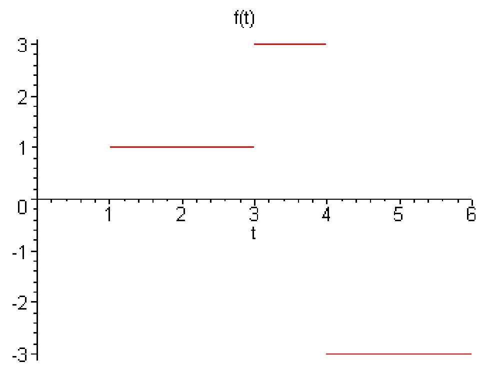
Section 6.3

1.



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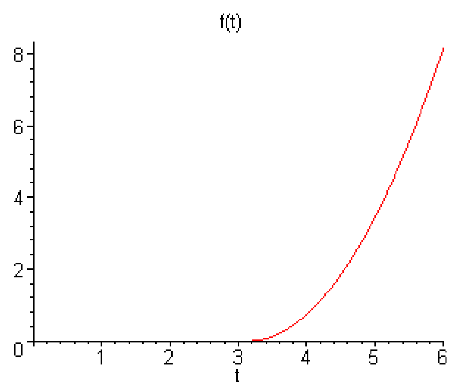

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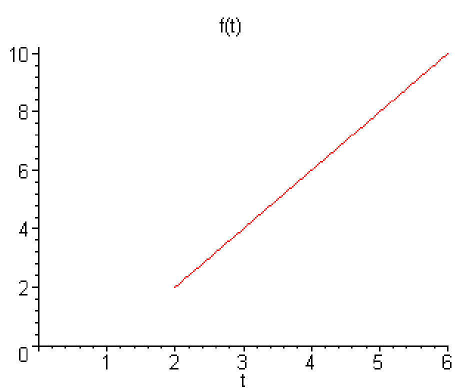

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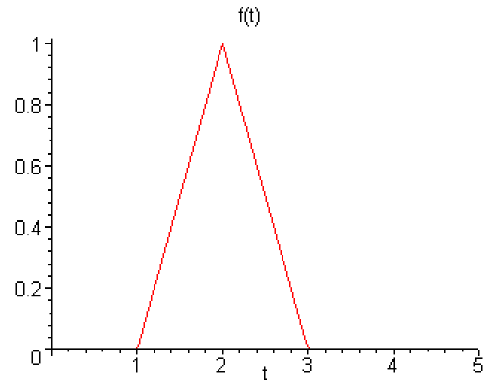


7. Using the Heaviside function, we can write

$$f(t) = (t-2)^2\, u_2(t)\,.$$

The Laplace transform has the property that

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Hence

$$\mathcal{L}\big[\,(t-2)^2\, u_2(t)\big] = \frac{2\,e^{-2s}}{s^2}.$$

9. The function can be expressed as

$$f(t) = (t-\pi)[u_\pi(t) - u_{2\pi}(t)].$$

Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t-\pi)\,u_\pi(t) - (t-2\pi)\,u_{2\pi}(t) - \pi\,u_{2\pi}(t)\,.$$

It follows that

$$\mathcal{L}[\,f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

10. It follows directly from the *translation property* of the transform that

$$\mathcal{L}[\,f(t)] = \frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s}.$$

11. Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t-2)\,u_2(t) - \,u_2(t) - (t-3)\,u_3(t) - u_3(t)\,.$$

It follows that

$$\mathcal{L}[\,f(t)] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

12. It follows directly from the $translation\, property$ of the transform that

$$\mathcal{L}[\,f(t)] = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

13. Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] = t^3 e^{2t}\,.$$

15. First consider the function

$$G(s) = \frac{2(s-1)}{s^2 - 2s + 2}\,.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2 + 1}\,.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = 2\,e^t \,cos\, t\,.$$

Hence

$$\mathcal{L}^{-1}\left[e^{-2s}G(s)\right] = 2\,e^{(t-2)} cos\,(t-2)\,u_2(t)\,.$$

16. The *inverse transform* of the function $2/(s^2-4)$ is $f(t) = sinh\,2t$. Using the $translation\, property$ of the transform,

$$\mathcal{L}^{-1}\left[\frac{2\,e^{-2s}}{s^2-4}\right] = sinh\,2(t-2)\cdot\,u_2(t)\,.$$

17. First consider the function

$$G(s) = \frac{(s-2)}{s^2 - 4s + 3}\,.$$

Completing the square in the denominator,

$$G(s) = \frac{(s-2)}{(s-2)^2 - 1}\,.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = e^{2t}\,cosh\,t\,.$$

