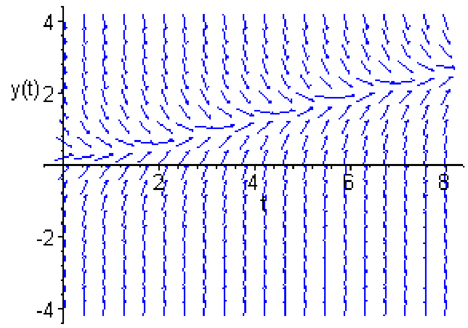
Chapter Two

Section 2.1

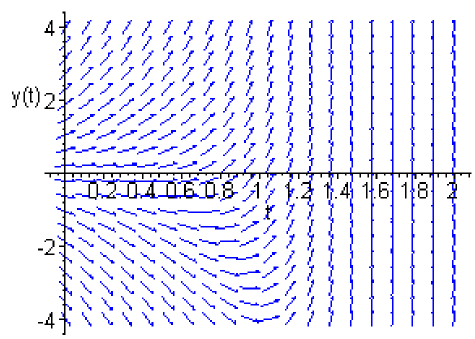
$1(a)$.



(b). Based on the direction field, all solutions seem to converge to a specific increasing function.

(c). The integrating factor is $\mu(t) = e^{3t}$, and hence $y(t) = t/3 - 1/9 + e^{-2t} + c\, e^{-3t}$. It follows that all solutions converge to the function $y_1(t) = t/3 - 1/9$.

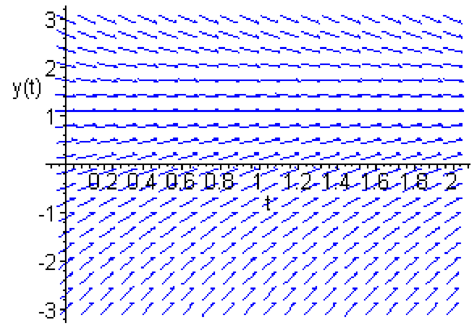
$2(a)$.



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = e^{-2t}$, and hence $y(t) = t^3 e^{2t}/3 + c\, e^{2t}$. It is evident that all solutions increase at an exponential rate.

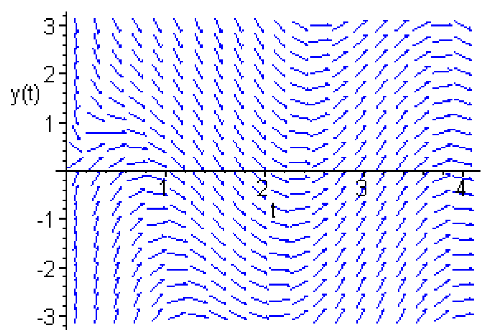
$3(a)$



(b). All solutions seem to converge to the function $y_0(t) = 1$.

(c). The integrating factor is $\mu(t) = e^{2t}$, and hence $y(t) = t^2 e^{-t}/2 + 1 + c\,e^{-t}$. It is clear that all solutions converge to the specific solution $y_0(t) = 1$.

$4(a)$.



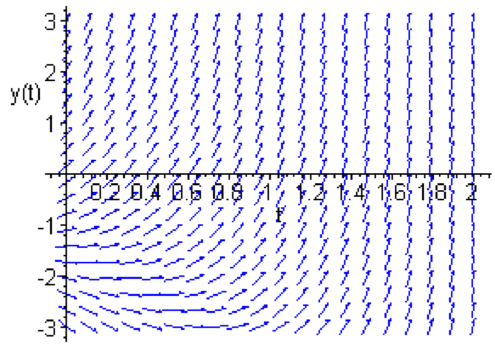
(b). Based on the direction field, the solutions eventually become oscillatory.

(c). The integrating factor is $\mu(t) = t$, and hence the general solution is

$$y(t) = \frac{3cos(2t)}{4t} + \frac{3}{2}sin(2t) + \frac{c}{t}$$

in which c is an arbitrary constant. As t becomes large, all solutions converge to the function $y_1(t) = 3sin(2t)/2$.

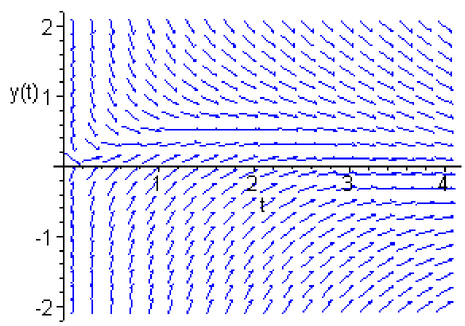
$5(a)$.



