Chapter Eleven

Section 11.1

1. Since the right hand sides of the ODE and the boundary conditions are all *zero*, the boundary value problem is *homogeneous*.

3. The right hand side of the ODE is *nonzero*. Therefore the boundary value problem is *nonhomogeneous*.

6. The ODE can also be written as

$$y'' + \lambda\big(1 + x^2\big)y = 0\,.$$

Although the second boundary condition has a more general form, the boundary value problem is *homogeneous*.

7. First assume that $\lambda = 0$. The general solution of the ODE is $y(x) = c_1 x + c_2$. The boundary condition at $x = 0$ requires that $c_2 = 0$. Imposing the second condition,

$$c_1(\pi + 1) + c_2 = 0\,.$$

It follows that $c_1 = c_2 = 0$. Hence there are no nontrivial solutions.

Suppose that $\lambda = -\mu^2$. In this case, the general solution of the ODE is

$$y(x) = c_1 cosh\, \mu x + c_2 sinh\, \mu x\,.$$

The first boundary condition requires that $c_1 = 0$. Imposing the second condition,

$$c_1\,(cosh\, \mu\pi + \mu sinh\, \mu\pi) + c_2(sinh\, \mu\pi + \mu cosh\, \mu\pi) = 0\,.$$

The two boundary conditions result in

$$c_2(tanh\, \mu\pi + \mu) = 0\,.$$

Since the *only* solution of the equation $tanh\, \mu\pi + \mu = 0$ is $\mu = 0$, we have $c_2 = 0$. Hence there are no nontrivial solutions.

Let $\lambda = \mu^2$, with $\mu > 0$. Then the general solution of the ODE is

$$y(x) = c_1 cos\, \mu x + c_2 sin\, \mu x\,.$$

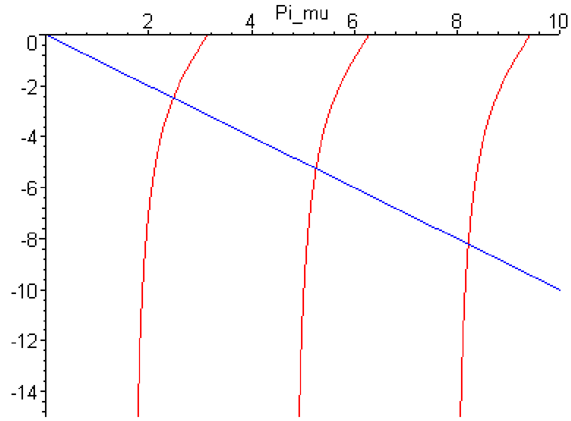
Imposing the boundary conditions, we obtain $c_1 = 0$ and

$$c_1\,(cos\, \mu\pi - \mu sin\, \mu\pi) + c_2(sin\, \mu\pi + \mu cos\, \mu\pi) = 0\,.$$

For a *nontrivial* solution of the ODE, we require that $sin\, \mu\pi + \mu cos\, \mu\pi = 0$. Note that

$$cos\, \mu\pi = 0 \Rightarrow sin\, \mu\pi = 0\,,$$

which is false. It follows that $tan\, \mu\pi = -\mu$. From a plot of $\pi\, tan\, \pi\mu$ and $-\pi\mu$,



we find that there is a sequence of solutions, $\mu_1 \approx 0.7876$, $\mu_2 \approx 1.6716$, $\cdots$; For large values of n,

$$\pi\,\mu_n \approx (2n-1)\frac{\pi}{2}\,.$$

Therefore the eigenfunctions are $\phi_n(x) = sin\,\mu_n x$, with corresponding eigenvalues

$$\lambda_1 \approx 0.6204\,,\ \lambda_2 \approx 2.7943\,,\cdots.$$

Asymptotically,

$$\lambda_n \approx \frac{(2n-1)^2}{4}\,.$$

8. With $\lambda = 0$, the general solution of the ODE is $y(x) = c_1 x + c_2$. Imposing the two boundary conditions, $c_1 = 0$ and $2c_1 + c_2 = 0$. It follows that $c_1 = c_2 = 0$. Hence there are no nontrivial solutions.

