Section 10.7

$2(a)$. The initial velocity is *zero*. Therefore the solution, as given by Eq. (20), is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n \, sin \frac{n\pi x}{L} \, cos \frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $f(x)$. That is,

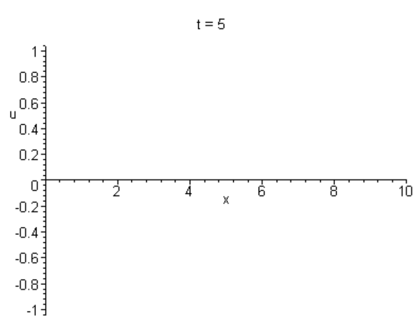
$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin \frac{n\pi x}{L} dx \\ &= \frac{2}{L}\left[\int_0^{L/4} \frac{4x}{L} sin \frac{n\pi x}{L} dx + \int_{L/4}^{3L/4} sin \frac{n\pi x}{L} dx + \int_{3L/4}^{L} \frac{4L-4x}{L} sin \frac{n\pi x}{L} dx\right] \\ &= 8\,\frac{sin\, n\pi/4 + sin\, 3n\pi/4}{n^2\pi^2}\,. \end{aligned}$$

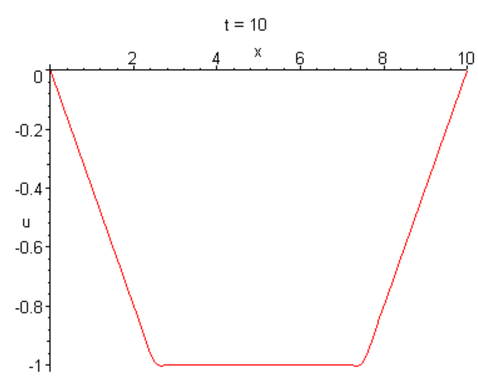
Therefore the displacement of the string is given by

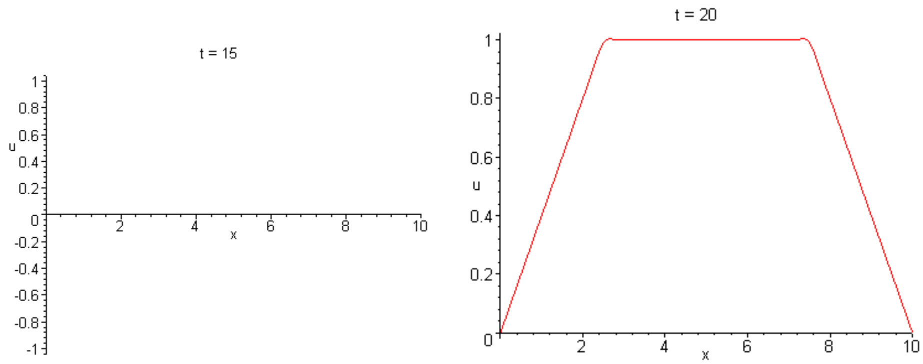
$$u(x\,,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[sin\frac{n\pi}{4} + sin\frac{3n\pi}{4}\right] sin\frac{n\pi x}{L}\, cos\frac{n\pi a\,t}{L}\,.$$

(b). With $a = 1$ and $L = 10$,

$$u(x\,,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[sin\frac{n\pi}{4} + sin\frac{3n\pi}{4}\right] sin\frac{n\pi x}{10}\, cos\frac{n\pi t}{10}\,.$$







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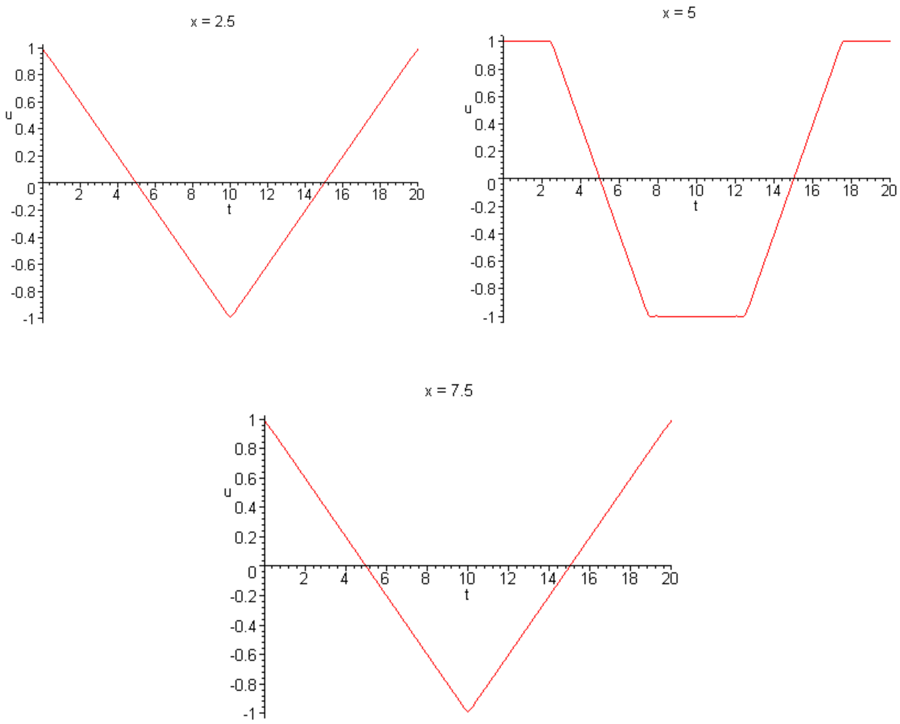

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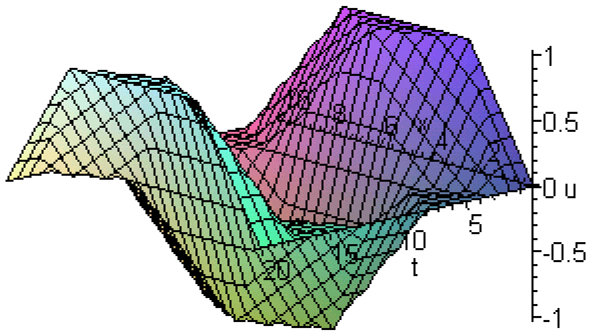
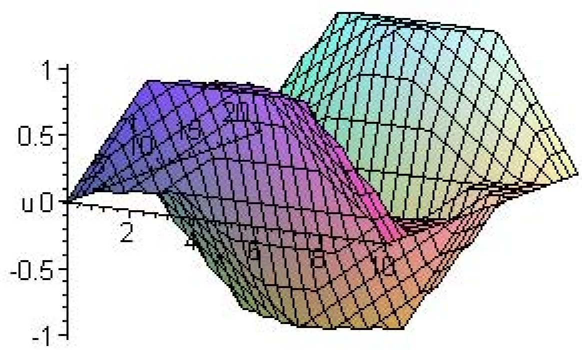




$3(a)$. The initial velocity is *zero*. As given by Eq. (20), the solution is

$$u(x\,,t)=\sum_{n=1}^{\infty}c_n\,sin\frac{n\pi x}{L}\,cos\frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $f(x)$. That is,

