**Section 5.2**

1. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n]x^n = 0\,.$$

Equating all the coefficients to *zero*,

$$(n+2)(n+1)a_{n+2} - a_n = 0\,, \qquad n = 0, 1, 2, \cdots.$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}\,, \qquad n = 0, 1, 2, \cdots.$$

The subscripts differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \cdots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots = \frac{a_1}{(2k+1)!}\,.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\,.$$

The linearly independent solutions are

$$y_1 = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) = a_0 cosh\, x$$

$$y_2 = a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) = a_1 sinh\, x\,.$$

4. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + k^2 x^2 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Rewriting the *second* summation,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + \sum_{n=2}^{\infty} k^2 a_{n-2}\, x^n = 0\,,$$

that is,

$$2a_2 + 3\cdot 2\, a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + k^2 a_{n-2}\right] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we have $a_2 = 0$, $a_3 = 0$, and

$$(n+2)(n+1)a_{n+2} + k^2 a_{n-2} = 0\,, \quad \text{for } n = 2, 3, 4, \cdots.$$

The recurrence relation can be written as

$$a_{n+2} = -\frac{k^2 a_{n-2}}{(n+2)(n+1)}\,, \quad n = 2, 3, 4, \cdots.$$

The indices differ by *four*, so $a_4$, $a_8$, $a_{12}$, $\cdots$ are defined by

$$a_4 = -\frac{k^2 a_0}{4\cdot 3}\,,\ a_8 = -\frac{k^2 a_4}{8\cdot 7}\,, a_{12} = -\frac{k^2 a_8}{12\cdot 11}\,, \cdots.$$

Similarly, $a_5$, $a_9$, $a_{13}$, $\cdots$ are defined by

$$a_5 = -\frac{k^2 a_1}{5\cdot 4}\,,\ a_9 = -\frac{k^2 a_5}{9\cdot 8}\,, a_{13} = -\frac{k^2 a_9}{13\cdot 12}\,, \cdots.$$

The remaining coefficients are *zero*. Therefore the general solution is

$$y = a_0\left[1 - \frac{k^2}{4\cdot 3}x^4 + \frac{k^4}{8\cdot 7\cdot 4\cdot 3}x^8 - \frac{k^6}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{12} + \cdots\right] + \\ + a_1\left[x - \frac{k^2}{5\cdot 4}x^5 + \frac{k^4}{9\cdot 8\cdot 5\cdot 4}x^9 - \frac{k^6}{13\cdot 12\cdot 9\cdot 8\cdot 4\cdot 4}x^{13} + \cdots\right].$$

Note that for the *even* coefficients,

$$a_{4m} = -\frac{k^2 a_{4m-4}}{(4m-1)4m}, \quad m = 1, 2, 3, \cdots$$

and for the *odd* coefficients,

$$a_{4m+1} = -\frac{k^2 a_{4m-3}}{4m(4m+1)}, \quad m = 1, 2, 3, \cdots.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(k^2 x^4\right)^{m+1}}{3 \cdot 4 \cdot 7 \cdot 8 \cdots (4m+3)(4m+4)}$$

$$y_2(x) = x\left[1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(k^2 x^4\right)^{m+1}}{4 \cdot 5 \cdot 8 \cdot 9 \cdots (4m+4)(4m+5)}\right].$$

6. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(2 + x^2\right)\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x + \sum_{n=2}^{\infty} \left[2(n+2)(n+1)a_{n+2} + n(n-1)a_n - n\, a_n + 4a_n\right]x^n = 0\,.$$

Equating the coefficients to *zero*, we find that $a_2 = -a_0$, $a_3 = -a_1/4$, and

$$a_{n+2} = -\frac{n^2-2n+4}{2(n+2)(n+1)}\,a_n\,,\quad n = 0,1,2,\cdots.$$

The indices differ by *two*, so for $k = 0,1,2,\cdots$

$$a_{2k+2} = -\frac{(2k)^2-4k+4}{2(2k+2)(2k+1)}\,a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2-4k+2}{2(2k+3)(2k+2)}\,a_{2k+1}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \cdots$$

$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \cdots.$$

7. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty}a_nx^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_nx^n.$$

We then obtain

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + n\,a_n + 2a_n]x^n = 0\,.$$