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = exp(-\int 2dt) = e^{-2t}$. The differential equation can
be written as $e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$, that is, $(e^{-2t}y)' = 3e^{-t}$. Integration of both sides of the equation results in the general solution $y(t) = -3e^{t} + c\,e^{2t}$. It follows that all solutions will increase exponentially.

$6(a)$



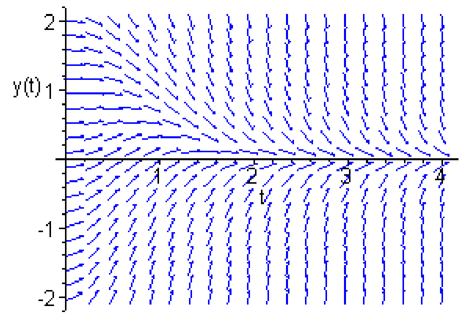
(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). The integrating factor is $\mu(t) = t^2$, and hence the general solution is

$$y(t) = -\frac{cos(t)}{t} + \frac{sin(2t)}{t^2} + \frac{c}{t^2}$$

in which c is an arbitrary constant. As t becomes large, all solutions converge to the function $y_0(t) = 0$.

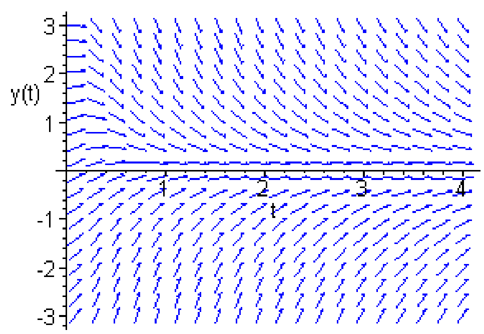
$7(a)$.



(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). The integrating factor is $\mu(t) = exp(t^2)$, and hence $y(t) = t^2 e^{-t^2} + c\, e^{-t^2}$. It is clear that all solutions converge to the function $y_0(t) = 0$.

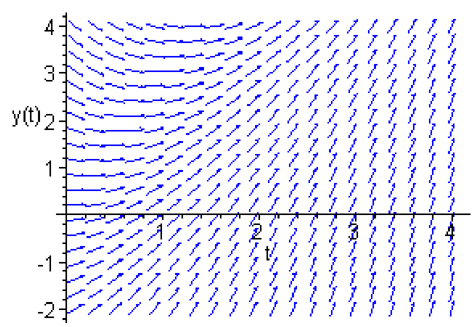
$8(a)$



(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). Since $\mu(t) = (1+t^2)^2$, the general solution is $y(t) = [tan^{-1}(t) + C]/(1+t^2)^2$. It follows that all solutions converge to the function $y_0(t) = 0$.

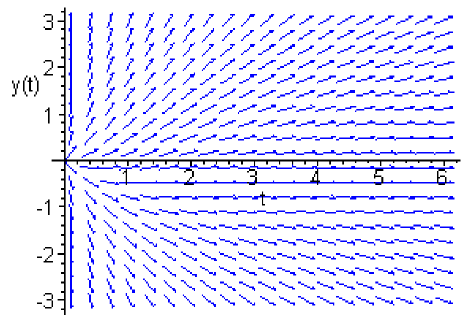
$9(a)$.



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = exp\big(\int \frac{1}{2}dt\big) = e^{t/2}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t\,e^{t/2}/2$, that is, $(e^{t/2}\,y/2)' = 3t\,e^{t/2}/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t - 6 + c\,e^{-t/2}$. All solutions approach the specific solution $y_0(t) = 3t - 6$.

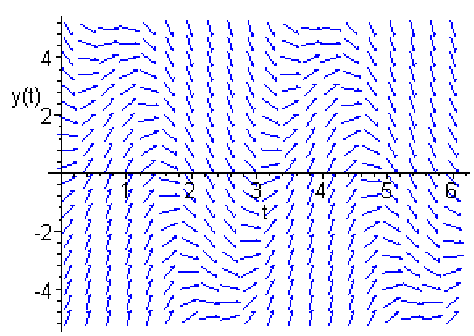
$10(a)$.



(b). For $y > 0$, the slopes are *all* positive, and hence the corresponding solutions increase
without bound. For $y < 0$, almost all solutions have negative slopes, and hence solutions tend to decrease without bound.

