Section 2.8

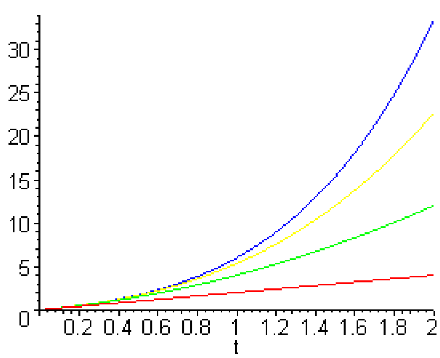
2. Let $z = y - 3$ and $\tau = t + 1$. It follows that $dz/d\tau = (dz/dt)(dt/d\tau) = dz/dt$. Furthermore, $dz/dt = dy/dt = 1 - y^3$. Hence $dz/d\tau = 1 - (z+3)^3$. The new initial condition is $z(\tau = 0) = 0$.

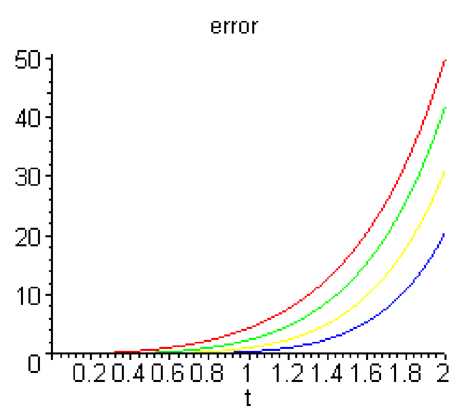
3. The approximating functions are defined recursively by $\phi_{n+1}(t) = \int_0^t 2[\phi_n(s) + 1]ds$. Setting $\phi_0(t) = 0$, $\phi_1(t) = 2t$. Continuing, $\phi_2(t) = 2t^2 + 2t$, $\phi_3(t) = \frac{4}{3}t^3 + 2t^2 + 2t$, $\phi_4(t) = \frac{2}{3}t^4 + \frac{4}{3}t^3 + 2t^2 + 2t$, $\cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)] \\ &= 2t + \sum_{k=2}^{\infty} \frac{a_k}{k\,!} t^k .\end{aligned}$$

Comparing coefficients, $a_3/3! = 4/3$, $a_4/4! = 2/3$, $\cdots$. It follows that $a_3 = 8$, $a_4 = 16$,
and so on. We find that in general, that $a_k = 2^k$. Hence

$$\begin{aligned}\phi(t) &= \sum_{k=1}^{\infty} \frac{2^k}{k\,!} t^k \\ &= e^{2t} - 1 .\end{aligned}$$





5. The approximating functions are defined recursively by

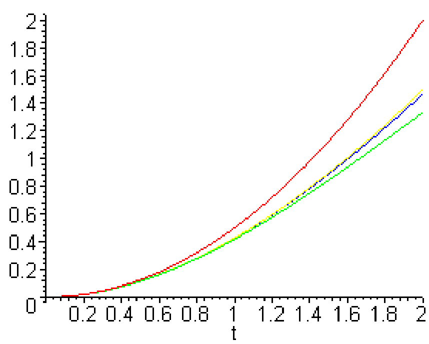
$$\phi_{n+1}(t) = \int_0^t [-\phi_n(s)/2 + s]ds .$$

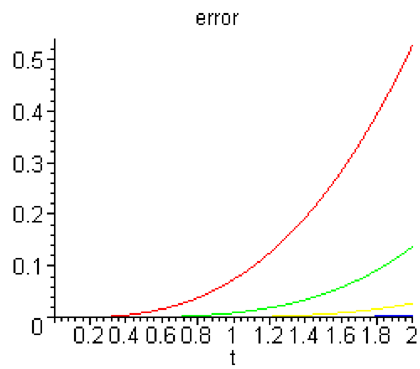
Setting $\phi_0(t) = 0$, $\phi_1(t) = t^2/2$. Continuing, $\phi_2(t) = t^2/2 - t^3/12$, $\phi_3(t) = t^2/2 - t^3/12 + t^4/96$, $\phi_4(t) = t^2/2 - t^3/12 + t^4/96 - t^5/960$, $\cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)] \\ &= t^2/2 + \sum_{k=3}^{\infty}\frac{a_k}{k\,!}t^k\,.\end{aligned}$$