It follows that $a_2 = -a_0$ and $a_{n+2} = -a_n/(n+1)$, $n = 0, 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = -\frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-3)(2k-1)} = \cdots = \frac{(-1)^k a_0}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$$

and

$$a_{2k+1} = -\frac{a_{2k-1}}{2k} = \frac{a_{2k-3}}{(2k-2)2k} = \cdots = \frac{(-1)^k a_1}{2 \cdot 4 \cdot 6 \cdots (2k)}\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} - \frac{x^6}{1 \cdot 3 \cdot 5} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)}\,.$$

9. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(1 + x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 4x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4n\, a_n + 6a_n\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain $a_2 = -3a_0$, $a_3 = -a_1/3$, and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}\, a_n\,, \quad n = 0, 1, 2, \cdots.$$

Observe that for $n = 2$ and $n = 3$, we obtain $a_4 = a_5 = 0$. Since the indices differ by *two*, we also have $a_n = 0$ for $n \geq 4$. Therefore the general solution is a polynomial

$$y = a_0 + a_1 x - 3a_0 x^2 - a_1 x^3/3\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3\,.$$

10. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(4 - x^2\right)\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

It follows that

$$2a_0 + 8a_2 + (2a_1 + 24a_3)x + \sum_{n=2}^{\infty} \left[4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n\right]x^n = 0\,.$$

We obtain $a_2 = -a_0/4$, $a_3 = -a_1/12$ and

$$4(n+2)a_{n+2} = (n-2)a_n\,, \quad n = 0, 1, 2, \cdots.$$

Note that for $n = 2$, $a_4 = 0$. Since the indices differ by *two*, we also have $a_{2k} = 0$ for $k = 2, 3, \cdots$. On the other hand, for $k = 1, 2, \cdots$,

$$a_{2k+1} = \frac{(2k-3)a_{2k-1}}{4(2k+1)} = \frac{(2k-5)(2k-3)a_{2k-3}}{4^2(2k-1)(2k+1)} = \cdots = \frac{-a_1}{4^k(2k-1)(2k+1)}\,.$$

Therefore the general solution is

$$y = a_0 + a_1 x - a_0 \frac{x^2}{4} - a_1 \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n (2n-1)(2n+1)} \,.$$

Hence the linearly independent solutions are $y_1(x) = 1 - x^2/4$ and

$$y_2(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \cdots = x - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n (2n-1)(2n+1)} \,.$$

11. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$\left(3 - x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 3x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

It follows that

$$6a_2 - a_0 + (-4a_1 + 18a_3)x + \sum_{n=2}^{\infty} \left[3(n+2)(n+1)a_{n+2} - n(n-1)a_n - 3n\, a_n - a_n\right] x^n = 0.$$

We obtain $a_2 = a_0/6$, $2a_3 = a_1/9$, and

$$3(n+2)a_{n+2} = (n+1)a_n \,, \quad n = 0, 1, 2, \cdots .$$

The indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{(2k-1)a_{2k-2}}{3(2k)} = \frac{(2k-3)(2k-1)a_{2k-4}}{3^2(2k-2)(2k)} = \cdots = \frac{3 \cdot 5 \cdots (2k-1)\, a_0}{3^k \cdot 2 \cdot 4 \cdots (2k)}$$

and

$$a_{2k+1} = \frac{(2k)a_{2k-1}}{3(2k+1)} = \frac{(2k-2)(2k)a_{2k-3}}{3^2(2k-1)(2k+1)} = \cdots = \frac{2\cdot 4\cdot 6\cdots(2k)\,a_1}{3^k\cdot 3\cdot 5\cdots(2k+1)}\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \cdots = 1 + \sum_{n=1}^{\infty}\frac{3\cdot 5\cdots(2n-1)\,x^{2n}}{3^n\cdot 2\cdot 4\cdots(2n)}$$

$$y_2(x) = x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \cdots = x + \sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2n)\,x^{2n+1}}{3^n\cdot 3\cdot 5\cdots(2n+1)}\,.$$

12. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_nx^n = 0\,.$$

Before proceeding, write

$$x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n = \sum_{n=1}^{\infty}(n+1)n\,a_{n+1}x^n$$

and

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_nx^n.$$

It follows that

$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + n\,a_n - a_n\right]x^n = 0.$$

