Section 3.4

2. $exp(2-3i) = e^2 e^{-3i} = e^2(cos\,3 - i\,sin\,3)$.

3. $e^{i\pi} = cos\,\pi + i\,sin\,\pi = -1$.

4. $exp\left(2-\frac{\pi}{2}i\right) = e^2\left(cos\,\frac{\pi}{2} - i\,sin\,\frac{\pi}{2}\right) = -e^2\,i$.

6. $\pi^{-1+2i} = exp[(-1+2i)ln\,\pi] = exp(-ln\,\pi)exp(2\,ln\,\pi\,i) = \frac{1}{\pi}\,exp(2\,ln\,\pi\,i) =$
$= \frac{1}{\pi}[cos\,(2\,ln\,\pi) + i\,sin\,(2\,ln\,\pi)]$.

8. The characteristic equation is $r^2 - 2r + 6 = 0$, with roots $r = 1 \pm i\sqrt{5}$. Hence the general solution is $y = c_1 e^t cos\,\sqrt{5}\,t + c_2\,e^t sin\,\sqrt{5}\,t$.

9. The characteristic equation is $r^2 + 2r - 8 = 0$, with roots $r = -4, 2$. The roots are *real* and different, hence the general solution is $y = c_1 e^{-4t} + c_2\,e^{2t}$.

10. The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Hence the general solution is $y = c_1 e^{-t} cos\,t + c_2\,e^{-t} sin\,t$.

12. The characteristic equation is $4r^2 + 9 = 0$, with roots $r = \pm\frac{3}{2}\,i$. Hence the general solution is $y = c_1 cos\,\frac{3}{2}t + c_2\,sin\,\frac{3}{2}t$.

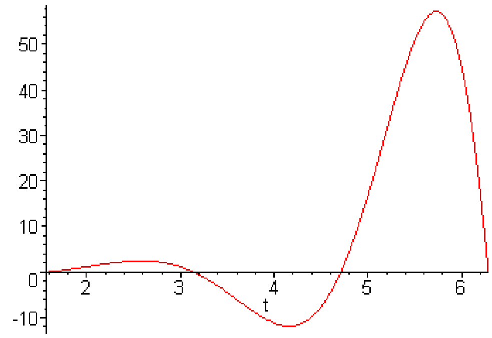
13. The characteristic equation is $r^2 + 2r + 1.25 = 0$, with roots $r = -1 \pm \frac{1}{2}i$. Hence the general solution is $y = c_1 e^{-t} cos\,\frac{1}{2}t + c_2\,e^{-t} sin\,\frac{1}{2}t$.

15. The characteristic equation is $r^2 + r + 1.25 = 0$, with roots $r = -\frac{1}{2} \pm i$. Hence the general solution is $y = c_1 e^{-t/2} cos\,t + c_2\,e^{-t/2} sin\,t$.

16. The characteristic equation is $r^2 + 4r + 6.25 = 0$, with roots $r = -2 \pm \frac{3}{2}\,i$. Hence the general solution is $y = c_1 e^{-2t} cos\,\frac{3}{2}t + c_2\,e^{-2t} sin\,\frac{3}{2}t$.

17. The characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence the general solution is $y = c_1 cos\,2t + c_2\,sin\,2t$. Its derivative is $y' = -2c_1 sin\,2t + 2c_2\,cos\,2t$. Based on the first condition, $y(0) = 0$, we require that $c_1 = 0$. In order to satisfy the condition $y'(0) = 1$, we find that $2c_2 = 1$. The constants are $c_1 = 0$ and $c_2 = 1/2$. Hence the specific solution is $y(t) = \frac{1}{2} sin\,2t$.