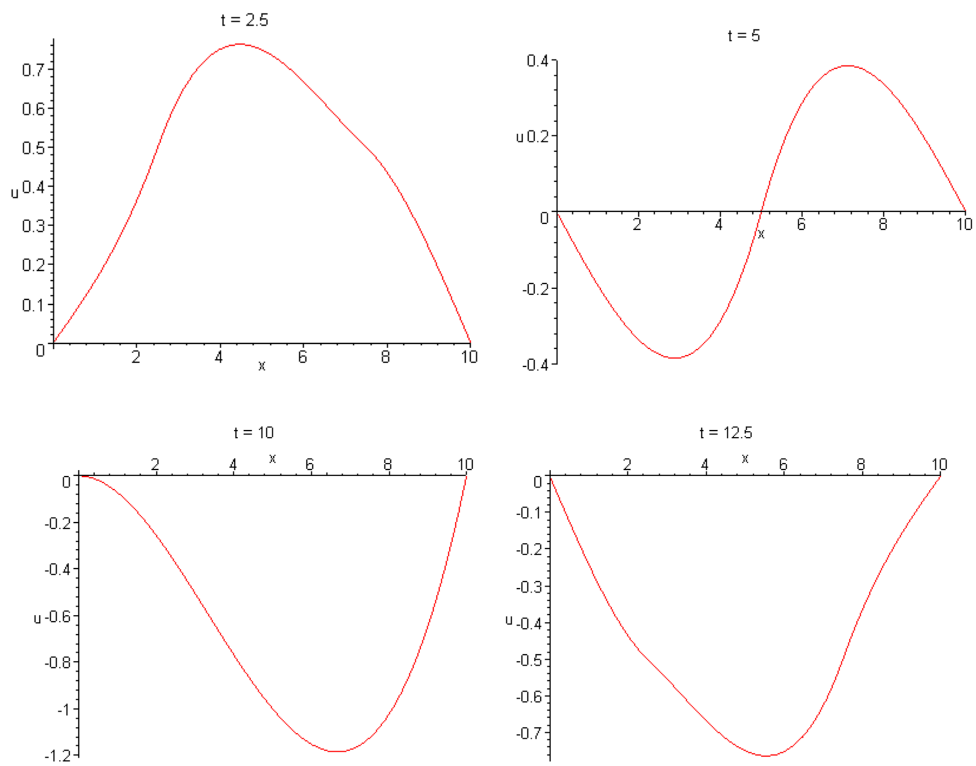
$$
\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\
&= \frac{2}{L}\int_0^L \frac{8x(L-x)^2}{L^3} sin\frac{n\pi x}{L} dx \\
&= 32\,\frac{2+cos\, n\pi}{n^3\pi^3}\,.
\end{aligned}
$$

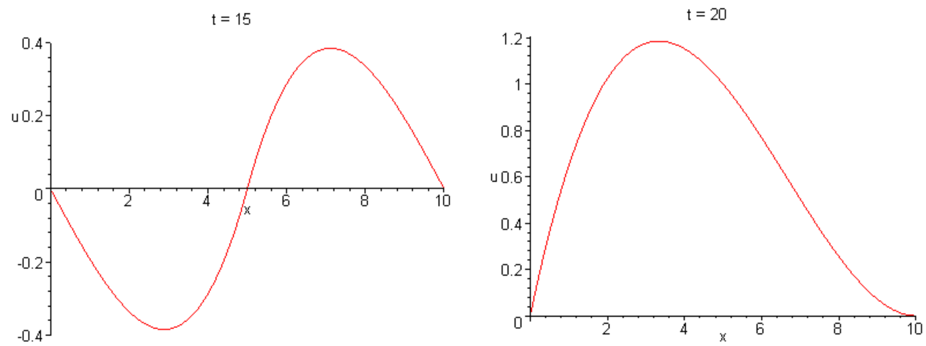
Therefore the displacement of the string is given by

$$
u(x\,,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{2+cos\, n\pi}{n^3} sin\frac{n\pi x}{L}\, cos\frac{n\pi a\, t}{L}\,.
$$

(b). With $a = 1$ and $L = 10$,

$$
u(x\,,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{2+cos\, n\pi}{n^3} sin\frac{n\pi x}{10}\, cos\frac{n\pi t}{10}\,.
$$





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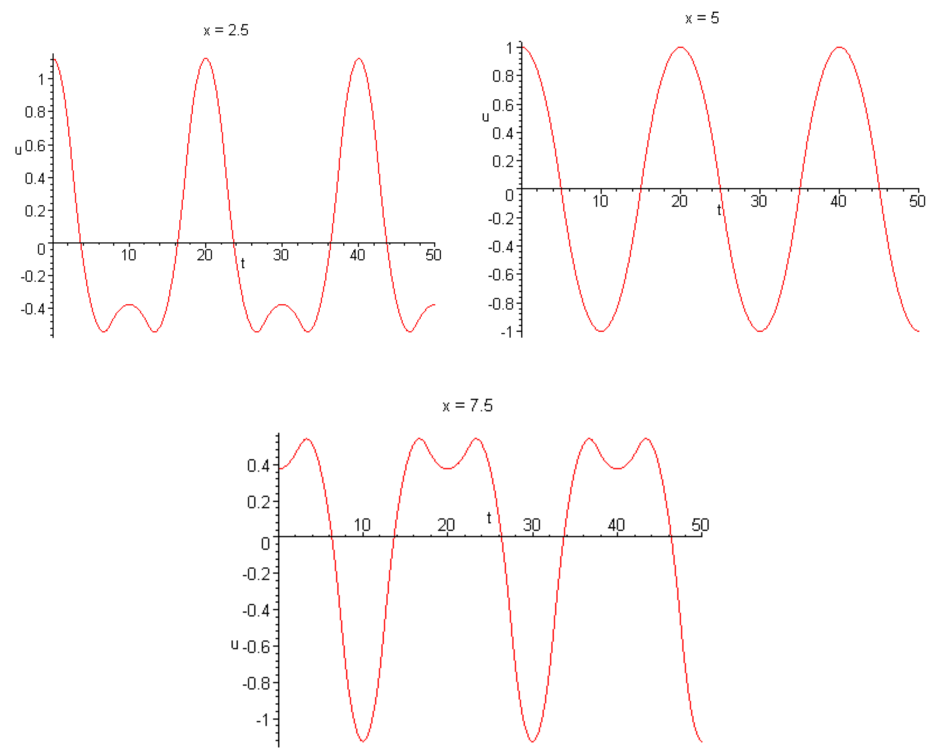

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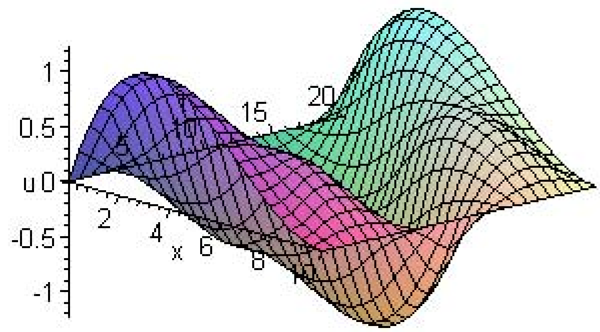
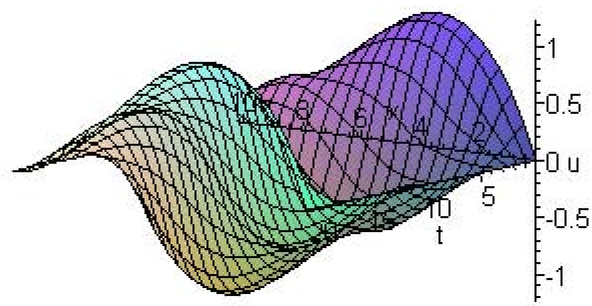




$4(a)$. As given by Eq. (20), the solution is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n \, sin \frac{n\pi x}{L} \, cos \frac{n\pi a\, t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $f(x)$. That is,

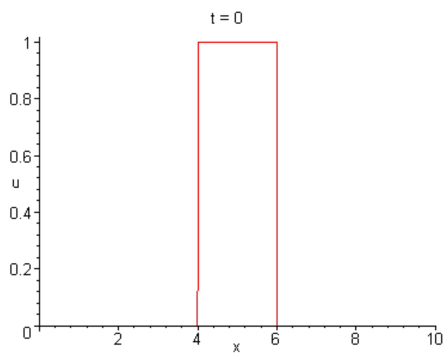
$$
\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L}dx \\
&= \frac{2}{L}\int_{L/2-1}^{L/2+1} sin\frac{n\pi x}{L}dx \\
&= 4\,\frac{sin\frac{n\pi}{2}\, sin\frac{n\pi}{L}}{n\pi}\,.
\end{aligned}
$$

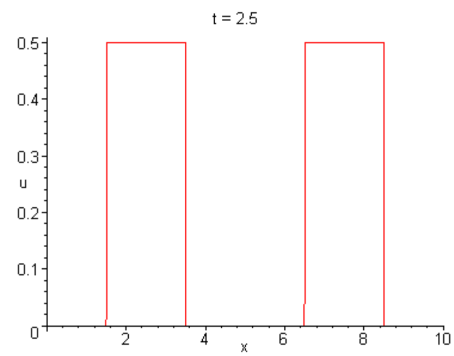
Therefore the displacement of the string is given by