Comparing coefficients, $a_3/3! = -1/12\,, a_4/4! = 1/96\,, a_5/5! = -1/960\,, \cdots$. We find that $a_3 = -1/2\,, a_4 = 1/4, a_5 = -1/8\,, \cdots$. In general, $a_k = 2^{-k+1}$. Hence

$$\begin{aligned}\phi(t) &= \sum_{k=2}^{\infty}\frac{2^{-k+2}}{k\,!}(-t)^k \\ &= 4\,e^{-t/2} + 2t - 4\,.\end{aligned}$$





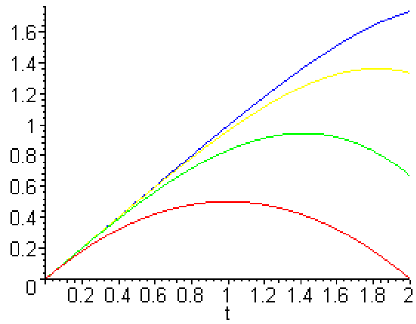
6. The approximating functions are defined recursively by

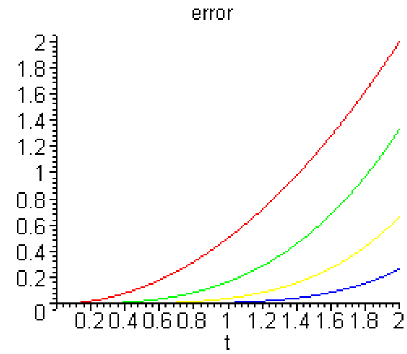
$$\phi_{n+1}(t) = \int_0^t [\phi_n(s) + 1 - s]ds\,.$$

Setting $\phi_0(t) = 0\,,\ \phi_1(t) = t - t^2/2\,, \phi_2(t) = t - t^3/6\,, \phi_3(t) = t - t^4/24\,, \phi_4(t) = = t - t^5/120\,, \cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)] \\ &= t - t^2/2 + \left[t^2/2 - t^3/6\right] + \left[t^3/6 - t^4/24\right] + \cdots \\ &= t + 0 + 0 + \cdots\,.\end{aligned}$$

Note that the terms can be rearranged, as long as the series converges *uniformly*.





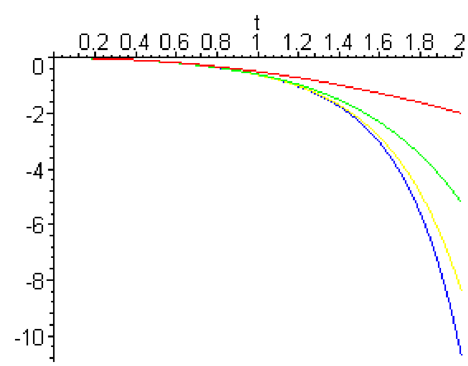
$8(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2\phi_n(s) - s\right] ds\,.$$

Set $\phi_0(t) = 0$. The iterates are given by $\phi_1(t) = -t^2/2$, $\phi_2(t) = -t^2/2 - t^5/10$, $\phi_3(t) = -t^2/2 - t^5/10 - t^8/80$, $\phi_4(t) = -t^2/2 - t^5/10 - t^8/80 - t^{11}/880, \cdots$. Upon inspection, it becomes apparent that

$$\begin{aligned}\phi_n(t) &= -t^2\left[\frac{1}{2} + \frac{t^3}{2\cdot 5} + \frac{t^6}{2\cdot 5\cdot 8} + \cdots + \frac{(t^3)^{n-1}}{2\cdot 5\cdot 8\cdots[2+3(n-1)]}\right] \\ &= -t^2\sum_{k=1}^{n}\frac{(t^3)^{k-1}}{2\cdot 5\cdot 8\cdots[2+3(k-1)]}\,.\end{aligned}$$

(b).



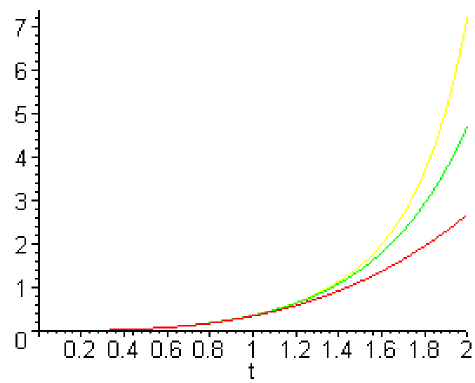
The iterates appear to be converging.

$9(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2 + \phi_n^2(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t^3/3$, $\phi_2(t) = t^3/3 + t^7/63$, $\phi_3(t) = t^3/3 + t^7/63 + 2t^{11}/2079 + t^{15}/59535$.

(b).