Setting $\lambda = -\mu^2$, the general solution of the ODE is

$$y(x) = c_1 cosh\,\mu x + c_2 sinh\,\mu x\,.$$

The first boundary condition requires that $c_2 = 0$. Imposing the second condition,

$$c_1\,(cosh\,\mu + \mu sinh\,\mu) + c_2(sinh\,\mu + \mu cosh\,\mu) = 0\,.$$

The two boundary conditions result in

$$c_1(1 + \mu\,tanh\,\mu) = 0\,.$$

Since $\mu\,tanh\,\mu \geq 0$, it follows that $c_1 = 0$, and there are no nontrivial solutions.

Let $\lambda = \mu^2$, with $\mu > 0$. Then the general solution of the ODE is

$$y(x) = c_1 cos\,\mu x + c_2 sin\,\mu x\,.$$

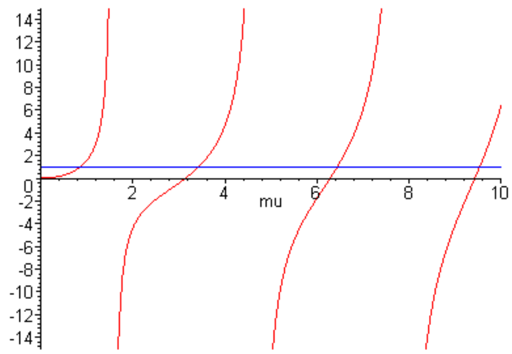
Imposing the boundary conditions, we obtain $c_2 = 0$ and

$$c_1\,(cos\,\mu-\mu sin\,\mu)+c_2(sin\,\mu+\mu cos\,\mu)=0\,.$$

For a *nontrivial* solution of the ODE, we require that $cos\,\mu-\mu sin\,\mu=0$. First note that

$$cos\,\mu=0\Rightarrow\mu=0\ \text{ or }\ sin\,\mu=0\,.$$

Therefore we find that $1-\mu\,tan\,\mu=0$. From a plot of $\mu\,tan\,\mu$, there is a sequence of



solutions, $\mu_1\approx 0.8603\,,\ \mu_2\approx 3.4256\,,\cdots$; For large n,

$$\mu_n\approx(n-1)\pi\,.$$

Therefore the eigenfunctions are $\phi_n(x)=cos\,\mu_n x$, with corresponding eigenvalues

$$\lambda_1\approx 0.7402\,,\ \lambda_2\approx 11.7349\,,\cdots\,.$$

Asymptotically,

$$\lambda_n\approx(n-1)^2\pi^2\,.$$

12. First note that $P(x)=1\,,\ Q(x)=-2x$ and $R(x)=\lambda$. Based on Prob. 11, the integrating factor is a solution of the ODE

$$\mu'(x)=-2x\,\mu(x)\,.$$

The differential equation is first order linear, with solution $\mu(x)=c\,exp(-x^2)$. It then follows that the *Hermite equation* can be written as

$$\left[e^{-x^2}y'\right]'+\lambda\,e^{-x^2}y=0\,.$$

14. For the *Laguerre equation*, $P(x)=x\,,\ Q(x)=1-x$ and $R(x)=\lambda$. Using the result of Prob. 11, the integrating factor is a solution of the ODE

$$x\,\mu'(x)=-x\,\mu(x)\,.$$

The general solution of $\mu'(x)=-\mu(x)$ is $\mu(x)=c\,e^{-x}$. Therefore the *Laguerre equation* can be written as

$$\left[x\, e^{-x} y'\right]' + \lambda\, e^{-x} y = 0\,.$$

15. For the *Chebyshev equation*, $P(x) = 1 - x^2$, $Q(x) = -x$ and $R(x) = \alpha^2$. The integrating factor is a solution of the ODE

$$\left(1 - x^2\right)\mu'(x) = x\,\mu(x)\,.$$

The differential equation is separable, with

$$\frac{d\mu}{\mu} = \frac{x}{1 - x^2}\,.$$

The general solution of the resulting ODE is

$$\mu(x) = \frac{c}{\sqrt{|1 - x^2|}}\,.$$

Recall that the *Chebyshev equation* is typically defined for $|x| \leq 1$. Therefore it can also be written as

$$\left[\sqrt{1 - x^2}\, y'\right]' + \frac{\alpha^2}{\sqrt{1 - x^2}}\, y = 0\,.$$