Hence

$$\mathcal{L}^{-1}\left[\frac{(s-2)e^{-s}}{s^2-4s+3}\right] = e^{2(t-1)}cosh\,(t-1)\,u_1(t)\,.$$

18. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}\,.$$

It follows from the $translation\,property$ of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}\right] = u_1(t)+u_2(t)-u_3(t)-u_4(t)\,.$$

19(a). By definition of the Laplace transform,

$$\mathcal{L}[\,f(ct)] = \int_0^\infty e^{-st}f(ct)dt\,.$$

Making a change of variable, $\tau = ct$, we have

$$\begin{aligned}\mathcal{L}[\,f(ct)] &= \frac{1}{c}\int_0^\infty e^{-s(\tau/c)}f(\tau)d\tau\\ &= \frac{1}{c}\int_0^\infty e^{-(s/c)\tau}f(\tau)d\tau\,.\end{aligned}$$

Hence $\mathcal{L}[\,f(ct)] = \frac{1}{c}\,F\left(\frac{s}{c}\right)$, where $s/c > a$.

(b). Using the result in Part (a),

$$\mathcal{L}\left[\,f\left(\frac{t}{k}\right)\right] = k\,F(ks).$$

Hence

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k}\,f\left(\frac{t}{k}\right).$$

(c). From Part (b),

$$\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Note that $as + b = a(s + b/a)$. Using the fact that $\mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)]_{s \to s-c}$,

$$\mathcal{L}^{-1}[F(as + b)] = e^{-bt/a}\, \frac{1}{a} f\left(\frac{t}{a}\right).$$

20. First write

$$F(s) = \frac{n!}{\left(\frac{s}{2}\right)^{n+1}}.$$

Let $G(s) = n!/s^{n+1}$. Based on the results in Prob. 19,

$$\frac{1}{2}\,\mathcal{L}^{-1}\left[G\left(\frac{s}{2}\right)\right] = g(2t),$$

in which $g(t) = t^n$. Hence

$$\mathcal{L}^{-1}[F(s)] = 2\,(2t)^n = 2^{n+1} t^n.$$

23. First write

$$F(s) = \frac{e^{-4(s-1/2)}}{2(s - 1/2)}.$$

Now consider

$$G(s) = \frac{e^{-2s}}{s}.$$

Using the result in Prob. $19(b)$,

$$\mathcal{L}^{-1}[G(2s)] = \frac{1}{2} g\left(\frac{t}{2}\right),$$

in which $g(t) = u_2(t)$. Hence $\mathcal{L}^{-1}[G(2s)] = \frac{1}{2}\, u_2(t/2) = \frac{1}{2}\, u_4(t)$. It follows that

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2} e^{t/2}\, u_4(t).$$

24. By definition of the Laplace transform,

$$\mathcal{L}[\,f(t)] = \int_0^\infty e^{-st} u_1(t) dt\,.$$

That is,

$$\begin{aligned}\mathcal{L}[\,f(t)] &= \int_0^1 e^{-st}\,dt \\ &= \frac{1-e^{-s}}{s}\,.\end{aligned}$$

25. First write the function as $f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t)$. It follows that

$$\mathcal{L}[\,f(t)] = \int_0^1 e^{-st} dt + \int_2^3 e^{-st} dt\,.$$

That is,

$$\begin{aligned}\mathcal{L}[\,f(t)] &= \frac{1-e^{-s}}{s} + \frac{e^{-2s}-e^{-3s}}{s} \\ &= \frac{1-e^{-s}+e^{-2s}-e^{-3s}}{s}\,.\end{aligned}$$

26. The transform may be computed directly. On the other hand, using the *translation property* of the transform,

$$\begin{aligned}\mathcal{L}[f(t)] &= \frac{1}{s} + \sum_{k=1}^{2n+1} (\,-1)^k\,\frac{e^{-ks}}{s} \\ &= \frac{1}{s}\left[\sum_{k=0}^{2n+1} (\,-e^{-s})^k\right] \\ &= \frac{1}{s}\,\frac{1-(\,-e^{-s})^{2n+2}}{1+e^{-s}}\,.\end{aligned}$$

That is,

$$\mathcal{L}[f(t)] = \frac{1-(e^{-2s})^{n+1}}{s(1+e^{-s})}\,.$$

29. The given function is *periodic*, with $T = 2$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t) dt = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st} dt\,.$$

That is,

$$\begin{aligned}\mathcal{L}[f(t)] &= \frac{1-e^{-s}}{s(1-e^{-2s})} \\ &= \frac{1}{s(1+e^{-s})}\,.\end{aligned}$$