(c). First divide both sides of the equation by t. From the resulting *standard form*, the integrating factor is $\mu(t) = exp\big(-\int \frac{1}{t}dt\big) = 1/t$. The differential equation can be written as $y'/t - y/t^2 = t\,e^{-t}$, that is, $(\,y/t)' = t\,e^{-t}$. Integration leads to the general solution $y(t) = -te^{-t} + c\,t$. For $c \neq 0$, solutions *diverge*, as implied by the direction field. For the case $c = 0$, the specific solution is $y(t) = -te^{-t}$, which evidently approaches *zero* as $t \to \infty$.

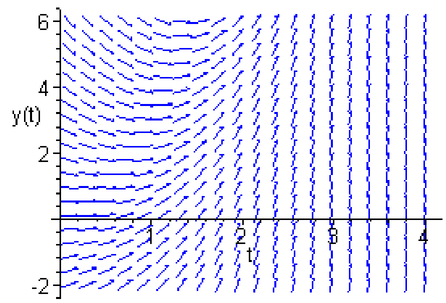
$11(a)$.



(b). The solutions appear to be oscillatory.

(c). The integrating factor is $\mu(t) = e^t$, and hence $y(t) = sin(2t) - 2\,cos(2t) + c\,e^{-t}$. It is evident that all solutions converge to the specific solution $y_0(t) = sin(2t) - 2$ $cos(2t)$.

$12(a)$.



(b). All solutions *eventually* have positive slopes, and hence increase without bound.

(c). The integrating factor is $\mu(t) = e^{2t}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t^2/2$, that is, $\left(e^{t/2}\,y/2\right)' = 3t^2/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t^2 - 12t + 24 + c\,e^{-t/2}$. It follows that all solutions converge to the specific solution $y_0(t) = 3t^2 - 12t + 24$.

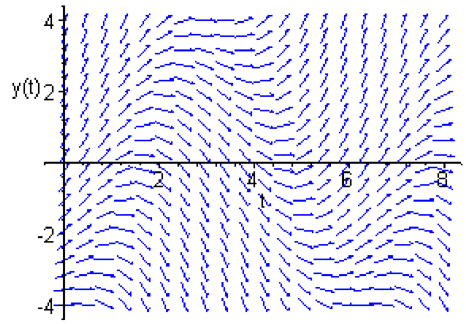
14. The integrating factor is $\mu(t) = e^{2t}$. After multiplying both sides by $\mu(t)$, the equation can be written as $\left(e^{2t}\,y\right)' = t$. Integrating both sides of the equation results in the general solution $y(t) = t^2e^{-2t}/2 + c\,e^{-2t}$. Invoking the specified condition, we require that $e^{-2}/2 + c\,e^{-2} = 0$. Hence $c = -1/2$, and the solution to the initial value problem is $y(t) = (t^2 - 1)e^{-2t}/2$.

16. The integrating factor is $\mu(t) = exp\left(\int \frac{2}{t}dt\right) = t^2$. Multiplying both sides by $\mu(t)$, the equation can be written as $(t^2\,y)' = cos(t)$. Integrating both sides of the equation results in the general solution $y(t) = sin(t)/t^2 + c\,t^{-2}$. Substituting $t = \pi$ and setting the value equal to *zero* gives $c = 0$. Hence the specific solution is $y(t) = sin(t)/t^2$.

17. The integrating factor is $\mu(t) = e^{-2t}$, and the differential equation can be written as $\left(e^{-2t}\,y\right)' = 1$. Integrating, we obtain $e^{-2t}\,y(t) = t + c$. Invoking the specified initial condition results in the solution $y(t) = (t+2)e^{2t}$.

19. After writing the equation in $standard\ form$, we find that the integrating factor is $\mu(t) = exp\left(\int \frac{4}{t}dt\right) = t^4$. Multiplying both sides by $\mu(t)$, the equation can be written as $\left(t^4\,y\right)' = t\,e^{-t}$. Integrating both sides results in $t^4y(t) = -(t+1)e^{-t} + c$. Letting $t = -1$ and setting the value equal to *zero* gives $c = 0$. Hence the specific solution of the initial value problem is $y(t) = -\left(t^{-3} + t^{-4}\right)e^{-t}$.

$21(a)$.

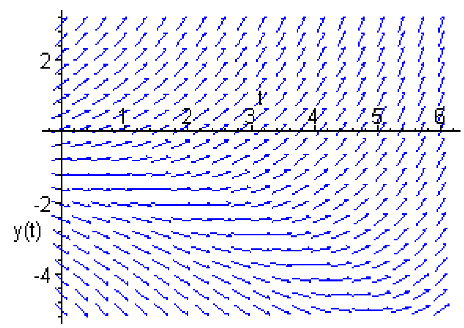


The solutions appear to diverge from an *apparent* oscillatory solution. From the direction
field, the critical value of the initial condition seems to be $a_0 = -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, solutions decrease without bound.