16. We consider solutions of the form $u(x\,, t) = X(x)T(t)$. Substitution into the PDE results in

$$XT'' + c\,XT' + k\,XT = \alpha^2 X''T\,.$$

Dividing both sides of the equation by XT, we obtain

$$\frac{XT''}{XT} + c\,\frac{XT'}{XT} + k = \alpha^2 \frac{X''T}{XT}\,,$$

that is,

$$\frac{T''}{T} + c\,\frac{T'}{T} = \alpha^2 \frac{X''}{X} - k\,.$$

Since both sides of the resulting equation are functions of different variables, each must be
equal to a constant, say $-\lambda$. Therefore we obtain two ordinary differential equations

$$\alpha^2 X'' + (\lambda - k)X = 0 \quad \text{and} \quad T'' + cT' + \lambda T = 0\,.$$

17(a). Setting $y = s(x)u$, we have $y' = s'u + s\,u'$ and $y'' = s''u + 2s'u' + s\,u''$. Substitution into the given ODE results in

$$s''u + 2s'u' + s\,u'' - 2(s'u + s\,u') + (1 + \lambda)su = 0\,.$$

Collecting the various terms,

$$s\,u'' + (2s' - 2s)u' + [s'' - 2s' + (1+\lambda)s]u = 0\,.$$

The second term on the left vanishes as long as $s' = s$.

(b). With $s(x) = e^x$, the transformed differential equation can be written as

$$u'' + \lambda u = 0\,.$$

Since the boundary conditions are *homogeneous*, we also have $u(0) = u(1) = 0$. It now follows that the eigenfunctions are $u_n = sin\,\sqrt{\lambda_n}\,x$, with corresponding eigenvalues

$$\lambda_n = n^2\pi^2.$$

Therefore the eigenfunctions for the original problem are $\phi_n(x) = e^x sin\,n\pi x$, with corresponding eigenvalues

$$1 + \lambda_n = 1 + n^2\pi^2.$$

(c). The given equation is a second order *constant coefficient* differential equation. The characteristic equation is

$$r^2 - 2r + (1+\lambda) = 0\,,$$

with roots $r_{1,2} = 1 \pm \sqrt{-\lambda}$.

If $\lambda = 0$, then the general solution is $y = c_1 e^x + c_2\,xe^x$. Imposing the two boundary conditions, we find that $c_1 = c_2 = 0$, and hence there are no nontrivial solutions. If $\lambda < 0$, then the general solution is

$$y = c_1 exp\Big(1 + \sqrt{-\lambda}\Big)x + c_2 exp\Big(1 - \sqrt{-\lambda}\Big)x\,.$$

It again follows that $c_1 = c_2 = 0$, and hence there are no nontrivial solutions.

Therefore $\lambda > 0$, and the general solution is

$$y = c_1 e^x cos\,\sqrt{\lambda}\,x + c_2\,e^x sin\,\sqrt{\lambda}\,x\,.$$

Invoking the boundary conditions, we have $c_1 = 0$ and $c_2 e\,sin\sqrt{\lambda} = 0$. For a nontrivial
solution, $\sqrt{\lambda} = n\pi$.

19. First write the differential equation as

$$y'' + (1+\lambda)y' + \lambda y = 0\,,$$

which is a second order *constant coefficient* differential equation. The characteristic equation is

$$r^2 + (1+\lambda)r + \lambda = 0\,,$$

with roots $r_1 = -1$ and $r_2 = -\lambda$. For $\lambda \neq 1$, the general solution is

$$y = c_1 e^{-x} + c_2\, e^{-\lambda x}.$$

Imposing the boundary conditions, we require that $c_1 + c_2 = 0$ and $c_1 e^{-1} + c_2\, e^{-\lambda} = 0$. For a nontrivial solution, it follows that $e^{-1} = e^{-\lambda}$, and hence $\lambda = 1$, which is contrary to the assumption.

If $\lambda = 1$, then the general solution is

$$y = c_1 e^{-x} + c_2\, x e^{-x}.$$

The boundary conditions require that $c_1 = 0$ and $c_1 e^{-1} + c_2\, e^{-1} = 0$. Hence there are no nontrivial solutions.

21. Suppose that $\lambda = 0$. In that case the general solution is $y = c_1 x + c_2$. The boundary
conditions require that $c_1 + 2c_2 = 0$ and $c_1 + c_2 = 0$. We find that $c_1 = c_2 = 0$, and hence there are no nontrivial solutions.