We obtain $a_2 = a_0/2$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + (n-1)a_n = 0$$

for $n = 0, 1, 2, \cdots$. Writing out the individual equations,

$$\begin{aligned} 3\cdot 2\,a_3 - 2\cdot 1\,a_2 &= 0 \\ 4\cdot 3\,a_4 - 3\cdot 2\,a_3 + a_2 &= 0 \\ 5\cdot 4\,a_5 - 4\cdot 3\,a_4 + 2\,a_3 &= 0 \\ 6\cdot 5\,a_6 - 5\cdot 4\,a_5 + 3\,a_4 &= 0 \\ &\vdots \end{aligned}$$

The coefficients can be calculated successively as $a_3 = a_0/(2\cdot 3)$, $a_4 = a_3/2 - a_2/12$ $= a_0/24$ , $a_5 = 3a_4/5 - a_3/10 = a_0/120$ , $\cdots$ . We can now see that for $n \geq 2$ , $a_n$ is proportional to $a_0$. In fact, for $n \geq 2$ , $a_n = a_0/(n!)$ . Therefore the general solution is

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \frac{a_0 x^4}{4!} + \cdots .$$

Hence the linearly independent solutions are $y_2(x) = x$ and

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} \,.$$

13. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ . Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3\sum_{n=0}^{\infty} a_n x^n = 0 \,.$$

First write

$$x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1) a_{n+2} + n\, a_n + 3a_n] x^n = 0 \,.$$

It follows that $a_2 = -3a_0/4$ and

$$2(n+2)(n+1) a_{n+2} + (n+3) a_n = 0$$

for $n = 0, 1, 2, \cdots$. The indices differ by *two*, so for $k = 1, 2, \cdots$

$$\begin{aligned} a_{2k} &= -\frac{(2k+1)a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1)a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \cdots \\ &= \frac{(-1)^k 3 \cdot 5 \cdots (2k+1)}{2^k\,(2k)!}\, a_0 \,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k+2)a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2)a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \cdots \\ &= \frac{(-1)^k\, 4 \cdot 6 \cdots (2k)(2k+2)}{2^k\,(2k+1)!}\, a_1 \,. \end{aligned}$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 5 \cdots (2n+1)}{2^n\,(2n)!} x^{2n}$$

$$y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cdot 6 \cdots (2n+2)}{2^n\,(2n+1)!} x^{2n+1}.$$

$15(a)$. From Prob. 2, we have

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\, n!} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n\, n!\, x^{2n+1}}{(2n+1)!} \,.$$

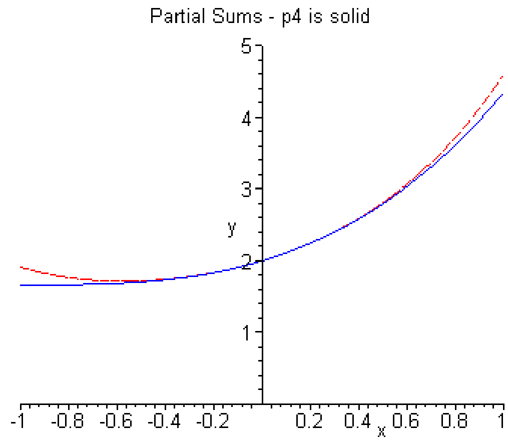
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 2\,y_1(x) + y_2(x)$. That is,

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \cdots.$$

The *four*- and *five*-term polynomial approximations are

$$\begin{aligned} p_4 &= 2 + x + x^2 + x^3/3 \\ p_5 &= 2 + x + x^2 + x^3/3 + x^4/4 \,. \end{aligned}$$

$(b)$.




$(c)$. The *four-term* approximation $p_4$ appears to be reasonably accurate (within $10\%$) on the interval $|x| < 0.7$.

$17(a)$. From Prob. 7, the linearly independent solutions are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

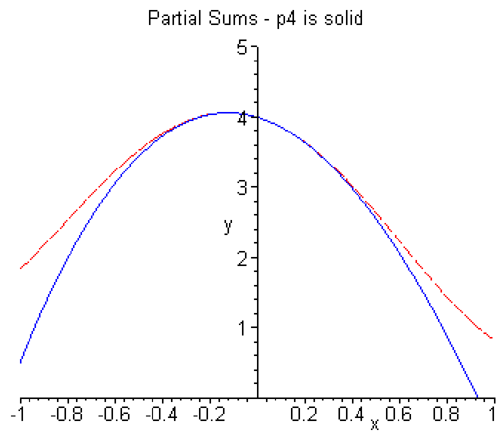
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 4\,y_1(x) - y_2(x)$. That is,

$$y(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 - \frac{1}{8}x^5 - \frac{4}{15}x^6 + \cdots.$$

The *four-* and *five*-term polynomial approximations are

$$p_4 = 4 - x - 4x^2 + \frac{1}{2}x^3$$
$$p_5 = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4.$$

$(b)$.




