Section 7.4

3. Eq. (14) states that the Wronskian satisfies the first order linear ODE

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})W.$$

The general solution is

$$W(t) = C\,exp\left[\int (p_{11} + p_{22} + \cdots + p_{nn})dt\right],$$

in which C is an arbitrary constant. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be matrices representing two sets of fundamental solutions. It follows that

$$\begin{aligned} det(\mathbf{X}_1) = W_1(t) = C_1 exp\left[\int (p_{11} + p_{22} + \cdots + p_{nn})dt\right] \\ det(\mathbf{X}_2) = W_2(t) = C_2 exp\left[\int (p_{11} + p_{22} + \cdots + p_{nn})dt\right]. \end{aligned}$$

Hence $det(\mathbf{X}_1)/det(\mathbf{X}_2) = C_1/C_2$. Note that $C_2 \neq 0$.

4. First note that $p_{11} + p_{22} = -p(t)$. As shown in Prob. (3),

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = c\,e^{-\int p(t)dt}.$$

For second order linear ODE, the Wronskian (as defined in Chap. 3) satisfies the first order differential equation $W' + p(t)W = 0$. It follows that

$$W\left[y^{(1)}, y^{(2)}\right] = c_1\,e^{-\int p(t)dt}.$$

Alternatively, based on the hypothesis,

$$\begin{aligned} y^{(1)} &= \alpha_{11}\,x_{11} + \alpha_{12}\,x_{12} \\ y^{(2)} &= \alpha_{21}\,x_{11} + \alpha_{22}\,x_{12}\,. \end{aligned}$$

Direct calculation shows that

$$\begin{aligned} W\left[y^{(1)}, y^{(2)}\right] &= \begin{vmatrix} \alpha_{11}\,x_{11} + \alpha_{12}\,x_{12} & \alpha_{21}\,x_{11} + \alpha_{22}\,x_{12} \\ \alpha_{11}\,x_{11}' + \alpha_{12}\,x_{12}' & \alpha_{21}\,x_{11}' + \alpha_{22}\,x_{12}' \end{vmatrix} \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{12}' - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{11}' \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{22} - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{21}\,. \end{aligned}$$

Here we used the fact that $x_1' = x_2$. Hence

$$W\left[y^{(1)}, y^{(2)}\right] = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right].$$

5. The *particular solution* satisfies the ODE $\left[\mathbf{x}^{(p)}\right]' = \mathbf{P}(t)\mathbf{x}^{(p)} + \mathbf{g}(t)$. Now let

$\mathbf{x} = \boldsymbol{\phi}(t)$ be any solution of the homogeneous equation. That is, $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. We know that $\mathbf{x} = \mathbf{x}^c$, in which $\mathbf{x}^c$ is a linear combination of some fundamental solution. By linearity of the differential equation, it follows that $\mathbf{x} = \mathbf{x}^{(p)} + \mathbf{x}^c$ is a solution of the ODE. Based on the *uniqueness theorem*, all solutions must have this form.

$7(a)$. By definition,

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = \left(t^2 - 2t\right)e^t.$$

(b). The Wronskian vanishes at $t_0 = 0$ and $t_0 = 2$. Hence the vectors are linearly independent on $\mathcal{D} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

(c). It follows from Theorem 7.4.3 that one or more of the coefficients of the ODE must be discontinuous at $t_0 = 0$ and $t_0 = 2$. If not, the Wronskian would not vanish.

(d). Let

$$\mathbf{x} = c_1\begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Then

$$\mathbf{x}' = c_1\begin{pmatrix} 2t \\ 2 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\mathbf{x} &= c_1\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} c_1[p_{11}t^2 + 2p_{12}t] + c_2[p_{11} + p_{12}]e^t \\ c_1[p_{21}t^2 + 2p_{22}t] + c_2[p_{21} + p_{22}]e^t \end{pmatrix}. \end{aligned}$$

Comparing coefficients, we find that

$$\begin{aligned} p_{11}t^2 + 2p_{12}t &= 2t \\ p_{11} + p_{12} &= 1 \\ p_{21}t^2 + 2p_{22}t &= 2 \\ p_{21} + p_{22} &= 1\,. \end{aligned}$$