(a). Let $\lambda = \mu^2$, with $\mu > 0$. Then the general solution of the ODE is

$$y(x) = c_1 cos\,\mu x + c_2 sin\,\mu x\,.$$

The boundary conditions require that

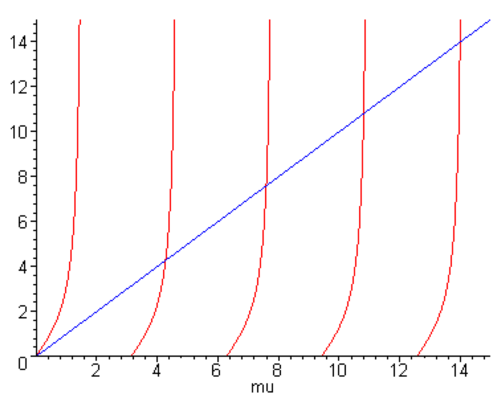
$$2c_1 + \mu\, c_2 = 0 \text{ and } c_1 cos\,\mu + c_2 sin\,\mu = 0\,.$$

These equations have a nonzero solution only if

$$2 sin\,\mu - \mu\, cos\,\mu = 0\,,$$

which can also be written as

$$2 tan\,\mu - \mu = 0\,.$$



Based on the graph, the positive roots of the determinantal equation are

$$\mu_1 \approx 4.2748\,,\ \mu_2 \approx 7.5965\,,\cdots;\ \text{for large } n,\ \mu_n \approx (2n+1)\frac{\pi}{2}\,.$$

Therefore the eigenvalues are

$$\lambda_1 \approx 18.2738\,,\ \lambda_2 \approx 57.7075\,,\cdots;\ \text{for large } n,\ \lambda_n \approx (2n+1)^2\frac{\pi^2}{4}\,.$$

(b). Setting $\lambda = -\mu^2 < 0$, the general solution of the ODE is

$$y(x) = c_1 cosh\,\mu x + c_2 sinh\,\mu x\,.$$

Imposing the boundary conditions, we obtain the equations

$$2c_1 + \mu\, c_2 = 0 \ \text{ and }\ c_1 cosh\,\mu + c_2 sinh\,\mu = 0\,.$$

These equations have a nonzero solution only if

$$2sinh\,\mu - \mu\, cosh\,\mu = 0\,.$$

The latter equation is satisfied only for $\mu = 0$ and $\mu = \pm 1.9150$. Hence the only *negative* eigenvalue is $\lambda_{-1} = 3.6673$.

24. Based on the physical problem, $\lambda = m\omega^2/EI > 0$. Let $\lambda = \mu^4$. The characteristic equation is $r^4 - \mu^4 = 0$, with roots $r_{1,2} = \pm\,\mu i$, $r_3 = -\mu$ and $r_4 = \mu$. Hence the general solution is

$$y(x) = c_1 cosh\,\mu x + c_2 sinh\,\mu x + c_3 cos\,\mu x + c_4 sin\,\mu x\,.$$

(a). Simply supported on both ends : $y(0) = y''(0) = 0$; $y(L) = y''(L) = 0$.
Invoking the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 - c_3 &= 0 \\ c_1 cosh\,\mu L + c_2 sinh\,\mu L + c_3 cos\,\mu L + c_4 sin\,\mu L &= 0 \\ c_1\mu^2 cosh\,\mu L + c_2\mu^2 sinh\,\mu L - c_3\mu^2 cos\,\mu L - c_4\mu^2 sin\,\mu L &= 0\,. \end{aligned}$$

The determinantal equation is

$$\mu^4 sinh\,\mu L\, sin\,\mu L = 0\,.$$

The nonzero roots are $\mu_n = n\pi/L$, $n = 1, 2, \cdots$. The first two equations result in $c_1 = c_3 = 0$. The last two equations,

$$\begin{aligned} c_2 sinh\,n\pi + c_4 sin\,n\pi &= 0 \\ c_2 sinh\,n\pi - c_4 sin\,n\pi &= 0\,, \end{aligned}$$

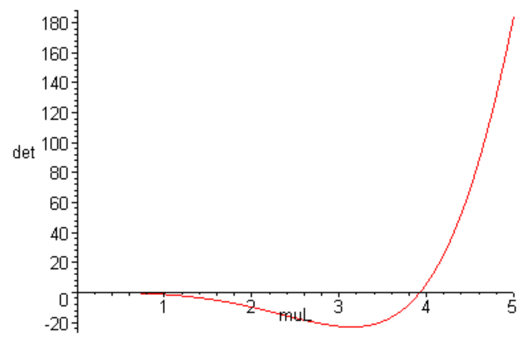
imply that $c_2 = 0$. Therefore the eigenfunctions are $\phi_n = sin\,\mu_n x$, with corresponding eigenvalues $\lambda_n = n^4\pi^4/L^4$.