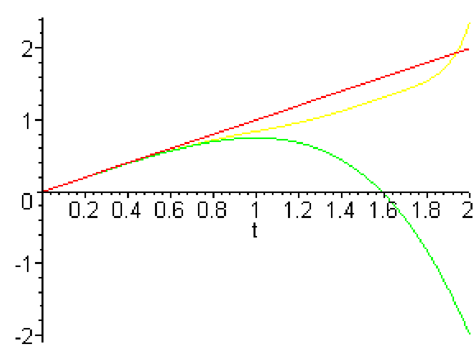
The iterates appear to be converging.

$10(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[1 - \phi_n^3(s)\right] ds \, .$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t$, $\phi_2(t) = t - t^4/4$, $\phi_3(t) = t - t^4/4 + 3t^7/28 - 3t^{10}/160 + t^{13}/833$.

(b).



The approximations appear to be diverging.

$12(a)$. The approximating functions are defined recursively by

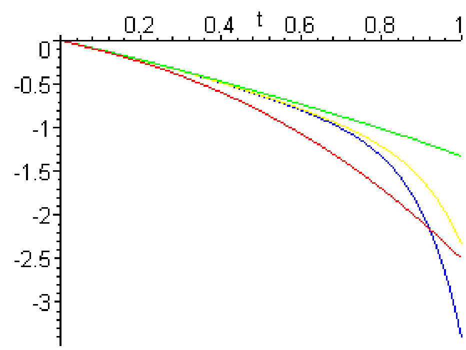
$$\phi_{n+1}(t) = \int_0^t \left[\frac{3s^2 + 4s + 2}{2(\phi_n(s) - 1)}\right] ds \, .$$

Note that $1/(2y - 2) = -\frac{1}{2} \sum_{k=0}^{6} y^k + O(y^7)$. For computational purposes, replace the above iteration formula by

$$\phi_{n+1}(t) = -\frac{1}{2}\int_0^t \left[\left(3s^2 + 4s + 2\right) \sum_{k=0}^{6} \phi_n^k(s) \right] ds \,.$$

Set $\phi_0(t) = 0$. The first four approximations are given by $\phi_1(t) = -t - t^2 - t^3/2$,
$\phi_2(t) = -t - t^2/2 + t^3/6 + t^4/4 - t^5/5 - t^6/24 + \cdots$,
$\phi_3(t) = -t - t^2/2 + t^4/12 - 3t^5/20 + 4t^6/45 + \cdots$,
$\phi_4(t) = -t - t^2/2 + t^4/8 - 7t^5/60 + t^6/15 + \cdots$

(b).



The approximations appear to be converging to the exact solution,

$$\phi(t) = 1 - \sqrt{1 + 2t + 2t^2 + t^3} \,.$$

13. Note that $\phi_n(0) = 0$ and $\phi_n(1) = 1, \forall\, n \geq 1$. Let $a \in (0\,,1)$. Then $\phi_n(a) = a^n$. Clearly, $\lim\limits_{n\to\infty} a^n = 0$. Hence the assertion is true.

14(a). $\phi_n(0) = 0, \forall\, n \geq 1$. Let $a \in (0\,,1]$. Then $\phi_n(a) = 2na\,e^{-na^2} = 2na/e^{na^2}$. Using l'Hospital's rule, $\lim\limits_{z\to\infty} 2az/e^{az^2} = \lim\limits_{z\to\infty} 1/ze^{az^2} = 0$. Hence $\lim\limits_{n\to\infty} \phi_n(a) = 0$.

(b). $\int_0^1 2nx\,e^{-nx^2}dx = -e^{-nx^2}\big|_0^1 = 1 - e^{-n}$. Therefore,

$$\lim_{n\to\infty}\int_0^1 \phi_n(x)dx \neq \int_0^1 \lim_{n\to\infty} \phi_n(x)dx \,.$$

15. Let t be fixed, such that $(t\,,y_1), (t\,,y_2) \in D$. Without loss of generality, assume that $y_1 < y_2$. Since f is differentiable with respect to y, the mean value theorem asserts that $\exists\, \xi \in (y_1\,,y_2)$ such that $f(t\,,y_1) - f(t\,,y_2) = f_y(t\,,\xi)(y_1 - y_2)$. Taking the absolute value of both sides, $|f(t\,,y_1) - f(t\,,y_2)| = |f_y(t\,,\xi)|\,|y_1 - y_2|$. Since, by assumption, $\partial f/\partial y$ is continuous in D, f_y attains a *maximum* on any closed and bounded subset of D .