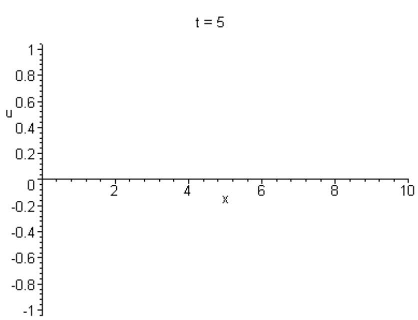
$$u(x\,,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[sin\frac{n\pi}{2}\, sin\frac{n\pi}{L}\right] sin\frac{n\pi x}{L}\, cos\frac{n\pi a\, t}{L}\,.$$

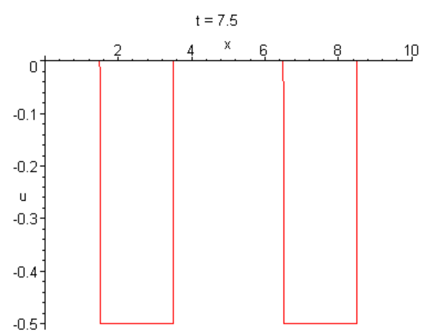
(b). With $a = 1$ and $L = 10$,

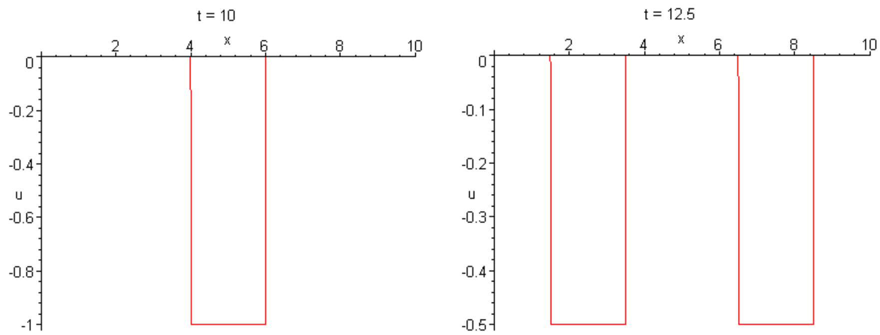
$$u(x\,,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[sin\frac{n\pi}{2}\, sin\frac{n\pi}{10}\right] sin\frac{n\pi x}{10}\, cos\frac{n\pi t}{10}\,.$$











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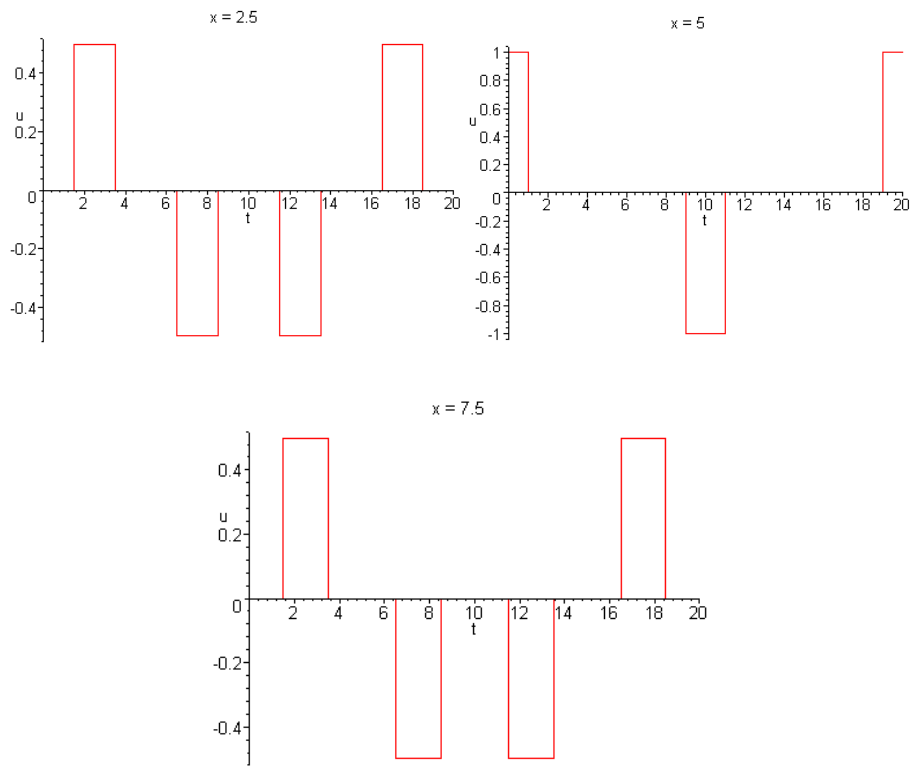

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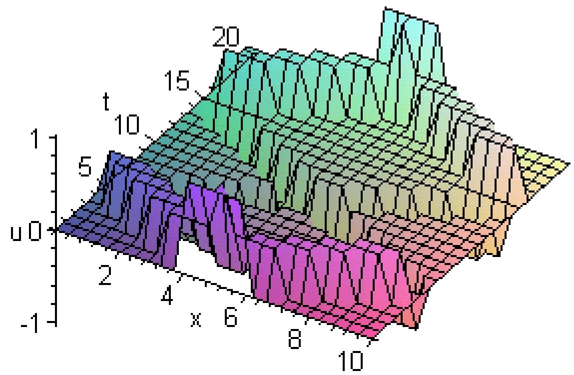
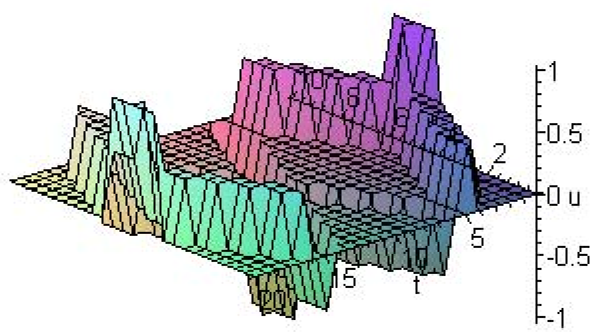




$5(a)$. The initial displacement is *zero.* Therefore the solution, as given by Eq. (34), is

$$u(x\,,t)=\sum_{n=1}^{\infty}k_n\,sin\frac{n\pi x}{L}\,sin\frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $u_t(x\,,0)=g(x)$. It follows that

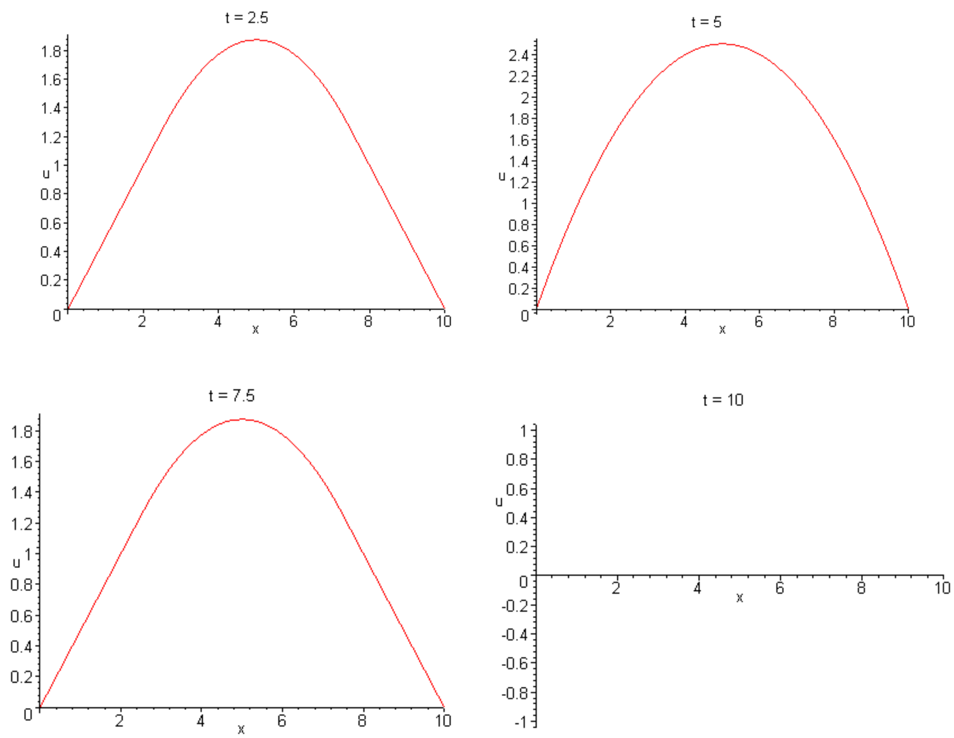
$$\begin{aligned} k_n &= \frac{2}{n\pi a}\int_0^L g(x) sin\frac{n\pi x}{L}dx \\ &= \frac{2}{n\pi a}\left[\int_0^{L/2}\frac{2x}{L}sin\frac{n\pi x}{L}dx + \int_{L/2}^{L}\frac{2(L-x)}{L}sin\frac{n\pi x}{L}dx\right] \\ &= 8L\,\frac{sin\,n\pi/2}{n^3\pi^3 a}\,. \end{aligned}$$