(b). Simply supported : $y(0) = y''(0) = 0$; clamped : $y(L) = y'(L) = 0$.
Invoking the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 - c_3 &= 0 \\ c_1 cosh\,\mu L + c_2 sinh\,\mu L + c_3 cos\,\mu L + c_4 sin\,\mu L &= 0 \\ c_1\,\mu sinh\,\mu L + c_2\,\mu cosh\,\mu L - c_3\,\mu sin\,\mu L + c_4\,\mu cos\,\mu L &= 0\,. \end{aligned}$$

The determinantal equation is

$$2\mu^3 sinh\,\mu L\,cos\,\mu L - 2\mu^3 cosh\,\mu L\,sin\,\mu L = 0\,.$$



Based on numerical analysis, $\mu_1 \approx 3.9266/L$ and $\mu_2 \approx 7.0686/L$.

The first two equations result in $c_1 = c_3 = 0$. The last two equations,

$$\begin{aligned} c_2 sinh\,\mu_n L + c_4 sin\,\mu_n L &= 0 \\ c_2\,cosh\,\mu_n L + c_4\,cos\,\mu_n L &= 0\,, \end{aligned}$$

imply that

$$c_2 = -\frac{sin\,\mu_n L}{sinh\,\mu_n L} c_4\,.$$

Therefore the eigenfunctions are

$$\phi_n = -\frac{sin\,\mu_n L}{sinh\,\mu_n L} sinh\,\mu_n x + sin\,\mu_n x\,,$$

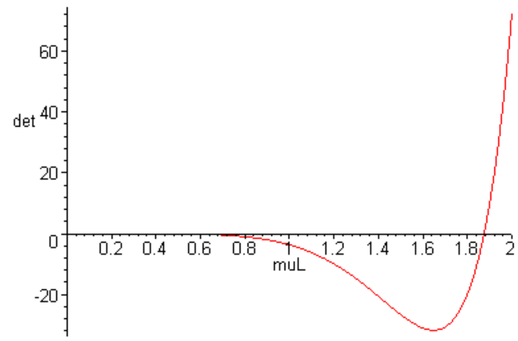
with corresponding eigenvalues $\lambda_n = \mu_n^4$.

(c). Clamped : $y(0) = y'(0) = 0$; free : $y''(L) = y'''(L) = 0$.
Invoking the boundary conditions, we obtain the system of equations

$$c_1 + c_3 = 0$$
$$\mu c_2 + \mu c_4 = 0$$
$$c_1\mu^2 cosh\,\mu L + c_2\mu^2 sinh\,\mu L - c_3\mu^2 cos\,\mu L - c_4\mu^2 sin\,\mu L = 0$$
$$c_1\,\mu^3 sinh\,\mu L + c_2\,\mu^3 cosh\,\mu L + c_3\,\mu^3 sin\,\mu L - c_4\,\mu^3 cos\,\mu L = 0\,.$$

The determinantal equation is

$$1 + cosh\,\mu L\,cos\,\mu L = 0\,.$$



The first two *nonzero* roots are $\mu_1 \approx 1.8751/L$ and $\mu_2 \approx 4.6941/L$. With $c_3 = -c_1$ and $c_4 = -c_2$, the system of equations reduce to

$$c_1(cosh\,\mu_n L + cos\,\mu_n L) + c_2(sinh\,\mu_n L + sin\,\mu_n L) = 0$$
$$c_1(sinh\,\mu_n L - sin\,\mu_n L) + c_2(cosh\,\mu_n L + cos\,\mu_n L) = 0\,.$$

Let $A_n = (cosh\,\mu_n L + cos\,\mu_n L)/(sinh\,\mu_n L + sin\,\mu_n L)$. The eigenfunctions are given by

$$\phi_n(x) = cosh\,\mu_n x - cos\,\mu_n x + A_n(sin\,\mu_n x - sinh\,\mu_n x)\,,$$

with corresponding eigenvalues $\lambda_n = \mu_n^4$.

