Section 5.6

1. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{1}{2x} = \frac{1}{2}.$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1}{2} = 0.$$

Hence $x = 0$ is a *regular* singular point. Let

$$y = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots\right) = \sum_{n=0}^{\infty} a_n\, x^{r+n}.$$

Then

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}.$$

Substitution into the ODE results in

$$2 \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n-1} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0.$$

That is,

$$2 \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2}\, x^{r+n} = 0.$$

It follows that

$$a_0[2r(r-1) + r]x^r + a_1[2(r+1)r + r + 1]x^{r+1} + \\ + \sum_{n=2}^{\infty} [2(r+n)(r+n-1)a_n + (r+n)a_n + a_{n-2}]x^{r+n} = 0.$$

Assuming that $a_0 \neq 0$, we obtain the *indicial equation* $2r^2 - r = 0$, with roots $r_1 = 1/2$

and $r_2 = 0$. It immediately follows that $a_1 = 0$. Setting the remaining coefficients equal
to *zero*, we have

$$a_n = \frac{-\,a_{n-2}}{(r+n)[2(r+n)-1]}\,, \quad n = 2, 3, \cdots.$$

For $r = 1/2$, the recurrence relation becomes

$$a_n = \frac{-\,a_{n-2}}{n(1+2n)}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*. Furthermore, for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(1+4k)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-3)(4k+1)} = \frac{(-1)^k a_0}{2^k\, k!\, 5\cdot 9\cdot 13\cdots(4k+1)}\,.$$

For $r = 0$, the recurrence relation becomes

$$a_n = \frac{-\,a_{n-2}}{n(2n-1)}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*, and for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(4k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-5)(4k-1)} = \frac{(-1)^k a_0}{2^k\, k!\, 3\cdot 7\cdot 11\cdots(4k-1)}\,.$$

The two linearly independent solutions are

$$y_1(x) = \sqrt{x}\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{2^k\, k!\, 5\cdot 9\cdot 13\cdots(4k+1)}\right]$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{2^k\, k!\, 3\cdot 7\cdot 11\cdots(4k-1)}\,.$$

3. Note that $x\,p(x) = 0$ and $x^2 q(x) = x$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,,$$

and after multiplying both sides of the equation by x,

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0\,.$$

It follows that

$$a_0[r(r-1)]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + a_{n-1}]x^{r+n} = 0\,.$$

Setting the coefficients equal to *zero*, the *indicial equation* is $r(r-1) = 0$. The roots are $r_1 = 1$ and $r_2 = 0$. Here $r_1 - r_2 = 1$. The recurrence relation is

$$a_n = \frac{-\,a_{n-1}}{(r+n)(r+n-1)}\,, \quad n = 1, 2, \cdots.$$

For $r = 1$,

$$a_n = \frac{-\,a_{n-1}}{n(n+1)}\,, \quad n = 1, 2, \cdots.$$

Hence for $n \geq 1$,

$$a_n = \frac{-\,a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \cdots = \frac{(-1)^n a_0}{n!(n+1)!}\,.$$

Therefore one solution is

$$y_1(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n\,x^n}{n!(n+1)!}\,.$$

5. Here $x\,p(x) = 2/3$ and $x^2q(x) = x^2/3$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$3\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + 2\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \sum_{n=0}^{\infty}a_nx^{r+n+2} = 0\,.$$

It follows that

$$a_0[3r(r-1)+2r]x^r + a_1[3(r+1)r + 2(r+1)]x^{r+1} + \\ + \sum_{n=2}^{\infty}[3(r+n)(r+n-1)a_n + 2(r+n)a_n + a_{n-2}]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $3r^2 - r = 0$, with roots $r_1 = 1/3$, $r_2 = 0$. Setting the remaining coefficients equal to *zero*, we have $a_1 = 0$, and

$$a_n = \frac{-\,a_{n-2}}{(r+n)[3(r+n)-1]}\,, \quad n = 2, 3, \cdots.$$

It immediately follows that the *odd* coefficients are equal to *zero*. For $r = 1/3$,

$$a_n = \frac{-\,a_{n-2}}{n(1+3n)}\,, \quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(6k+1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-5)(6k+1)} = \frac{(-1)^k a_0}{2^k\,k!\,7\cdot 13\cdots(6k+1)}\,.$$

For $r = 0$,

$$a_n = \frac{-\,a_{n-2}}{n(3n-1)}\,,\quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(6k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-7)(6k-1)} = \frac{(-1)^k a_0}{2^k\,k!\,5\cdot 11\cdots(6k-1)}\,.$$