19. The characteristic equation is $r^2 - 2r + 5 = 0$, with roots $r = 1 \pm 2i$. Hence the general solution is $y = c_1 e^t cos\,2t + c_2\,e^t sin\,2t$. Based on the condition, $y(\pi/2) = 0$, we require that $c_1 = 0$. It follows that $y = c_2\,e^t sin\,2t$, and so the first derivative is $y' = c_2\,e^t sin\,2t + 2c_2\,e^t cos\,2t$. In order to satisfy the condition $y'(\pi/2) = 2$, we find that $-2e^{\pi/2}c_2 = 2$. Hence we have $c_2 = -e^{-\pi/2}$. Therefore the specific solution is $y(t) = -e^{t-\pi/2}\,sin\,2t$.

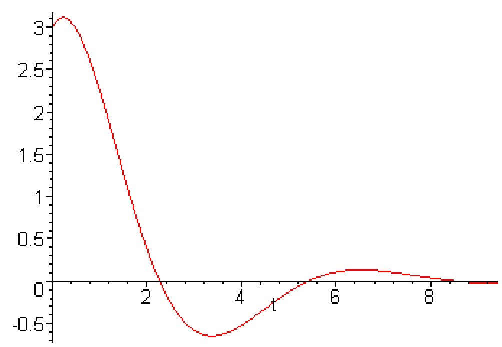


20. The characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$. Hence the general solution is $y = c_1 cos\, t + c_2\, sin\, t$. Its derivative is $y' = -c_1 sin\, t + c_2\, cos\, t$. Based on the first condition, $y(\pi/3) = 2$, we require that $c_1 + \sqrt{3}\, c_2 = 4$. In order to satisfy the condition $y'(\pi/3) = -4$, we find that $-\sqrt{3}\, c_1 + c_2 = -8$. Solving these for the constants, $c_1 = 1 + 2\sqrt{3}$ and $c_2 = \sqrt{3} - 2$. Hence the specific solution is a steady oscillation, given by $y(t) = \left(1 + 2\sqrt{3}\right) cos\, t + \left(\sqrt{3} - 2\right) sin\, t$.

21. From Prob. 15, the general solution is $y = c_1 e^{-t/2} cos\, t + c_2\, e^{-t/2} sin\, t$. Invoking the first initial condition, $y(0) = 3$, which implies that $c_1 = 3$. Substituting, it follows that $y = 3e^{-t/2} cos\, t + c_2\, e^{-t/2} sin\, t$, and so the first derivative is

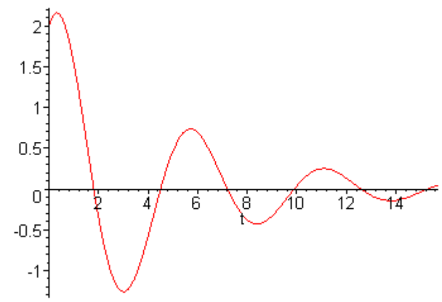
$$y' = -\frac{3}{2} e^{-t/2} cos\, t - 3e^{-t/2} sin\, t + c_2\, e^{-t/2} cos\, t - \frac{c_2}{2} e^{-t/2} sin\, t\,.$$

Invoking the initial condition, $y'(0) = 1$, we find that $-\frac{3}{2} + c_2 = 1$, and so $c_2 = \frac{5}{2}$. Hence the specific solution is $y(t) = 3e^{-t/2} cos\, t + \frac{5}{3}\, e^{-t/2} sin\, t$.



24(a). The characteristic equation is $5r^2 + 2r + 7 = 0$, with roots $r = -\frac{1}{5} \pm i \frac{\sqrt{34}}{5}$. The solution is $u = c_1 e^{-t/5} cos\, \frac{\sqrt{34}}{5} t + c_2 e^{-t/5} sin\, \frac{\sqrt{34}}{5} t$. Invoking the given initial conditions, we obtain the equations for the coefficients: $c_1 = 2$, $-2 + \sqrt{34}\, c_2 = 5$. That is, $c_1 = 2$, $c_2 = 7/\sqrt{34}$. Hence the specific solution is

$$u(t) = 2e^{-t/5} cos\, \frac{\sqrt{34}}{5} t + \frac{7}{\sqrt{34}} e^{-t/5} sin\, \frac{\sqrt{34}}{5} t\,.$$



(b). Based on the graph of $u(t)$, T is in the interval $14 < t < 16$. A numerical solution on that interval yields $T \approx 14.5115$.