(b). The integrating factor is $\mu(t) = e^{-t/2}$. The general solution of the differential equation is $y(t) = (8sin(t) - 4cos(t))/5 + c\, e^{t/2}$. The solution is sinusoidal as long as $c = 0$. The *initial value* of this sinusoidal solution is
$a_0 = (8sin(0) - 4cos(0))/5 = -4/5$.

(c). See part (b).

$22(a)$.



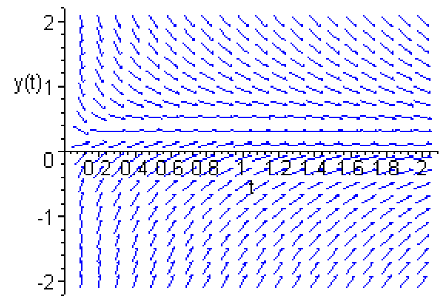
All solutions appear to *eventually* increase without bound. The solutions *initially* increase
or decrease, depending on the initial value a. The critical value seems to be $a_0 = -1$.

(b). The integrating factor is $\mu(t) = e^{-t/2}$, and the general solution of the differential equation is $y(t) = -3e^{t/3} + c\, e^{t/2}$. Invoking the initial condition $y(0) = a$, the solution
may also be expressed as $y(t) = -3e^{t/3} + (a+3)\, e^{t/2}$. Differentiating, follows that $y'(0) = -1 + (a+3)/2 = (a+1)/2$. The critical value is evidently $a_0 = -1$.

(c). For $a_0 = -1$, the solution is $y(t) = -3e^{t/3} + 2\,e^{t/2}$, which (for large t) is dominated by the term containing $e^{t/2}$.

is $y(t) = (8sin(t) - 4cos(t))/5 + c\,e^{t/2}$.

$23(a)$.

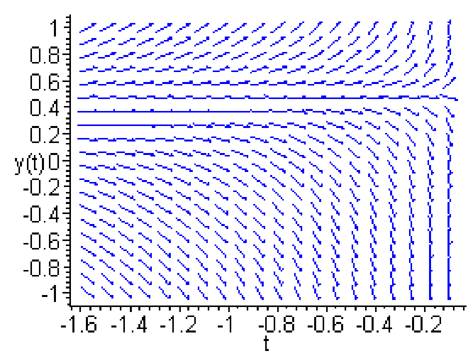


As $t \to 0$, solutions increase without bound if $y(1) = a > .4$, and solutions decrease without bound if $y(1) = a < .4$.

(b). The integrating factor is $\mu(t) = exp\left(\int \frac{t+1}{t}dt\right) = t\,e^t$. The general solution of the differential equation is $y(t) = t\,e^{-t} + c\,e^{-t}/t$. Invoking the specified value $y(1) = a$, we have $1 + c = a\,e$. That is, $c = a\,e - 1$. Hence the solution can also be expressed as $y(t) = t\,e^{-t} + (a\,e - 1)\,e^{-t}/t$. For *small* values of t, the second term is dominant. Setting $a\,e - 1 = 0$, critical value of the parameter is $a_0 = 1/e$.

(c). For $a > 1/e$, solutions increase without bound. For $a < 1/e$, solutions decrease without bound. When $a = 1/e$, the solution is $y(t) = t\,e^{-t}$, which approaches 0 as $t \to 0$
.

$24(a)$.



As $t \to 0$, solutions increase without bound if $y(1) = a > .4$, and solutions decrease without bound if $y(1) = a < .4$.

(b). Given the initial condition, $y(-\pi/2) = a$, the solution is $y(t) = (a\,\pi^2/4 - cos\,t)/t$.
Since $\lim_{t\to 0} cos\,t = 1$, solutions increase without bound if $a > 4/\pi^2$, and solutions decrease without bound if $a < 4/\pi^2$. Hence the critical value is
$a_0 = 4/\pi^2 = 0.452847....$

(c). For $a = 4/\pi^2$, the solution is $y(t) = (1 - cos\,t)/t$, and $\lim_{t\to 0} y(t) = 1/2$. Hence the solution is bounded.