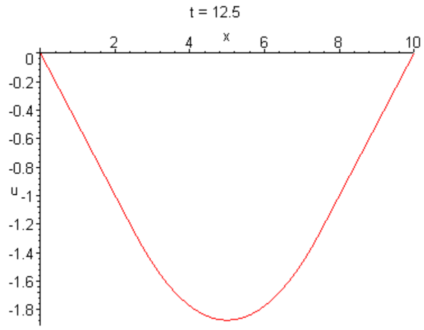
Therefore the displacement of the string is given by

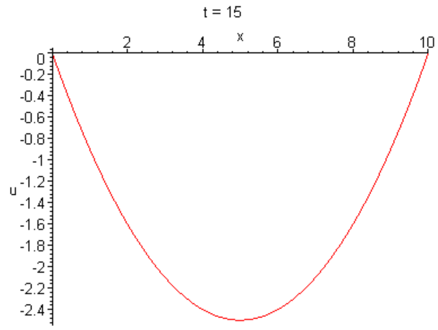
$$u(x\,,t) = \frac{8L}{a\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}sin\frac{n\pi}{2}\;sin\frac{n\pi x}{L}\;sin\frac{n\pi a\,t}{L}\,.$$

(b). With $a = 1$ and $L = 10$,

$$u(x\,,t) = \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}sin\frac{n\pi}{2}\;sin\frac{n\pi x}{10}\;sin\frac{n\pi t}{10}\,.$$







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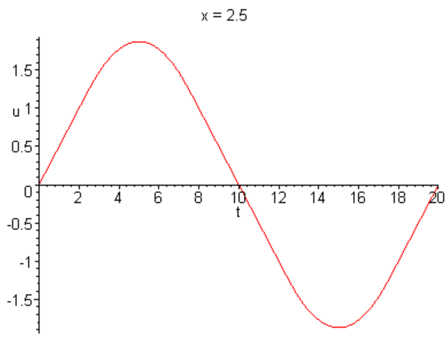

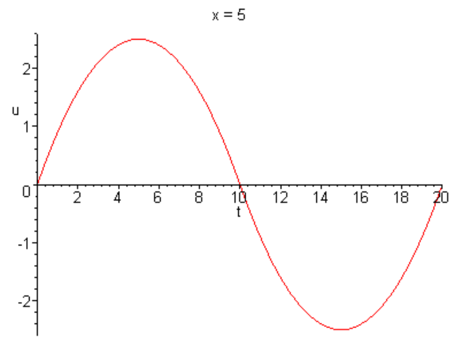

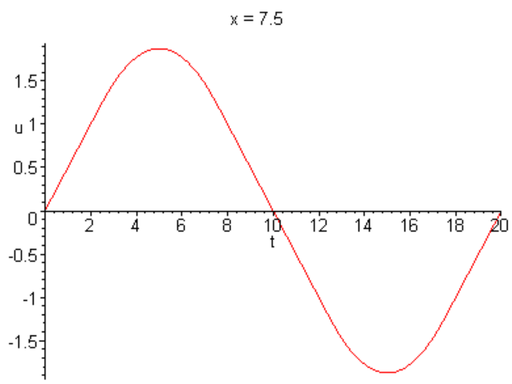

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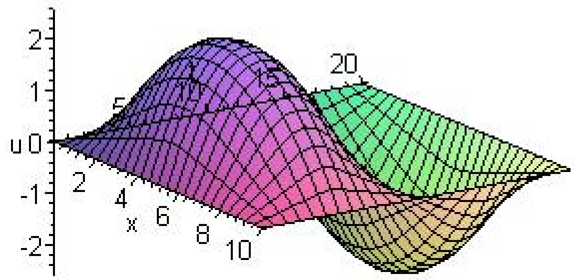
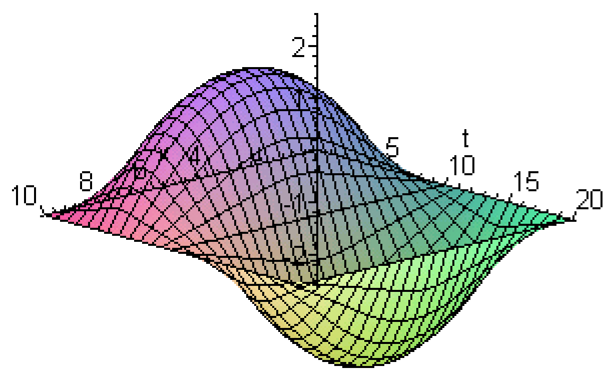




$7(a)$. The initial displacement is *zero.* As given by Eq. (34), the solution is

$$u(x\,,t) = \sum_{n=1}^{\infty} k_{\,n}\, sin\frac{n\pi x}{L}\; sin\frac{n\pi a\, t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $u_t(x\,,0) = g(x)$. It follows that

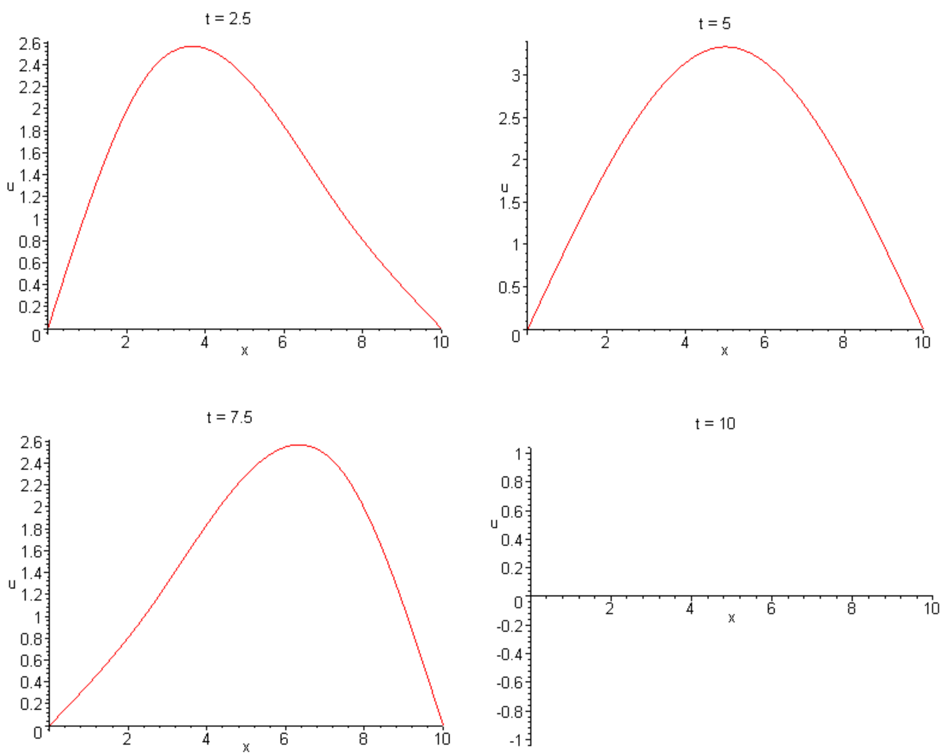
$$\begin{aligned} k_n &= \frac{2}{n\pi a}\int_0^L g(x) sin\frac{n\pi x}{L} dx \\ &= \frac{2}{n\pi a}\int_0^L \frac{8x(L-x)^2}{L^3} sin\frac{n\pi x}{L} dx \\ &= 32L\,\frac{2+cos\,n\pi}{n^4\pi^4 a}\,. \end{aligned}$$