$(c)$. The *four-term* approximation $p_4$ appears to be reasonably accurate (within $10\%$) on the interval $|x| < 0.5$.

$18(a)$. From Prob. $12$, we have

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad y_2(x) = x\,.$$

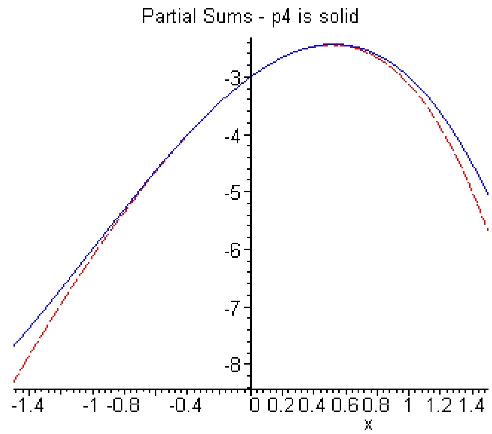
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = -3\,y_1(x) + 2\,y_2(x)$. That is,

$$y(x) = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 - \frac{1}{40}x^5 - \frac{1}{240}x^6 + \cdots.$$

The *four-* and *five*-term polynomial approximations are

$$p_4 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$p_5 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4\,.$$

$(b)$.




$(c)$. The *four-term* approximation $p_4$ appears to be reasonably accurate (within $10\%$) on the interval $|x| < 0.9$.

20. Two linearly independent solutions of *Airy's equation* (about $x_0 = 0$) are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)}.$$

Applying the *ratio test* to the terms of $y_1(x)$,

$$\lim_{n\to\infty} \frac{|2 \cdot 3 \cdots (3n-1)(3n)\, x^{3n+3}|}{|2 \cdot 3 \cdots (3n+2)(3n+3)\, x^{3n}|} = \lim_{n\to\infty} \frac{1}{(3n+1)(3n+2)(3n+3)} |x|^3 = 0\,.$$

Similarly, applying the *ratio test* to the terms of $y_2(x)$,

$$\lim_{n\to\infty} \frac{|3 \cdot 4 \cdots (3n)(3n+1)\, x^{3n+4}|}{|3 \cdot 4 \cdots (3n+3)(3n+4)\, x^{3n+1}|} = \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)(3n+4)} |x|^3 = 0\,.$$

Hence both series converge *absolutely* for all $x$.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \lambda\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$2a_2 + \lambda\, a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - 2n\, a_n + \lambda\, a_n]x^n = 0\,.$$

Setting the coefficients equal to *zero*, it follows that

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)}\, a_n$$

for $n = 0, 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$\begin{aligned} a_{2k} &= \frac{(4k-4-\lambda)a_{2k-2}}{(2k-1)2k} = \frac{(4k-8-\lambda)(4k-4-\lambda)a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \cdots \\ &= (-1)^k \frac{\lambda\cdots(\lambda-4k+8)(\lambda-4k+4)}{(2k)!}\, a_0\,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= \frac{(4k-2-\lambda)a_{2k-1}}{2k(2k+1)} = \frac{(4k-6-\lambda)(4k-2-\lambda)a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots \\ &= (-1)^k \frac{(\lambda-2)\cdots(\lambda-4k+6)(\lambda-4k+2)}{(2k+1)!}\, a_1. \end{aligned}$$

Hence the linearly independent solutions of the *Hermite equation* (about $x_0 = 0$) are

$$y_1(x) = 1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda-4)}{4!}x^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6\,!}x^6 + \cdots$$

$$y_2(x) = x - \frac{\lambda-2}{3\,!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5\,!}x^5 - \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7\,!}x^7 + \cdots.$$