The two linearly independent solutions are

$$y_1(x) = x^{1/3}\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,7\cdot 13\cdots(6k+1)}\left(\frac{x^2}{2}\right)^k\right]$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,5\cdot 11\cdots(6k-1)}\left(\frac{x^2}{2}\right)^k.$$

6. Note that $x\,p(x) = 1$ and $x^2 q(x) = x - 2$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \\ + \sum_{n=0}^{\infty}a_n x^{r+n+1} - 2\sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1)+r-2]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n - 2\,a_n + a_{n-1}]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - 2 = 0$, with roots $r = \pm\sqrt{2}$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{-\,a_{n-1}}{(r+n)^2 - 2}\,,\quad n = 1, 2, \cdots.$$

First note that $(r+n)^2 - 2 = \left(r + n + \sqrt{2}\right)\left(r + n - \sqrt{2}\right)$. So for $r = \sqrt{2}$,

$$a_n = \frac{-\,a_{n-1}}{n\left(n + 2\sqrt{2}\right)}\,,\quad n = 1, 2, \cdots.$$

It follows that

$$a_n = \frac{(-1)^n a_0}{n!\left(1+2\sqrt{2}\right)\left(2+2\sqrt{2}\right)\cdots\left(n+2\sqrt{2}\right)}\,, \quad n = 1, 2, \cdots.$$

For $r = -\sqrt{2}$,

$$a_n = \frac{-a_{n-1}}{n\left(n-2\sqrt{2}\right)}\,, \quad n = 1, 2, \cdots,$$

and therefore

$$a_n = \frac{(-1)^n a_0}{n!\left(1-2\sqrt{2}\right)\left(2-2\sqrt{2}\right)\cdots\left(n-2\sqrt{2}\right)}\,, \quad n = 1, 2, \cdots.$$

The two linearly independent solutions are

$$y_1(x) = x^{\sqrt{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!\left(1+2\sqrt{2}\right)\left(2+2\sqrt{2}\right)\cdots\left(n+2\sqrt{2}\right)}\right]$$

$$y_2(x) = x^{-\sqrt{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!\left(1-2\sqrt{2}\right)\left(2-2\sqrt{2}\right)\cdots\left(n-2\sqrt{2}\right)}\right].$$

7. Here $x\,p(x) = 1 - x$ and $x^2 q(x) = -x$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n-1} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n-1} - \\ - \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} - \sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After multiplying both sides by x ,

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} - \\ - \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n+1} - \sum_{n=0}^{\infty}a_n x^{r+n+1} = 0.$$

After adjusting the indices in the *last two* series, we obtain

$$a_0[r(r-1)+r]x^r+\sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n+(r+n)a_n-(r+n)a_{n-1}]x^{r+n}=0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2=0$, with roots $r_1=r_2=0$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n=\frac{a_{n-1}}{r+n}\,,\quad n=1,2,\cdots.$$

With $r=0$,

$$a_n=\frac{a_{n-1}}{n}\,,\quad n=1,2,\cdots.$$

Hence one solution is

$$y_1(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots=e^x.$$

8. Note that $x\,p(x)=3/2$ and $x^2q(x)=x^2-1/2$, which are *both* analytic at $x=0$. Set $y=x^r(a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots)$. Substitution into the ODE results in

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n}+\,3\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}+$$
$$+\,2\sum_{n=0}^{\infty}a_nx^{r+n+2}-\sum_{n=0}^{\infty}a_nx^{r+n}=0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[2r(r-1)+3r-1]x^r+a_1[2(r+1)r+3(r+1)-1]+$$
$$+\sum_{n=2}^{\infty}[2(r+n)(r+n-1)a_n+3(r+n)a_n-a_n+2\,a_{n-2}]x^{r+n}=0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $2r^2+r-1=0$, with roots $r_1=1/2$ and $r_2=-1$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n=\frac{-\,2a_{n-2}}{(r+n+1)[2(r+n)-1]}\,,\quad n=2,3,\cdots.$$

Setting the remaining coefficients equal to *zero*, we have $a_1=0$, which implies that all of the *odd* coefficients are *zero*. With $r=1/2$,

$$a_n=\frac{-\,2a_{n-2}}{n(2n+3)}\,,\quad n=2,3,\cdots.$$

So for $k=1,2,\cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{k(4k+3)} = \frac{a_{2k-4}}{(k-1)k(4k-5)(4k+3)} = \frac{(\,-\,1)^k a_0}{k!\,7\cdot 11\cdots(4k+3)}\ .$$