25. The integrating factor is $\mu(t) = exp\left(\int \frac{1}{2} dt\right) = e^{t/2}$. Therefore general solution is $y(t) = [4cos(t) + 8sin(t)]/5 + c\,e^{-t/2}$. Invoking the initial condition, the specific solution is $y(t) = [4cos(t) + 8sin(t) - 9\,e^{t/2}]/5$. Differentiating, it follows that

$$y'(t) = \left[-4sin(t) + 8cos(t) + 4.5\,e^{-t/2}\right]/5$$
$$y''(t) = \left[-4cos(t) - 8sin(t) - 2.25\,e^{-t/2}\right]/5$$

Setting $y'(t) = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the *first* stationary point. Since $y''(t_1) < 0$, the first stationary point in a local *maximum*. The coordinates of the point are $(1.3643\,, .82008)$.

26. The integrating factor is $\mu(t) = exp\left(\int \frac{2}{3} dt\right) = e^{2t/3}$, and the differential equation can
be written as $(e^{2t/3}\,y)' = e^{2t/3} - t\,e^{2t/3}/2$. The general solution is $y(t) = (21 - 6t)/8 +$ $+ c\,e^{-2t/3}$. Imposing the initial condition, we have $y(t) = (21 - 6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Since the solution is smooth, the desired intersection will be a point of tangency. Taking the derivative, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = \frac{3}{2} ln[(21 - 8y_0)/9]$. Substituting into the solution, the respective *value* at the stationary point is $y(t_1) = \frac{3}{2} + \frac{9}{4} ln\,3 - \frac{9}{8} ln(21 - 8y_0)$. Setting this result equal to *zero*, we obtain the required initial value $y_0 = (21 - 9\,e^{4/3})/8 = -1.643$.

27. The integrating factor is $\mu(t) = e^{t/4}$, and the differential equation can be written as $(e^{t/4}\,y)' = 3\,e^{t/4} + 2\,e^{t/4} cos(2t)$. The general solution is

$$y(t) = 12 + [8cos(2t) + 64sin(2t)]/65 + c\;e^{-t/4}.$$

Invoking the initial condition, $y(0) = 0$, the specific solution is

$$y(t) = 12 + \left[8cos(2t) + 64sin(2t) - 788\,e^{-t/4}\right]/65\,.$$

As $t \to \infty$, the exponential term will decay, and the solution will oscillate about an *average*
value of 12, with an *amplitude* of $8/\sqrt{65}$.

29. The integrating factor is $\mu(t) = e^{-3t/2}$, and the differential equation can be written as $(e^{-3t/2}\, y)' = 3t\, e^{-3t/2} + 2\, e^{-t/2}$. The general solution is $y(t) = -2t - 4/3 - 4\, e^t + {}$ $+ c\, e^{3t/2}$. Imposing the initial condition, $y(t) = -2t - 4/3 - 4\, e^t + (y_0 + 16/3)\, e^{3t/2}$. As $t \to \infty$, the term containing $e^{3t/2}$ will *dominate* the solution. Its *sign* will determine the divergence properties. Hence the critical value of the initial condition is
$y_0 = -16/3$.
The corresponding solution, $y(t) = -2t - 4/3 - 4\, e^t$, will also decrease without bound.

Note on Problems 31-34 :

Let $g(t)$ be *given*, and consider the function $y(t) = y_1(t) + g(t)$, in which $y_1(t) \to \infty$ as $t \to \infty$. Differentiating, $y'(t) = y_1'(t) + g'(t)$. Letting a be a *constant*, it follows that $y'(t) + ay(t) = y_1'(t) + ay_1(t) + g'(t) + ag(t)$. Note that the hypothesis on the function $y_1(t)$ will be satisfied, if $y_1'(t) + ay_1(t) = 0$. That is, $y_1(t) = c\, e^{-at}$. Hence $y(t) = c\, e^{-at} + g(t)$, which is a solution of the equation $y' + ay = g'(t) + ag(t)$. For convenience, choose $a = 1$.

31. Here $g(t) = 3$, and we consider the linear equation $y' + y = 3$. The integrating factor is $\mu(t) = e^t$, and the differential equation can be written as $(e^t\, y)' = 3e^t$. The general solution is $y(t) = 3 + c\, e^{-t}$.

33. $g(t) = 3 - t$. Consider the linear equation $y' + y = -1 + 3 - t$ .The integrating factor is $\mu(t) = e^t$, and the differential equation can be written as $(e^t\, y)' = (2 - t)e^t$. The general solution is $y(t) = 3 - t + c\, e^{-t}$.

34. $g(t) = 4 - t^2$. Consider the linear equation $y' + y = 4 - 2t - t^2$ .The integrating factor is $\mu(t) = e^t$, and the equation can be written as $(e^t\, y)' = (4 - 2t - t^2)e^t$. The general solution is $y(t) = 4 - t^2 + c\, e^{-t}$.