26(a). The characteristic equation is $r^2 + 2a\, r + (a^2 + 1) = 0$, with roots $r = -a \pm i$. Hence the general solution is $y(t) = c_1 e^{-at} cos\, t + c_2 e^{-at} sin\, t$. Based on the initial conditions, we find that $c_1 = 1$ and $c_2 = a$. Therefore the specific solution is given by

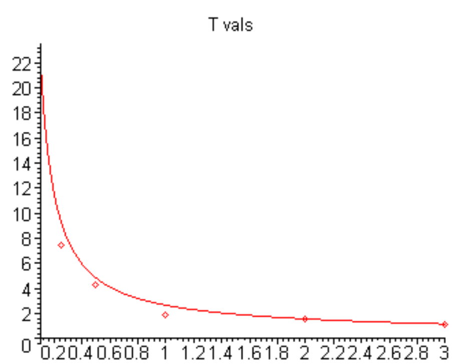
$$\begin{aligned} y(t) &= e^{-at} cos\, t + a\, e^{-at} sin\, t \\ &= \sqrt{1 + a^2}\; e^{-at} cos\,(t - \phi), \end{aligned}$$

in which $\phi = tan^{-1}(a)$.

(b). For estimation, note that $|y(t)| \leq \sqrt{1 + a^2}\; e^{-at}$. Now consider the inequality $\sqrt{1 + a^2}\; e^{-at} \leq 1/10$. The inequality holds for $t \geq \frac{1}{a} ln \left[10\sqrt{1 + a^2} \right]$. Therefore $T \leq \frac{1}{a} ln \left[10\sqrt{1 + a^2} \right]$. Setting $a = 1$, numerical analysis gives $T \approx 1.8763$.

(c). Similarly, $T_{1/4} \approx 7.4284$, $T_{1/2} \approx 4.3003$, $T_2 \approx 1.5116$, $T_3 \approx 1.1496$.

(d).



Note that the estimates T_a approach the graph of $\frac{1}{a}ln\left[10\sqrt{1+a^2}\right]$ as a gets large.

27. Direct calculation gives the result. On the other hand, it was shown in Prob. 3.3.23 that $W(f\,g\,, f\,h) = f^2 W(g\,, h)$. Hence

$$\begin{aligned} W\left(e^{\lambda t}cos\,\mu t\,, e^{\lambda t}sin\,\mu t\right) &= e^{2\lambda t}\,W(cos\,\mu t\,, sin\,\mu t) \\ &= e^{2\lambda t}\left[cos\,\mu t(sin\,\mu t)' - (cos\,\mu t)' sin\,\mu t\right] \\ &= \mu\, e^{2\lambda t}. \end{aligned}$$

28(a). Clearly, y_1 and y_2 are solutions. Also, $W(cos\,t, sin\,t) = cos^2 t + sin^2 t = 1$.

(b). $y' = i\,e^{it}\,, y'' = i^2\,e^{it} = \,-\,e^{it}$. Evidently, y is a solution and so $y = c_1 y_1 + c_2 y_2$.

(c). Setting $t = 0\,,\ 1 = c_1 cos\,0 + c_2\,sin\,0$, and $c_1 = 0$. Differentiating, $i\,e^{it} = c_2\,cos\,t$. Setting $t = 0\,,\ i = c_2\,cos\,0$ and hence $c_2 = i$. Therefore $e^{it} = cos\,t + i\,sin\,t$.

29. *Euler's formula* is $e^{it} = cos\,t + i\,sin\,t$. It follows that $e^{-it} = cos\,t - i\,sin\,t$. Adding these equation, $e^{it} + e^{-it} = 2\,cos\,t$. Subtracting the two equations results in $e^{it} - e^{-it} = 2i\,sin\,t$.