25(a). Assume that the solution has the form $u(x\,,t) = X(x)T(t)$. Substitution into the
PDE results in

$$\frac{E}{\rho}X''T = XT''\,.$$

Dividing both sides of the equation by XT, we obtain

$$\frac{E}{\rho}\frac{X''T}{XT} = \frac{XT''}{XT}\,,$$

that is,

$$\frac{X''}{X} = \frac{\rho}{E}\frac{T''}{T}\,.$$

Since both sides of the resulting equation are functions of different variables, each must be
equal to a constant, say $-\lambda$. Therefore we obtain two ordinary differential equations

$$X'' + \lambda X = 0 \quad \text{and} \quad T'' + \lambda\frac{E}{\rho}T = 0\,.$$

(b). Given that $u(0\,,t) = X(0)T(t)$ for $t > 0$, it follows that $X(0) = 0$. The second boundary condition can be expressed as

$$EAX'(L)T(t) + mX(L)T''(t) = 0\,,\;\; t > 0\,.$$

From the result in Part (a),

$$EAX'(L)T(t) - \lambda m\frac{E}{\rho}X(L)T(t) = 0\,,\;\; t > 0\,.$$

Since the condition is to be satisfied for all $t > 0$, we arrive at the boundary condition

$$X'(L) - \lambda\frac{m}{\rho A}X(L) = 0\,.$$

(c). If $\lambda = 0$, the general solution of the spatial equation is

$$X(x) = c_1 x + c_2\,.$$

The boundary condition require that $c_1 = c_2 = 0$. Hence there are no nontrivial solutions.
If $\lambda = -\mu^2 < 0$, then the general solution is

$$X(x) = c_1 cosh\,\mu x + c_2 sinh\,\mu x\,.$$

The first boundary condition implies that $c_1 = 0$. The second boundary condition requires
that

$$c_2 cosh\,\mu L + c_2\,\mu\frac{m}{\rho A}sinh\,\mu L = 0\,.$$

The solution is nontrivial only if

$$\mu\,tanh\,\mu L = -\frac{\rho A}{m}\,.$$

Since $\mu\,tanh\,\mu L \geq 0$, there are no nontrivial solutions.

Let $\lambda = \mu^2 > 0$. The general solution of the spatial equation is

$$X(x) = c_1 cos\,\mu x + c_2 sin\,\mu x\,.$$

The first boundary condition implies that $c_1 = 0$. The second boundary condition requires
that

$$c_2 \, cos \, \mu L - c_2 \, \mu \frac{m}{\rho A} sin \, \mu L = 0 \, .$$

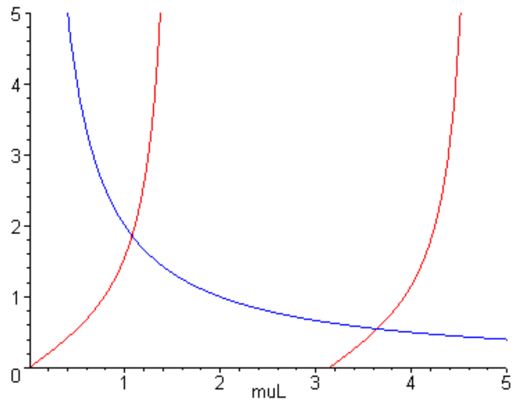
For a nontrivial solution, it is necessary that

$$cos \, \mu L - \mu \frac{m}{\rho A} sin \, \mu L = 0 \, ,$$

or

$$tan \, \mu L = \frac{\rho A}{m \mu} \, .$$

For the case $(m/\rho AL) = 0.5$,



we find that $\mu_1 L \approx 1.0769$ and $\mu_2 L \approx 3.6436$. Therefore the eigenfunctions are given by $\phi_n(x) = sin \, \mu_n x$. The corresponding eigenvalues are solutions of

$$cos \, \sqrt{\lambda_n} L - \frac{L}{2} \sqrt{\lambda_n} sin \, \sqrt{\lambda_n} L = 0 \, .$$

The first two eigenvalues are approximated as $\lambda_1 \approx 1.1597/L^2$ and $\lambda_2 \approx 13.276/L^2$.