$(b)$. Based on the recurrence relation

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)}\, a_n \,,$$

the series solution will *terminate* as long as $\lambda$ is a *nonnegative* even integer. If $\lambda = 2m$, then *one or the other* of the solutions in Part $(b)$ will contain at most $m/2+1$ terms. In particular, we obtain the polynomial solutions corresponding to $\lambda = 0, 2, 4, 6, 8, 10$ :

| | |
|---|---|
| $\lambda = 0$ | $y_1(x) = 1$ |
| $\lambda = 2$ | $y_2(x) = x$ |
| $\lambda = 4$ | $y_1(x) = 1 - 2x^2$ |
| $\lambda = 6$ | $y_2(x) = x - 2x^3/3$ |
| $\lambda = 8$ | $y_1(x) = 1 - 4x^2 + 4x^4/3$ |
| $\lambda = 10$ | $y_2(x) = x - 4x^3/3 + 4x^5/15$ |

$(c)$. Observe that if $\lambda = 2n$, and $a_0 = a_1 = 1$, then

$$a_{2k} = (-1)^k \frac{2n\cdots(2n-4k+8)(2n-4k+4)}{(2k)!}$$

and

$$a_{2k+1} = (-1)^k \frac{(2n-2)\cdots(2n-4k+6)(2n-4k+2)}{(2k+1)!}\,.$$

for $k = 1, 2, \cdots [n/2]$. It follows that the *coefficient* of $x^n$, in $y_1$ and $y_2$, is

$$a_n = \begin{cases} (-1)^k \frac{4^k\, k!}{(2k)!} & \text{for } n = 2k \\ (-1)^k \frac{4^k\, k!}{(2k+1)!} & \text{for } n = 2k+1 \end{cases}$$

Then by definition,

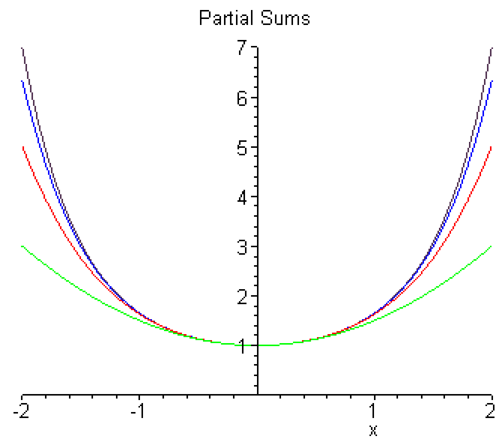
$$H_n(x) = \begin{cases} (-1)^k\, 2^n \frac{(2k)!}{4^k\, k!} y_1(x) = (-1)^k \frac{(2k)!}{k!} y_1(x) & \text{for } n = 2k \\ (-1)^k\, 2^n \frac{(2k+1)!}{4^k\, k!} y_2(x) = (-1)^k \frac{2\,(2k+1)!}{k!} y_2(x) & \text{for } n = 2k+1 \end{cases}$$

Therefore the first six *Hermite polynomials* are

$$\begin{aligned} &H_0(x) = 1 \\ &H_1(x) = 2x \\ &H_2(x) = 4x^2 - 2 \\ &H_3(x) = 8x^8 - 12x \\ &H_4(x) = 16x^4 - 48x^2 + 12 \\ &H_5(x) = 32x^5 - 160x^3 + 120x \end{aligned}$$

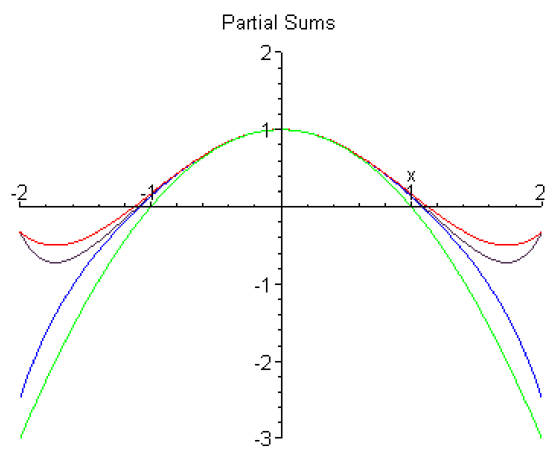
23. The series solution is given by

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{2^2\,2!}x^4 + \frac{1}{2^3 3!}x^6 + \frac{1}{2^4 4!}x^8 + \cdots.$$