Hence $|f(t\,,y_1)-f(t\,,y_2)|\le K\,|y_1-y_2|$.

16. For a *sufficiently small* interval of t, $\phi_{n-1}(t)\,,\phi_n(t)\in D$. Since f satisfies a Lipschitz condition, $|f(t\,,\phi_n(t))-f(t\,,\phi_{n-1}(t))|\le K\,|\phi_n(t)-\phi_{n-1}(t)|$. Here $K=max\,|f_y|$.

17(a). $\phi_1(t)=\int_0^t f(s\,,0)ds$. Hence $|\phi_1(t)|\le\int_0^{|t|}|f(s\,,0)|ds\le\int_0^{|t|}Mds=M|t|$, in which M is the maximum value of $|f(t\,,y)|$ on D.

(b). By definition, $\phi_2(t)-\phi_1(t)=\int_0^t[f(s\,,\phi_1(s))-f(s\,,0)]ds$. Taking the absolute value of both sides, $|\phi_2(t)-\phi_1(t)|\le\int_0^{|t|}|[f(s\,,\phi_1(s))-f(s\,,0)]|ds$. Based on the results in Problems 16 and 17, $|\phi_2(t)-\phi_1(t)|\le\int_0^{|t|}K|\phi_1(s)-0|ds\le KM\int_0^{|t|}|s|ds$. Evaluating the last integral, we obtain $|\phi_2(t)-\phi_1(t)|\le MK|t|^2/2$.

(c). Suppose that

$$|\phi_i(t)-\phi_{i-1}(t)|\le\frac{MK^{i-1}|t|^i}{i\,!}$$

for some $i\ge 1$. By definition, $\phi_{i+1}(t)-\phi_i(t)=\int_0^t[f(t\,,\phi_i(s))-f(s\,,\phi_{i-1}(s))]ds$. It follows that

$$\begin{aligned}|\phi_{i+1}(t)-\phi_i(t)|\;&\le\int_0^{|t|}|f(s\,,\phi_i(s))-f(s\,,\phi_{i-1}(s))|ds\\&\le\int_0^{|t|}K|\phi_i(s)-\phi_{i-1}(s)|ds\\&\le\int_0^{|t|}K\frac{MK^{i-1}|s|^i}{i\,!}ds\\&=\frac{MK^i|t|^{i+1}}{(i+1)\,!}\le\frac{MK^ih^{i+1}}{(i+1)\,!}\,.\end{aligned}$$

Hence, by mathematical induction, the assertion is true.

18(a). Use the triangle inequality, $|a+b|\le|a|+|b|$.

(b). For $|t|\le h$, $|\phi_1(t)|\le Mh$, and $|\phi_n(t)-\phi_{n-1}(t)|\le MK^{n-1}h^n/(n\,!)$. Hence

$$\begin{aligned}|\phi_n(t)|\;&\le M\sum_{i=1}^{n}\frac{K^{i-1}h^i}{i\,!}\\&=\frac{M}{K}\sum_{i=1}^{n}\frac{(Kh)^i}{i\,!}\,.\end{aligned}$$

(c). The sequence of partial sums in (b) converges to $\frac{M}{K}\left(e^{Kh}-1\right)$. By the *comparison test*, the sums in (a) also converge. Furthermore, the sequence $|\phi_n(t)|$ is *bounded*, and hence has a convergent subsequence. Finally, since individual terms of the series must tend to zero, $|\phi_n(t)-\phi_{n-1}(t)| \to 0$, and it follows that the sequence $|\phi_n(t)|$ is convergent.

19(a). Let $\phi(t)=\int_0^t f(s\,,\phi(s))ds$ and $\psi(t)=\int_0^t f(s\,,\psi(s))ds$. Then by *linearity* of the integral, $\phi(t)-\psi(t)=\int_0^t [f(s\,,\phi(s))-f(s\,,\psi(s))]ds$.

(b). It follows that $|\phi(t)-\psi(t)| \leq \int_0^t |f(s\,,\phi(s))-f(s\,,\psi(s))|ds$.

(c). We know that f satisfies a Lipschitz condition,

$$|f(t\,,y_1)-f(t\,,y_2)| \leq K\,|y_1-y_2|\,,$$

based on $|\partial f/\partial y| \leq K$ in D. Therefore,

$$\begin{aligned}|\phi(t)-\psi(t)| &\leq \int_0^t |f(s\,,\phi(s))-f(s\,,\psi(s))|ds \\ &\leq \int_0^t K|\phi(s)-\psi(s)|ds\,.\end{aligned}$$