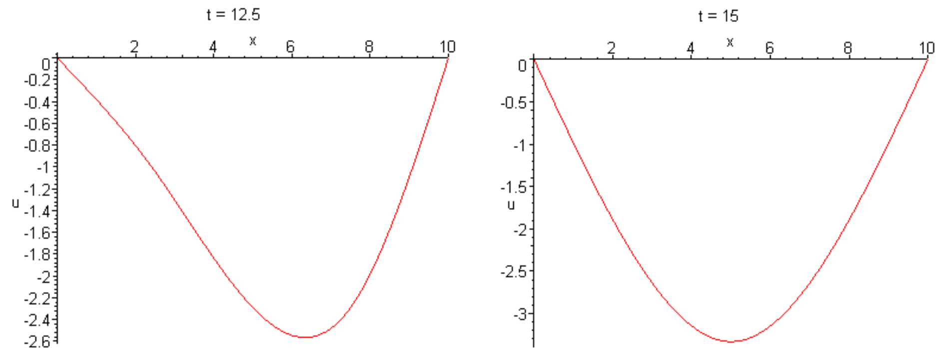
Therefore the displacement of the string is given by

$$u(x\,,t) = \frac{32L}{a\pi^4}\sum_{n=1}^{\infty}\frac{2+cos\,n\pi}{n^4} sin\frac{n\pi x}{L}\,sin\frac{n\pi a\,t}{L}\,.$$

(b). With $a = 1$ and $L = 10$,

$$u(x\,,t) = \frac{320}{\pi^4}\sum_{n=1}^{\infty}\frac{2+cos\,n\pi}{n^4} sin\frac{n\pi x}{10}\,sin\frac{n\pi t}{10}\,.$$





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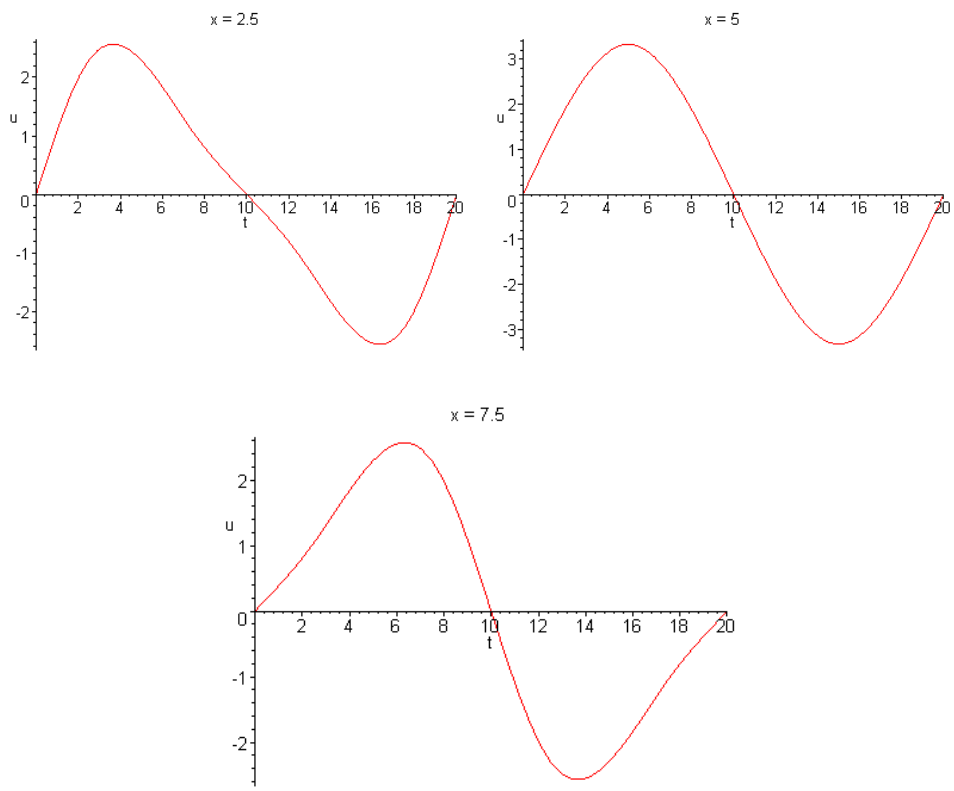

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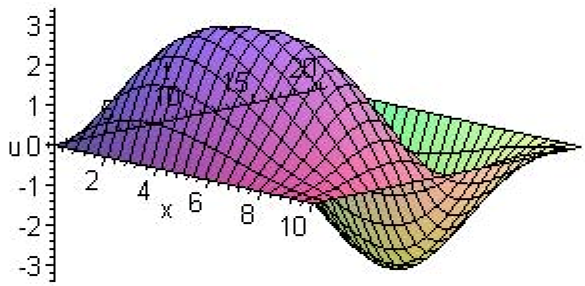
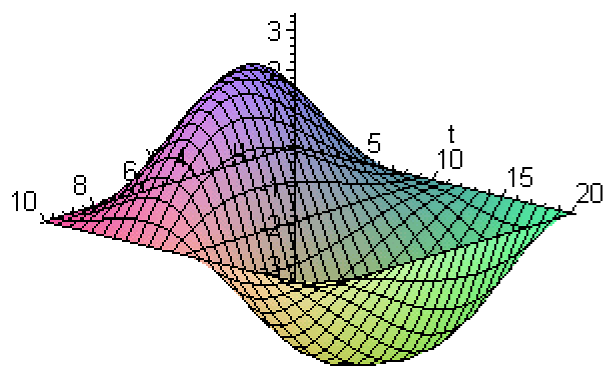




$8(a)$. As given by Eq. (34), the solution is

$$u(x\,,t)=\sum_{n=1}^{\infty}k_n\,sin\frac{n\pi x}{L}\,sin\frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $u_t(x\,,0)=g(x)$. It follows that

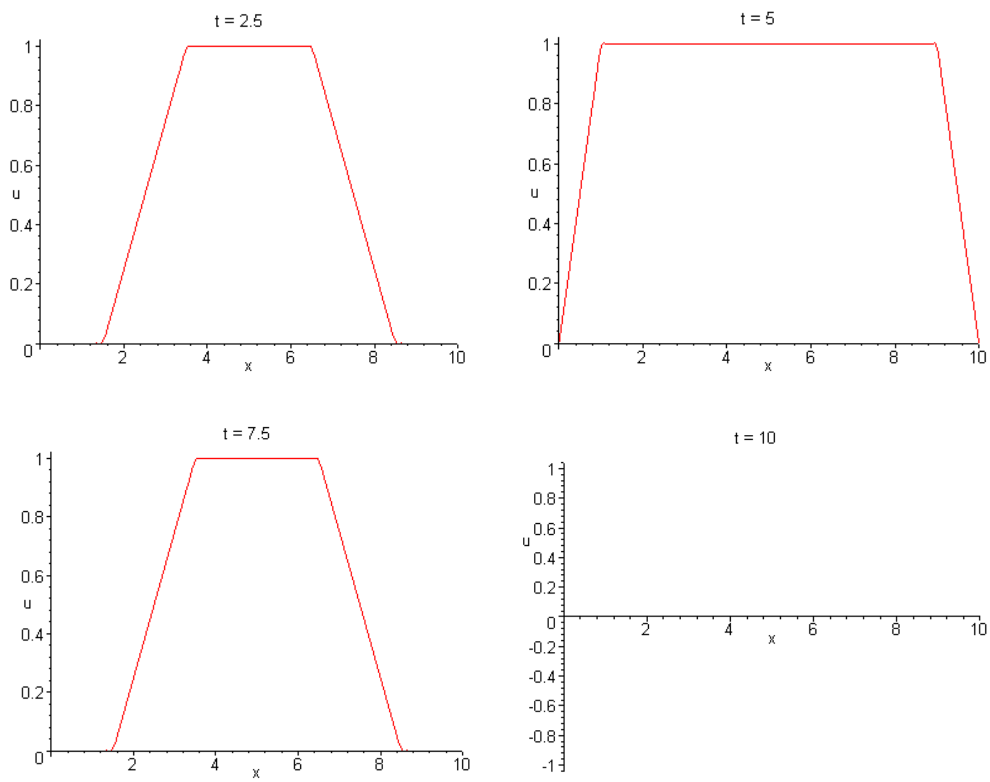
$$
\begin{aligned}
k_n &= \frac{2}{n\pi a}\int_0^L g(x) sin \frac{n\pi x}{L} dx \\
&= \frac{2}{n\pi a}\int_{L/2-1}^{L/2+1} sin \frac{n\pi x}{L} dx \\
&= 4L\,\frac{sin\frac{n\pi}{2}\, sin\frac{n\pi}{L}}{n^2\pi^2 a}\,.
\end{aligned}
$$