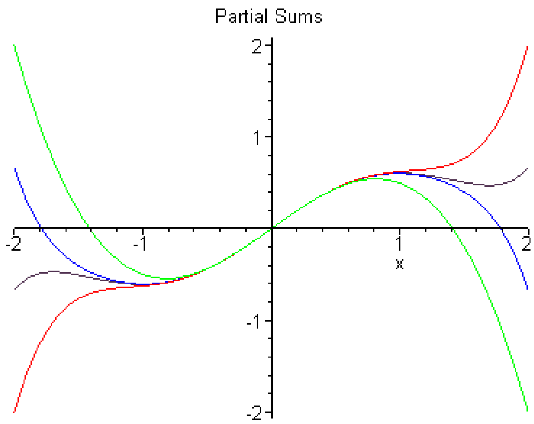

24. The series solution is given by

$$y(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \frac{x^8}{120} + \cdots.$$



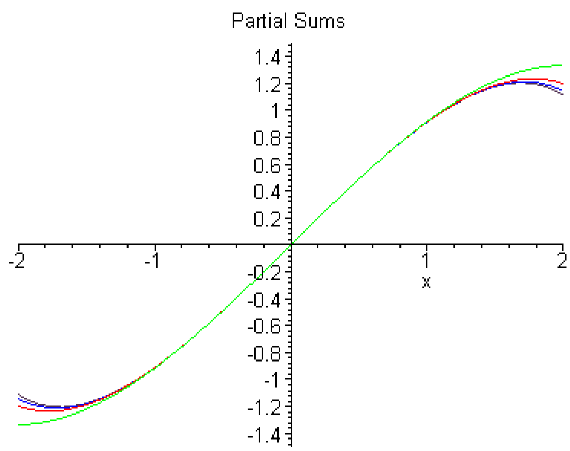

25. The series solution is given by

$$y(x) = x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \frac{x^9}{2\cdot 4\cdot 6\cdot 8} - \cdots.$$



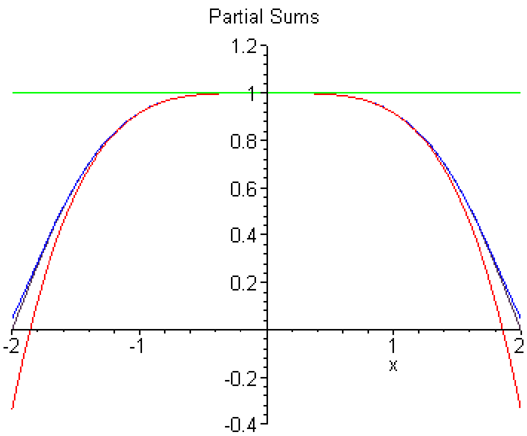

26. The series solution is given by

$$y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \frac{x^9}{16128} - \cdots.$$




27. The series solution is given by

$$y(x) = 1 - \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^{12}}{88704} + \cdots.$$




28. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

After appropriately shifting the indices, it follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - (n+1)n\, a_{n+1} + n\, a_n - 2\, a_n] x^n = 0.$$

We find that $a_2 = a_0$ and

$$(n+2)(n+1) a_{n+2} - (n+1)n\, a_{n+1} + (n-2) a_n = 0$$

for $n = 1, 2, \cdots$. Writing out the individual equations,

$$\begin{aligned} 3 \cdot 2\, a_3 - 2 \cdot 1\, a_2 - a_1 &= 0 \\ 4 \cdot 3\, a_4 - 3 \cdot 2\, a_3 &= 0 \\ 5 \cdot 4\, a_5 - 4 \cdot 3\, a_4 + a_3 &= 0 \\ 6 \cdot 5\, a_6 - 5 \cdot 4\, a_5 + 2\, a_4 &= 0 \\ &\vdots \end{aligned}$$

Since $a_0 = 0$ and $a_1 = 1$, the remaining coefficients satisfy the equations

$$\begin{aligned} 3 \cdot 2\, a_3 - 1 &= 0 \\ 4 \cdot 3\, a_4 - 3 \cdot 2\, a_3 &= 0 \\ 5 \cdot 4\, a_5 - 4 \cdot 3\, a_4 + a_3 &= 0 \\ 6 \cdot 5\, a_6 - 5 \cdot 4\, a_5 + 2\, a_4 &= 0 \\ &\vdots \end{aligned}$$

That is, $a_3 = 1/6\,, a_4 = 1/12\,, a_5 = 1/24\,, a_6 = 1/45\,, \cdots$. Hence the series solution of the initial value problem is

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \cdots .$$