Solution of this system of equations results in

$$p_{11}(t) = 0\ ,\ p_{12}(t) = 1\ ,\ p_{21}(t) = \frac{2 - 2t}{t^2 - 2t}\ ,\ p_{22}(t) = \frac{t^2 - 2}{t^2 - 2t}\,.$$

Hence the vectors are solutions of the ODE

$$\mathbf{x}' = \frac{1}{t^2 - 2t}\begin{pmatrix} 0 & t^2 - 2t \\ 2 - 2t & t^2 - 2 \end{pmatrix}\mathbf{x}\,.$$

8. Suppose that the solutions $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(m)}$ are linearly *dependent* at $t = t_0$. Then there are constants $c_1\,, c_2, \cdots, c_m$ (not all zero) such that

$$c_1\mathbf{x}^{(1)}(t_0) + c_2\,\mathbf{x}^{(2)}(t_0) + \cdots + c_m\,\mathbf{x}^{(m)}(t_0) = \mathbf{0}\,.$$

Now let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\,\mathbf{x}^{(2)}(t) + \cdots + c_m\,\mathbf{x}^{(m)}(t)$. Then clearly, $\mathbf{z}(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, with $\mathbf{z}(t_0) = 0$. Furthermore, $\mathbf{y}(t) \equiv \mathbf{0}$ is also a solution, with $\mathbf{y}(t_0) = 0$. By the *uniqueness theorem*, $\mathbf{z}(t) = \mathbf{y}(t) = \mathbf{0}$. Hence

$$c_1\mathbf{x}^{(1)}(t) + c_2\,\mathbf{x}^{(2)}(t) + \cdots + c_m\,\mathbf{x}^{(m)}(t) = \mathbf{0}$$

on the entire interval $\alpha < t < \beta$. Going in the other direction is trivial.

$9(a)$. Let $\mathbf{y}(t)$ be *any* solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. It follows that

$$\mathbf{z}(t) + \mathbf{y}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\,\mathbf{x}^{(2)}(t) + \cdots + c_n\,\mathbf{x}^{(n)}(t) + \mathbf{y}(t)$$

is also a solution. Now let $t_0 \in (\alpha\,, \beta)$. Then the collection of vectors

$$\mathbf{x}^{(1)}(t_0), \mathbf{x}^{(2)}(t_0), \cdots,\ \mathbf{x}^{(n)}(t_0),\ \mathbf{y}(t_0)$$

constitutes $n+1$ vectors, each with n components. Based on the assertion in Prob. 11, Section 7.3, these vectors are necessarily linearly *dependent*. That is, there are $n+1$ constants $b_1\,, b_2, \cdots, b_n, b_{n+1}$ (not all zero) such that

$$b_1\mathbf{x}^{(1)}(t_0) + b_2\,\mathbf{x}^{(2)}(t_0) + \cdots + b_n\,\mathbf{x}^{(n)}(t_0) + b_{n+1}\,\mathbf{y}(t_0) = \mathbf{0}\,.$$

From Prob. 8, we have

$$b_1\mathbf{x}^{(1)}(t) + b_2\,\mathbf{x}^{(2)}(t) + \cdots + b_n\,\mathbf{x}^{(n)}(t) + b_{n+1}\,\mathbf{y}(t) = \mathbf{0}$$

for all $t \in (\alpha\,, \beta)$. Now $b_{n+1} \neq 0$, otherwise that would contradict the fact that the first n vectors are linearly independent. Hence

$$\mathbf{y}(t) = -\frac{1}{b_{n+1}}\big(b_1\mathbf{x}^{(1)}(t) + b_2\,\mathbf{x}^{(2)}(t) + \cdots + b_n\,\mathbf{x}^{(n)}(t)\big),$$

and the assertion is true.

(b). Consider $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\,\mathbf{x}^{(2)}(t) + \cdots + c_n\,\mathbf{x}^{(n)}(t)$, and suppose that we also have

$$\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + k_2\,\mathbf{x}^{(2)}(t) + \cdots + k_n\,\mathbf{x}^{(n)}(t)\,.$$

Based on the assumption,

$$(k_1 - c_1)\mathbf{x}^{(1)}(t) + (k_2 - c_2)\mathbf{x}^{(2)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}.$$

The collection of vectors

$$\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \cdots, \ \mathbf{x}^{(n)}(t)$$

is linearly *independent* on $\alpha < t < \beta$. It follows that $k_i - c_i = 0$, for $i = 1, 2, \cdots, n$.