249997	2.920×10^{-6}
$t = 0.75$	0.5625	0.562464	3.579×10^{-5}
$t = 1.0$	1.0	0.999564	4.362×10^{-4}

$\lambda = 20$:

	$\phi(t)$	y_n	$\lvert y_n - \phi(t_n)\rvert$
$t = 0.25$	0.0625	$0.062889..$	1.10×10^{-5}
$t = 0.5$	0.25	0.248342	1.658×10^{-3}
$t = 0.75$	0.5625	0.316458	0.246042
$t = 1.0$	1.0	-35.5139	36.5139

$\lambda = 50$:

	$\phi(t)$	y_n	$\lvert y_n - \phi(t_n)\rvert$
$t = 0.25$	0.0625	$-0.044803..$	0.107303
$t = 0.5$	0.25	-28669.55	28669.804
$t = 0.75$	0.5625	-7.66014×10^{9}	7.66014×10^{9}
$t = 1.0$	1.0	-2.04668×10^{15}	2.04668×10^{15}

The following table shows the calculated value, y_1, at the *first* time step :

$\phi(t)$	$y_1(\lambda = 1)$	$y_1(\lambda = 10)$	$y_1(\lambda = 20)$	$y_1(\lambda = 50)$
10^{-4}	9.99999×10^{-5}	9.99979×10^{-5}	9.99833×10^{-5}	9.97396×10^{-5}

(c). Referring back to the *exact* solution given in Part(a), if a *nonzero* initial condition, say $y(0) = \varepsilon$, is specified, the solution of the IVP becomes

$$\phi_\varepsilon(t) = \varepsilon\, e^{\lambda t} + t^2.$$

We then have $\lvert\phi(t) - \phi_\varepsilon(t)\rvert = \lvert\varepsilon\rvert\, e^{\lambda t}$. It is evident that for any $t > 0$,

$$\lim_{\lambda \to \infty} |\phi(t) - \phi_\varepsilon(t)| = \infty\,.$$

This implies that virtually any error introduced early in the calculations will be magnified as $\lambda \to \infty$. The initial value problem is inherently *unstable*.