With $r = \,-\,1\,$,

$$a_n = \frac{-\,2a_{n-2}}{n(2n-3)}\,,\quad n = 2,3,\cdots.$$

So for $k = 1,2,\cdots,$

$$a_{2k} = \frac{-\,a_{2k-2}}{k(4k-3)} = \frac{a_{2k-4}}{(k-1)k(4k-11)(4k-3)} = \frac{(\,-\,1)^k a_0}{k!\,5\cdot 9\cdots(4k-3)}\ .$$

The two linearly independent solutions are

$$y_1(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty}\frac{(\,-\,1)^n\,x^{2n}}{n!\,7\cdot 11\cdots(4n+3)}\right]$$

$$y_2(x) = x^{-1}\left[1 + \sum_{n=1}^{\infty}\frac{(\,-\,1)^n\,x^{2n}}{n!\,5\cdot 9\cdots(4n-3)}\right].$$

9. Note that $x\,p(x) = \,-\,x - 3$ and $x^2q(x) = x+3$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n+1} - 3\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}$$
$$+\sum_{n=0}^{\infty}a_nx^{r+n+1} + 3\sum_{n=0}^{\infty}a_nx^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1) - 3r + 3]x^r +$$
$$+\sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n - (r+n-2)a_{n-1} - 3(r+n-1)a_n]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - 4r + 3 = 0$, with roots $r_1 = 3$ and $r_2 = 1$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{(r+n-2)a_{n-1}}{(r+n-1)(r+n-3)}\,,\quad n = 1,2,\cdots.$$

With $r = 3$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)}\,, \quad n = 1, 2, \cdots.$$

It follows that for $n \geq 1$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)} = \frac{a_{n-2}}{(n-1)(n+2)} = \cdots = \frac{2\,a_0}{n!\,(n+2)}\,.$$

Therefore one solution is

$$y_1(x) = x^3\left[1 + \sum_{n=1}^{\infty}\frac{2\,x^n}{n!\,(n+2)}\right].$$

10. Here $x\,p(x) = 0$ and $x^2q(x) = x^2 + 1/4$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}a_nx^{r+n+2} + \frac{1}{4}\sum_{n=0}^{\infty}a_nx^{r+n} = 0\,.$$

After adjusting the indices in the *second* series, we obtain

$$a_0\left[r(r-1) + \frac{1}{4}\right]x^r \;+ a_1\left[(r+1)r + \frac{1}{4}\right]x^{r+1} + \\ + \sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + \frac{1}{4}a_n + a_{n-2}\right]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - r + \frac{1}{4} = 0$, with roots $r_1 = r_2 = 1/2$. Setting the remaining coefficients equal to *zero*, we find that $a_1 = 0$. The recurrence relation is

$$a_n = \frac{-\,4a_{n-2}}{(2r+2n-1)^2}\,, \quad n = 2, 3, \cdots.$$

With $r = 1/2$,

$$a_n = \frac{-\,a_{n-2}}{n^2}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*. So for $k \geq 1$,

$$a_{2k} = \frac{-\,a_{2k-2}}{4k^2} = \frac{a_{2k-4}}{4^2(k-1)^2k^2} = \cdots = \frac{(-1)^k a_0}{4^k(k!)^2}\,.$$

Therefore one solution is

$$y_1(x) = \sqrt{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right].$$

$12(a)$. Dividing through by the leading coefficient, the ODE can be written as

$$y'' - \frac{x}{1-x^2}\,y' + \frac{\alpha^2}{1-x^2}\,y = 0\,.$$

For $x = 1$,

$$p_0 = \lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{x}{x+1} = \frac{1}{2}\,.$$

$$q_0 = \lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}\frac{\alpha^2(1-x)}{x+1} = 0\,.$$

For $x = -1$,

$$p_0 = \lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}\frac{x}{x-1} = \frac{1}{2}\,.$$

$$q_0 = \lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}\frac{\alpha^2(x+1)}{(1-x)} = 0\,.$$

Hence both $x = -1$ and $x = 1$ are *regular* singular points. As shown in Example **1**, the indicial equation is given by

$$r(r-1) + p_0 r + q_0 = 0\,.$$

In this case, *both* sets of roots are $r_1 = 1/2$ and $r_2 = 0$.