31. The function is *periodic*, with $T = 1$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-s}}\int_0^1 t\,e^{-st}dt\,.$$

It follows that

$$\mathcal{L}[f(t)] = \frac{1-e^{-s}(1+s)}{s^2(1-e^{-s})}\,.$$

32. The function is *periodic*, with $T = \pi$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-\pi s}}\int_0^\pi sin\,t\cdot e^{-st}dt\,.$$

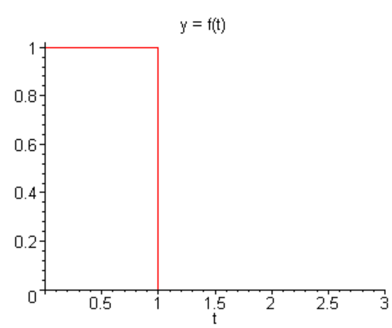
We first calculate

$$\int_0^\pi sin\,t\cdot e^{-st}dt = \frac{1+e^{-\pi s}}{1+s^2}\,.$$

Hence

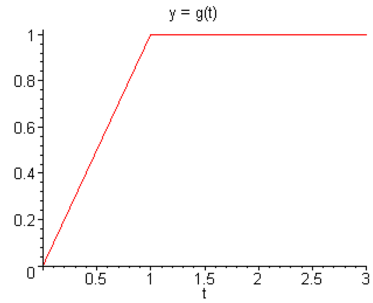
$$\mathcal{L}[f(t)] = \frac{1+e^{-\pi s}}{(1-e^{-\pi s})(1+s^2)}\,.$$

33(a).



$$\begin{aligned}\mathcal{L}[f(t)] &= \mathcal{L}[1] - \mathcal{L}[u_1(t)] \\ &= \frac{1}{s} - \frac{e^{-s}}{s}\,.\end{aligned}$$

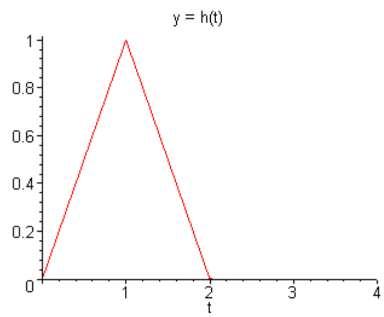
(b).



Let $F(s) = \mathcal{L}[1 - u_1(t)]$. Then

$$\mathcal{L}\left[\int_0^t [1 - u_1(\tau)]d\tau\right] = \frac{1}{s}\,F(s) = \frac{1 - e^{-s}}{s^2}\,.$$

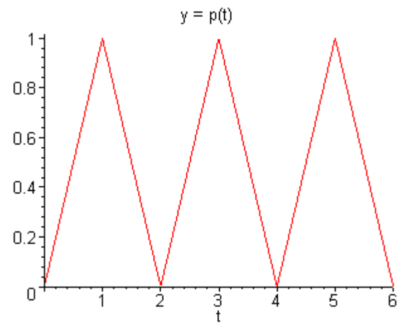
(c).



Let $G(s) = \mathcal{L}[g(t)]$. Then

$$\begin{aligned}\mathcal{L}[h(t)] &= G(s) - e^{-s}\,G(s)\\ &= \frac{1 - e^{-s}}{s^2} - e^{-s}\,\frac{1 - e^{-s}}{s^2}\\ &= \frac{(1 - e^{-s})^2}{s^2}\,.\end{aligned}$$

$34(a)$.



(b). The given function is *periodic*, with $T = 2$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} p(t) dt \,.$$

Based on the piecewise definition of $p(t)$,

$$\begin{aligned} \int_0^2 e^{-st} p(t) dt &= \int_0^1 t\, e^{-st} dt + \int_1^2 (2-t) e^{-st} dt \\ &= \frac{1}{s^2} (1 - e^{-s})^2. \end{aligned}$$

Hence

$$\mathcal{L}[p(t)] = \frac{(1 - e^{-s})}{s^2 (1 + e^{-s})} \,.$$

(c). Since $p(t)$ satisfies the hypotheses of Theorem 6.2.1 ,

$$\mathcal{L}[p'(t)] = s\, \mathcal{L}[p(t)] - p(0) \,.$$

Using the result of Prob. 30,

$$\mathcal{L}[p'(t)] = \frac{(1 - e^{-s})}{s(1 + e^{-s})} \,.$$

We note the $p(0) = 0$, hence

$$\mathcal{L}[p(t)] = \frac{1}{s} \left[\frac{(1 - e^{-s})}{s(1 + e^{-s})} \right].$$