30. Let $r_1 = \lambda_1 + i\mu_1$, and $r_2 = \lambda_2 + i\mu_2$. Then

$$\begin{aligned} exp(r_1 + r_2)t &= exp[(\lambda_1 + \lambda_2)t + i(\mu_1 + \mu_2)t] \\ &= e^{(\lambda_1+\lambda_2)t}[cos(\mu_1 + \mu_2)t + i\,sin(\mu_1 + \mu_2)t] \\ &= e^{(\lambda_1+\lambda_2)t}[(cos\,\mu_1 t + \ isin\,\mu_1 t)(cos\,\mu_2 t + \ isin\,\mu_2 t)] \\ &= e^{\lambda_1 t}(cos\,\mu_1 t + \ isin\,\mu_1 t)\cdot e^{\lambda_2 t}(cos\,\mu_1 t + \ isin\,\mu_1 t) \end{aligned}$$

Hence $e^{(r_1+r_2)t} = e^{r_1 t}\,e^{r_2 t}$.

32. If $\phi(t) = u(t) + i\,v(t)$ is a solution, then

$$(u + iv)'' + p(t)(u + iv)' + q(t)(u + iv) = 0\,,$$

and $(u'' + iv'') + p(t)(u' + iv') + q(t)(u + iv) = 0$. After expanding the equation and separating the *real* and *imaginary* parts,

$$\begin{aligned} u'' + p(t)u' + q(t)u = 0 \\ v'' + p(t)v' + q(t)v = 0 \end{aligned}$$

Hence both $u(t)$ and $v(t)$ are solutions.

34(a). By the *chain rule*, $y(x)' = \frac{dy}{dx}\,x'$. In general, $\frac{dz}{dt} = \frac{dz}{dx}\,\frac{dx}{dt}$. Setting $z = \frac{dy}{dt}$, we have $\frac{d^2y}{dt^2} = \frac{dz}{dx}\,\frac{dx}{dt} = \frac{d}{dx}\left[\frac{dy}{dx}\,\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2y}{dx^2}\,\frac{dx}{dt}\right]\frac{dx}{dt} + \frac{dy}{dx}\,\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt}$. However, $\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2x}{dt^2}\right]\frac{dt}{dx}\cdot\frac{dx}{dt} = \frac{d^2x}{dt^2}$. Hence $\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\,\frac{d^2x}{dt^2}$.

(b). Substituting the results in Part(a) into the general ODE, $y'' + p(t)y' + q(t)y = 0$, `we find that

$$\frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} + p(t)\frac{dy}{dx}\frac{dx}{dt} + q(t)y = 0\,.$$

Collecting the terms,

$$\left[\frac{dx}{dt}\right]^2\frac{d^2y}{dx^2} + \left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\frac{dy}{dx} + q(t)y = 0\,.$$

(c). Assuming $\left[\frac{dx}{dt}\right]^2 = k\,q(t)$, and $q(t) > 0$, we find that $\frac{dx}{dt} = \sqrt{k\,q(t)}$, which can be integrated. That is, $x = \xi(t) = \int \sqrt{k\,q(t)}\;dt$.

(d). Let $k = 1$. It follows that $\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} = \frac{d\xi}{dt} + p(t)\xi(t) = \frac{q'}{2\sqrt{q}} + p\,\sqrt{q}$. Hence

$$\left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\Big/\left[\frac{dx}{dt}\right]^2 = \frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}\,.$$

As long as $dx/dt \neq 0$, the differential equation can be expressed as

$$\frac{d^2y}{dx^2} + \left[\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}\right]\frac{dy}{dx} + y = 0\,.$$

* For the case $q(t) < 0$, write $q(t) = -[-q(t)]$, and set $\left[\frac{dx}{dt}\right]^2 = -q(t)$.

36. $p(t) = 3t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \left(1 + 3t^2\right)/t^2\,.$$

The ratio is *not* constant, and therefore the equation cannot be transformed.