(b). Let $t = x - 1$, and $u(t) = y(t+1)$. Under this change of variable, the differential equation becomes

$$\left(t^2 + 2t\right)u'' + (t+1)u' - \alpha^2 u = 0\,.$$

Based on Part (a), $t = 0$ is a *regular* singular point. Set $u = \sum_{n=0}^{\infty} a_n\, t^{r+n}$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, t^{r+n} + 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, t^{r+n-1} +$$
$$+\sum_{n=0}^{\infty}(r+n)a_n\, t^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\, t^{r+n-1} - \alpha^2\sum_{n=0}^{\infty}a_n t^{r+n} = 0\,.$$

Upon inspection, we can also write

$$\sum_{n=0}^{\infty}(r+n)^2 a_n\, t^{r+n} + 2\sum_{n=0}^{\infty}(r+n)\left(r+n-\frac{1}{2}\right)a_n\, t^{r+n-1} - \alpha^2\sum_{n=0}^{\infty}a_n t^{r+n} = 0.$$

After adjusting the indices in the *second* series, it follows that

$$a_0\left[2r\left(r-\frac{1}{2}\right)\right]t^{r-1} + \sum_{n=0}^{\infty}\left[(r+n)^2 a_n + 2(r+n+1)\left(r+n+\frac{1}{2}\right)a_{n+1} - \alpha^2 a_n\right]t^{r+n} = 0.$$

Assuming that $a_0 \neq 0$, the *indicial equation* is $2r^2 - r = 0$, with roots $r = 0\,,\, 1/2$. The recurrence relation is

$$(r+n)^2 a_n + 2(r+n+1)\left(r+n+\frac{1}{2}\right)a_{n+1} - \alpha^2 a_n = 0\,, \quad n = 0, 1, 2, \cdots.$$

With $r_1 = 1/2$, we find that for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{4\alpha^2 - (2n-1)^2}{4n(2n+1)}\, a_{n-1} \\ &= (-1)^n \frac{[1-4\alpha^2][9-4\alpha^2]\cdots\left[(2n-1)^2 - 4\alpha^2\right]}{2^n(2n+1)!}\, a_0\,. \end{aligned}$$

With $r_2 = 0$, we find that for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{\alpha^2 - (n-1)^2}{n(2n-1)}\, a_{n-1} \\ &= (-1)^n \frac{\alpha(-\alpha)[1-\alpha^2][4-\alpha^2]\cdots\left[(n-1)^2-\alpha^2\right]}{n!\cdot 3\cdot 5\cdots(2n-1)}\, a_0\,. \end{aligned}$$

The two linearly independent solutions of the *Chebyshev equation* are

$$y_1(x) = |x-1|^{1/2}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{[1-4\alpha^2][9-4\alpha^2]\cdots\left[(2n-1)^2-4\alpha^2\right]}{2^n(2n+1)!}(x-1)^n\right]$$

$$y_2(x) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{\alpha(-\alpha)[1-\alpha^2][4-\alpha^2]\cdots\left[(n-1)^2-\alpha^2\right]}{n!\cdot 3\cdot 5\cdots(2n-1)}(x-1)^n\,.$$

13. Here $x\,p(x) = 1 - x$ and $x^2 q(x) = \lambda\, x$, which are *both* analytic at $x = 0$. In fact,

$$p_0 = \lim_{x\to 0} x\,p(x) = 1 \text{ and } q_0 = \lim_{x\to 0} x^2 q(x) = 0\,.$$

Hence the *indicial equation* is $r(r-1) + r = 0$, with roots $r_{1,2} = 0$. Set

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

Substitution into the ODE results in

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} + \sum_{n=1}^{\infty} na_n\, x^{n-1} - \\ - \sum_{n=0}^{\infty} na_n\, x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}\, x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}\, x^n - \\ - \sum_{n=1}^{\infty} na_n\, x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

It follows that

$$a_1 + \lambda\, a_0 + \sum_{n=1}^{\infty} \big[(n+1)^2 a_{n+1} - (n-\lambda)a_n\big] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we find that $a_1 = -\lambda a_0$, and

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1}\,, \quad n = 2, 3, \cdots .$$

That is, for $n \geq 2$,

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1} = \cdots = \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, a_0\,.$$

Therefore one solution of the *Laguerre equation* is

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, x^n.$$

Note that if $\lambda = m$, a *positive integer*, then $a_n = 0$ for $n \geq m+1$. In that case, the solution is a *polynomial*

$$y_1(x) = 1 + \sum_{n=1}^{m} \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, x^n\,.$$