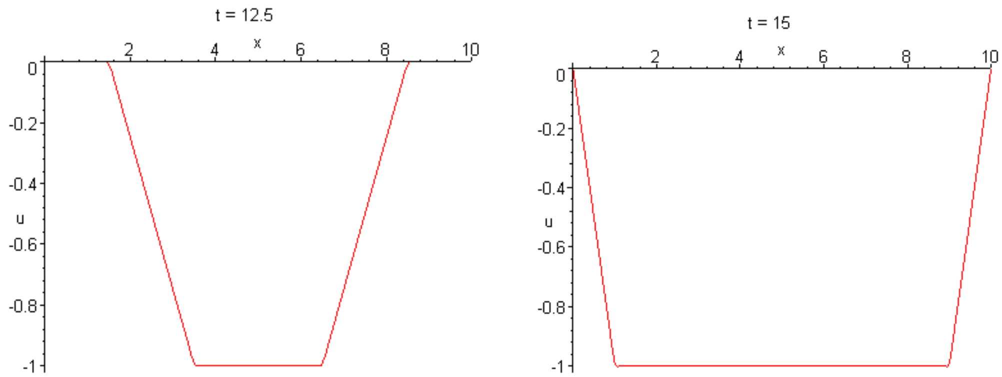
Therefore the displacement of the string is given by

$$
u(x\,,t) = \frac{4L}{a\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[sin\frac{n\pi}{2}\, sin\frac{n\pi}{L}\right] sin\frac{n\pi x}{L}\, sin\frac{n\pi a\, t}{L}\,.
$$

(b). With $a = 1$ and $L = 10$,

$$
u(x\,,t) = \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[sin\frac{n\pi}{2}\, sin\frac{n\pi}{10}\right] sin\frac{n\pi x}{10}\, sin\frac{n\pi t}{10}\,.
$$





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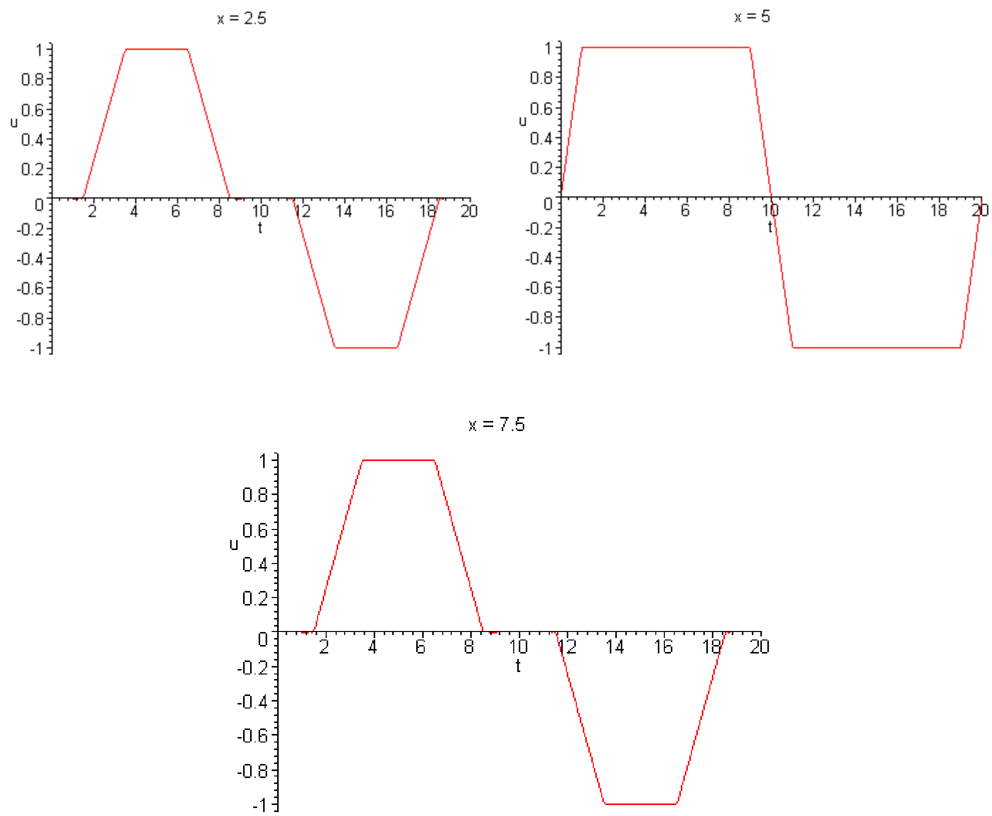

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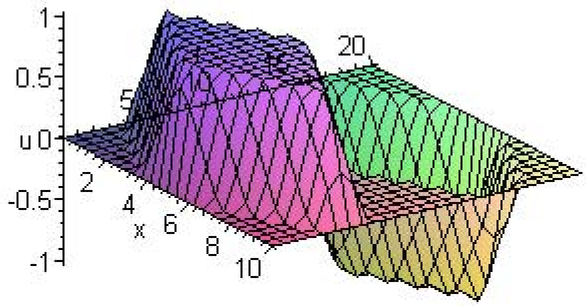
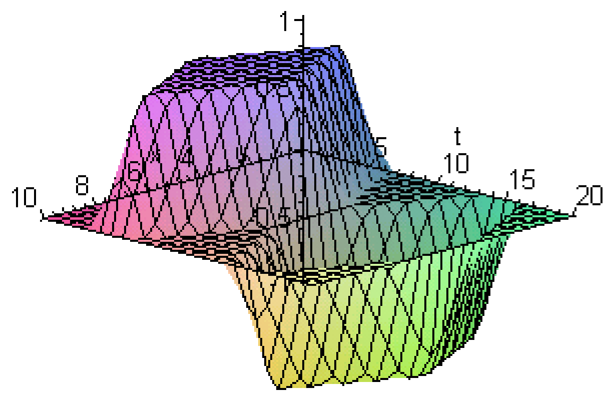




$11(a)$. As shown in Prob. 9, the solution is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n \, sin \frac{(2n-1)\pi x}{2L} \, cos \frac{(2n-1)\pi a\, t}{2L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $f(x)$. It follows that

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin \frac{(2n-1)\pi x}{2L} dx \\ &= \frac{2}{L}\int_0^L \frac{8x(L-x)^2}{L^3} sin \frac{(2n-1)\pi x}{2L} dx \\ &= 512\,\frac{3cos\, n\pi + (2n-1)\pi}{(2n-1)^4\pi^4}\,. \end{aligned}$$