37. $p(t) = t - 1/t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = 1\,.$$

The ratio is constant, and therefore the equation can be transformed. From Prob. 35, the transformed equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $r^2 + r + 1 = 0$, with roots $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution is

$$y(x) = c_1 e^{-x/2} cos\,\sqrt{3}\,x/2 + c_2\,e^{-x/2} sin\,\sqrt{3}\,x/2\,.$$

Since $x = t^2/2$, the solution in the original variable t is

$$y(t) = e^{-t^2/4}\Big[c_1 cos\,\Big(\sqrt{3}\,t^2/4\Big) + c_2\,sin\,\Big(\sqrt{3}\,t^2/4\Big)\Big]\,.$$

40. $p(t) = 4/t$ and $q(t) = 2/t^2$. We have $x = \sqrt{2}\int t^{-1}dt = \sqrt{2}\,ln\,t$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \frac{3}{\sqrt{2}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{3}{\sqrt{2}}\frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $\sqrt{2}\;r^2 + 3r + \sqrt{2} = 0$, with roots $r = -\sqrt{2}\,,\ -1/\sqrt{2}$. The general solution is

$$y(x) = c_1 e^{-\sqrt{2}x} + c_2\,e^{-x/\sqrt{2}}.$$

Since $x = \sqrt{2}\;ln\,t$, the solution in the original variable t is

$$\begin{aligned} y(t)\ &= c_1 e^{-2\,ln\,t} + c_2\,e^{-ln\,t} \\ &= c_1 t^{-2} + c_2\,t^{-1}. \end{aligned}$$

41. $p(t) = 3/t$ and $q(t) = 1.25/t^2$. We have $x = \sqrt{1.25}\int t^{-1}dt = \sqrt{1.25}\,ln\,t$. Checking the feasibility of the transformation,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \frac{4}{\sqrt{5}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{4}{\sqrt{5}}\frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $\sqrt{5}\,r^2 + 4r + \sqrt{5} = 0$, with roots $r = -\frac{2}{\sqrt{5}} \pm i\frac{1}{\sqrt{5}}$. The general solution is

$$y(x) = c_1 e^{-2x/\sqrt{5}} cos\,x/\sqrt{5} + c_2\,e^{-2x/\sqrt{5}} sin\,x/\sqrt{5}\,.$$

Since $2x/\sqrt{5} = ln\,t$, the solution in the original variable t is

$$\begin{aligned} y(t) &= c_1 e^{-ln\,t} cos\left(ln\sqrt{t}\right) + c_2\, e^{-ln\,t} sin\left(ln\sqrt{t}\right) \\ &= t^{-1}\left[c_1 cos\left(ln\sqrt{t}\right) + c_2\, sin\left(ln\sqrt{t}\right)\right]. \end{aligned}$$

42. $p(t) = -4/t$ and $q(t) = -6/t^2$. Set $x = \sqrt{6}\int t^{-1}dt = \sqrt{6}\,ln\,t$.
Checking the feasibility of the transformation (**see Prob. 34 d, with* $q < 0$),

$$\frac{-q'(t) - 2p(t)q(t)}{2[-q(t)]^{3/2}} = \frac{-5}{\sqrt{6}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{-5}{\sqrt{6}}\frac{dy}{dx} - y = 0\,.$$

Based on the methods in this section, the characteristic equation is $\sqrt{6}\,r^2 - 5$
$r - \sqrt{6} = 0$,
with roots $r = \sqrt{6}\,,\ -1/\sqrt{6}$. The general solution is

$$y(x) = c_1 e^{\sqrt{6}\,x} + c_2\, e^{-x/\sqrt{6}}.$$

Since $x = \sqrt{6}\,ln\,t$, the solution in the original variable t is

$$\begin{aligned} y(t) &= c_1 e^{6ln\,t} + c_2\, e^{-ln\,t} \\ &= c_1 t^6 + c_2\, t^{-1}. \end{aligned}$$