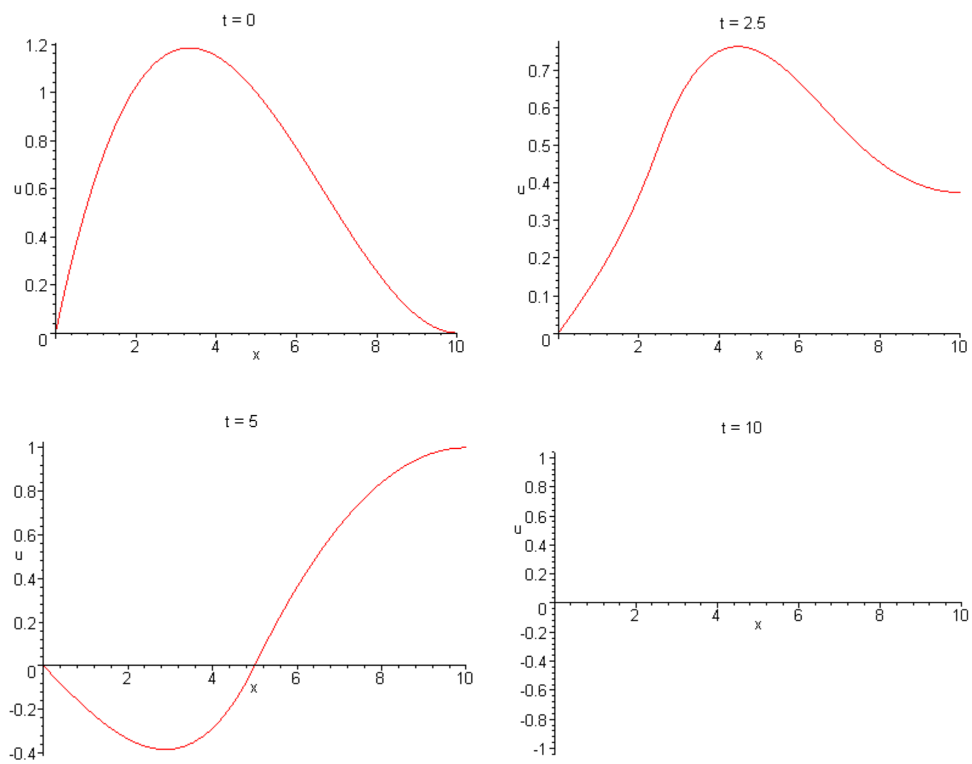
Therefore the displacement of the string is given by

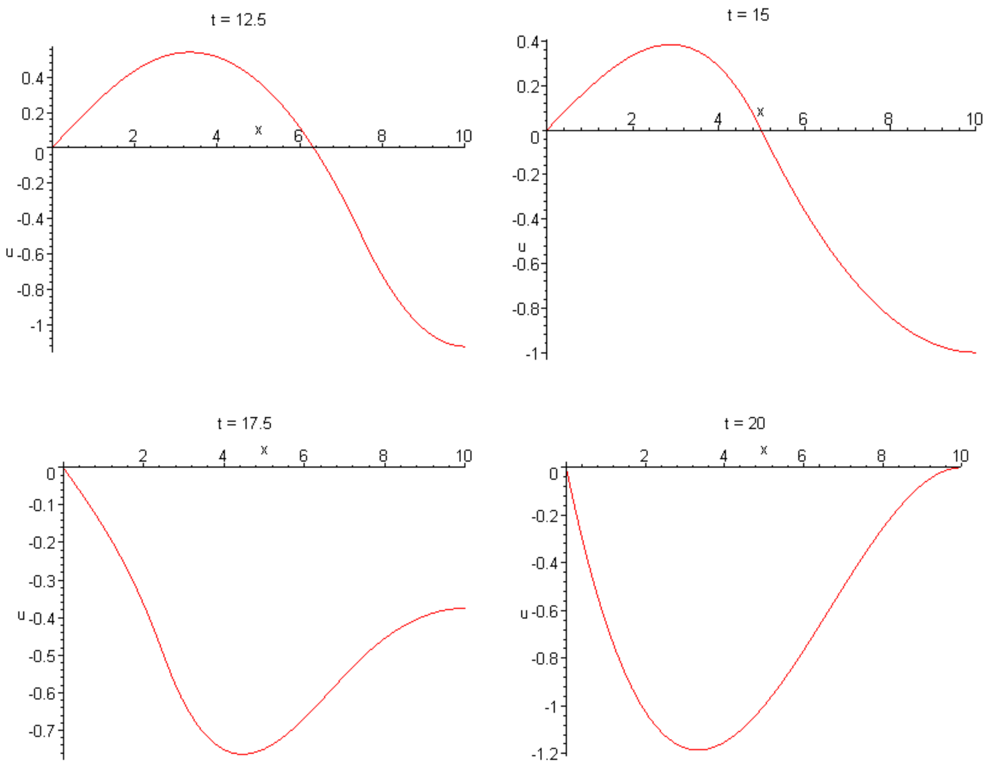
$$u(x\,,t) = \frac{512}{\pi^4}\sum_{n=1}^{\infty}\frac{3cos\, n\pi + (2n-1)\pi}{(2n-1)^4} sin \frac{(2n-1)\pi x}{2L}\, cos \frac{(2n-1)\pi a\, t}{2L}\,.$$

Note that the period is $T = 4L/a$.

(b). With $a = 1$ and $L = 10$,

$$u(x\,,t) = \frac{512}{\pi^4}\sum_{n=1}^{\infty}\frac{3cos\, n\pi + (2n-1)\pi}{(2n-1)^4} sin \frac{(2n-1)\pi x}{20}\, cos \frac{(2n-1)\pi t}{20}\,.$$





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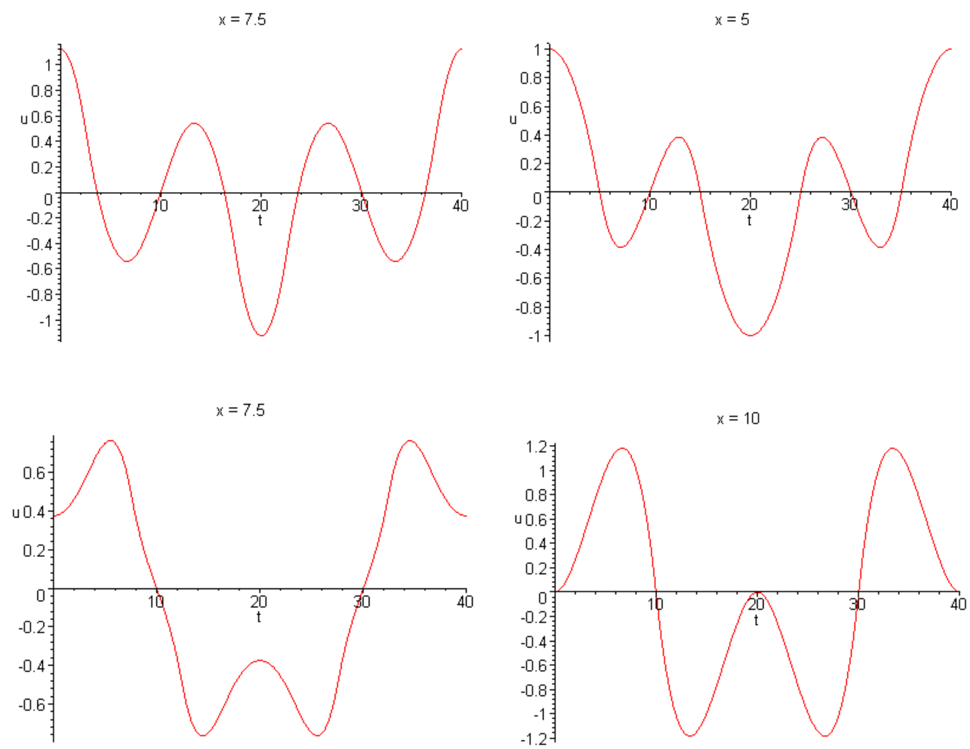

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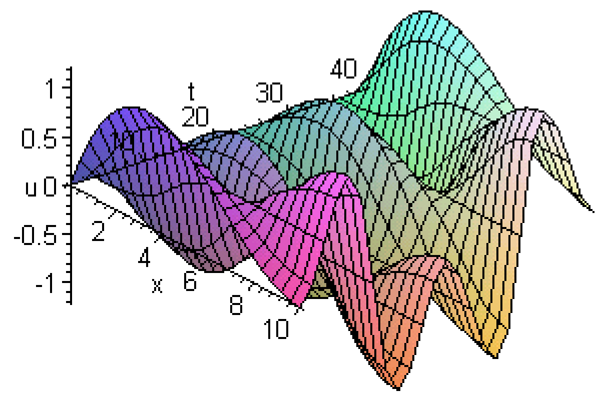

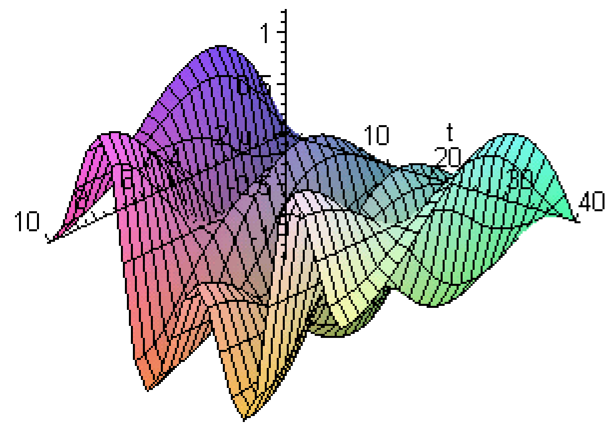

12. The *wave equation* is given by

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \,.$$

Setting $s = x/L$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \frac{ds}{dx} = \frac{1}{L} \frac{\partial u}{\partial s} \,.$$

It follows that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{L^2}\frac{\partial^2 u}{\partial s^2}\,.$$

Likewise, with $\tau = at/L$,

$$\frac{\partial u}{\partial t} = \frac{a}{L}\frac{\partial u}{\partial \tau} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = \frac{a^2}{L^2}\frac{\partial^2 u}{\partial \tau^2}\,.$$

Substitution into the original equation results in

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial \tau^2}\,.$$

15. The given specifications are $L = 5\,ft$, $T = 50\,lb$, and *weight* per unit length $\gamma = 0.026$ *lb/ft* . It follows that $\rho = \gamma/32.2 = 80.75 \times 10^{-5}$ *slugs/ft* .

(a). The transverse waves propagate with a speed of $a = \sqrt{T/\rho} = 248$ *ft/sec* .

(b). The *natural frequencies* are $\omega_n = n\pi a/L = 49.8\,\pi n$ *rad/sec* .

(c). The new wave speed is $a = \sqrt{(T + \Delta T)/\rho}$. For a string with fixed ends, the natural modes are proportional to the functions

$$M_n(x) = sin\frac{n\pi x}{L}\,,$$

which are independent of a .

19. The solution of the wave equation

$$a^2 v_{xx} = v_{tt}$$

in an infinite one-dimensional medium subject to the initial conditions

$$v(x\,,0) = f(x)\,, \quad v_t(x\,,0) = 0\,, \quad -\infty < x < \infty$$

is given by

$$v(x\,,t) = \frac{1}{2}[f(x - at) + f(x + at)].$$

The solution of the wave equation

$$a^2 w_{xx} = w_{tt}\,,$$

on the same domain, subject to the initial conditions

$$w(x\,,0) = 0\,, \quad w_t(x\,,0) = g(x)\,, \quad -\infty < x < \infty$$

is given by

$$w(x\,,t)=\frac{1}{2a}\int_{x-at}^{x+at}g(\xi)d\xi\,.$$

Let $u(x\,,t)=v(x\,,t)+w(x\,,t)$. Since the PDE is *linear*, it is easy to see that $u(x\,,t)$ is a solution of the wave equation $a^2u_{xx}=u_{tt}$. Furthermore, we have

$$u(x\,,0)=v(x\,,0)+w(x\,,0)=f(x)$$

and

$$u_t(x\,,0)=v_t(x\,,0)+w_t(x\,,0)=g(x)\,.$$

Hence $u(x\,,t)$ is a solution of the general wave propagation problem.

20. The solution of the specified wave propagation problem is

$$u(x\,,t)=\sum_{n=1}^{\infty}c_n\,sin\frac{n\pi x}{L}\,cos\frac{n\pi a\,t}{L}\,.$$

Using a standard trigonometric identity,

$$\begin{aligned} sin\frac{n\pi x}{L}\,cos\frac{n\pi a\,t}{L}\; &=\frac{1}{2}\left[sin\left(\frac{n\pi x}{L}+\frac{n\pi a\,t}{L}\right)+sin\left(\frac{n\pi x}{L}-\frac{n\pi a\,t}{L}\right)\right] \\ &=\frac{1}{2}\left[sin\frac{n\pi}{L}(x+at)+sin\frac{n\pi}{L}(x-at)\right]. \end{aligned}$$

We can therefore also write the solution as

$$u(x\,,t)=\frac{1}{2}\sum_{n=1}^{\infty}c_n\left[sin\frac{n\pi}{L}(x+at)+sin\frac{n\pi}{L}(x-at)\right].$$

Assuming that the series can be split up,

$$u(x\,,t)=\frac{1}{2}\left[\sum_{n=1}^{\infty}c_n\,sin\frac{n\pi}{L}(x-at)+\sum_{n=1}^{\infty}c_n\,sin\frac{n\pi}{L}(x+at)\right].$$

Comparing the solution to the one given by Eq. (28), we can infer that

$$h(x)=\sum_{n=1}^{\infty}c_n\,sin\frac{n\pi x}{L}\,.$$

21. Let $h(\xi)$ be a $2L$-periodic function defined by

$$h(\xi)=\begin{cases} f(\xi)\,, & 0\le\xi\le L\,; \\ -f(-\xi)\,, & -L\le\xi\le 0\,. \end{cases}$$

Set $u(x\,,t)=\frac{1}{2}[h(x-at)+h(x+at)]$. Assuming the appropriate differentiability conditions on h,

$$\frac{\partial u}{\partial x} = \frac{1}{2}[h'(x-at) + h'(x+at)]$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}[h''(x-at) + h''(x+at)].$$

Likewise,

$$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{2}[h''(x-at) + h''(x+at)].$$

It follows immediately that

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\,.$$

Let $t \geq 0$. Checking the first boundary condition,

$$u(0\,,t) = \frac{1}{2}[h(-at) + h(at)] = \frac{1}{2}[-h(at) + h(at)] = 0\,.$$

Checking the other boundary condition,

$$\begin{aligned} u(L\,,t) &= \frac{1}{2}[h(L-at) + h(L+at)] \\ &= \frac{1}{2}[-h(at-L) + h(at+L)]. \end{aligned}$$

Since h is $2L$-periodic, $h(at-L) = h(at-L+2L)$. Therefore $u(L\,,t) = 0$. Furthermore, for $0 \leq x \leq L$,

$$u(x\,,0) = \frac{1}{2}[h(x) + h(x)] = h(x) = f(x).$$

Hence $u(x\,,t)$ is a solution of the problem.

23. Assuming that we can differentiate term-by-term,

$$\frac{\partial u}{\partial t} = -\pi a \sum_{n=1}^{\infty} \frac{c_n\, n}{L} sin \frac{n\pi x}{L}\, sin \frac{n\pi a\, t}{L}$$

and

$$\frac{\partial u}{\partial x} = \pi \sum_{n=1}^{\infty} \frac{c_n\, n}{L} cos \frac{n\pi x}{L}\, cos \frac{n\pi a\, t}{L}\,.$$

Formally,

$$\left(\frac{\partial u}{\partial t}\right)^2 = \pi^2 a^2 \sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 sin^2\,\frac{n\pi x}{L}\,sin^2\,\frac{n\pi a\,t}{L} + \\ + \pi^2 a^2 \sum_{n\neq m}^{\infty} F_{nm}(x,t)$$

and

$$\left(\frac{\partial u}{\partial x}\right)^2 = \pi^2 \sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 cos^2\,\frac{n\pi x}{L}\,cos^2\,\frac{n\pi a\,t}{L} + \\ + \pi^2 \sum_{n\neq m}^{\infty} G_{nm}(x,t)\,,$$

in which $F_{nm}(x,t)$ and $G_{nm}(x,t)$ contain *products* of the natural modes and their derivatives. Based on the *orthogonality* of the natural modes,

$$\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx = \pi^2 a^2 \frac{L}{2}\sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 sin^2\,\frac{n\pi a\,t}{L}$$

and

$$\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx = \pi^2 \frac{L}{2}\sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 cos^2\,\frac{n\pi a\,t}{L}\,.$$

Recall that $a^2 = T/\rho$. It follows that

$$\int_0^L \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 + T\left(\frac{\partial u}{\partial x}\right)^2\right] dx = \pi^2 \frac{TL}{2}\sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 sin^2\,\frac{n\pi a\,t}{L} + \\ + \pi^2 \frac{TL}{2}\sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 cos^2\,\frac{n\pi a\,t}{L}\,.$$

Therefore,

$$\int_0^L \left[\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial u}{\partial x}\right)^2\right] dx = \pi^2 \frac{T}{4L}\sum_{n=1}^{\infty} n^2 c_n^2\,.$$