Section 3.5

2. The characteristic equation is $9r^2 + 6r + 1 = 0$, with the *double* root $r = -1/3$. Based on the discussion in this section, the general solution is $y(t) = c_1 e^{-t/3} + c_2 t\, e^{-t/3}$.

3. The characteristic equation is $4r^2 - 4r - 3 = 0$, with roots $r = -1/2\,, 3/2$. The general solution is $y(t) = c_1 e^{-t/2} + c_2 e^{3t/2}$.

4. The characteristic equation is $4r^2 + 12r + 9 = 0$, with the *double* root $r = -3/2$. Based on the discussion in this section, the general solution is $y(t) = (c_1 + c_2\, t)e^{-3t/2}$.

5. The characteristic equation is $r^2 - 2r + 10 = 0$, with complex roots $r = 1 \pm 3i$. The general solution is $y(t) = c_1 e^t cos\, 3t + c_2 e^t sin\, 3t$.

6. The characteristic equation is $r^2 - 6r + 9 = 0$, with the *double* root $r = 3$. The general solution is $y(t) = c_1 e^{3t} + c_2 t\, e^{3t}$.

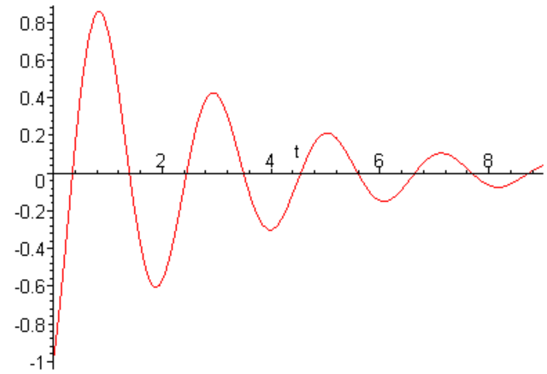
7. The characteristic equation is $4r^2 + 17r + 4 = 0$, with roots $r = -1/4\,, -4$. The general solution is $y(t) = c_1 e^{-t/4} + c_2 e^{-4t}$.

8. The characteristic equation is $16r^2 + 24r + 9 = 0$, with the *double* root $r = -3/4$. The general solution is $y(t) = c_1 e^{-3t/4} + c_2 t\, e^{-3t/4}$.

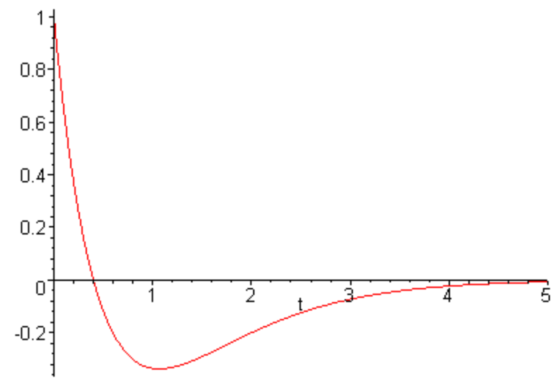
10. The characteristic equation is $2r^2 + 2r + 1 = 0$, with complex roots $r = -\frac{1}{2} \pm \frac{1}{2}\, i$. The general solution is $y(t) = c_1 e^{-t/2} cos\, t/2 + c_2 e^{-t/2} sin\, t/2$.

11. The characteristic equation is $9r^2 - 12r + 4 = 0$, with the *double* root $r = 2/3$. The general solution is $y(t) = c_1 e^{2t/3} + c_2 t\, e^{2t/3}$. Invoking the first initial condition, it follows that $c_1 = 2$. Now $y'(t) = (4/3 + c_2)e^{2t/3} + 2c_2 t\, e^{2t/3}/3$. Invoking the second initial condition, $4/3 + c_2 = -1$, or $c_2 = -7/3$. Hence $y(t) = 2e^{2t/3} - \frac{7}{3} t\, e^{2t/3}$. Since the *second* term dominates for large t, $y(t) \to -\infty$.

13. The characteristic equation is $9r^2 + 6r + 82 = 0$, with complex roots $r = -\frac{1}{3} \pm 3\, i$. The general solution is $y(t) = c_1 e^{-t/3} cos\, 3t + c_2 e^{-t/3} sin\, 3t$. Based on the first initial condition, $c_1 = -1$. Invoking the second initial condition, $1/3 + 3c_2 = 2$, or $c_2 = \frac{5}{9}$. Hence $y(t) = -e^{-t/3} cos\, 3t + \frac{5}{9} e^{-t/3} sin\, 3t$.



$15(a)$. The characteristic equation is $4r^2 + 12r + 9 = 0$, with the *double* root $r = -\frac{3}{2}$. The general solution is $y(t) = c_1 e^{-3t/2} + c_2 t\, e^{-3t/2}$. Invoking the first initial condition, it follows that $c_1 = 1$. Now $y'(t) = (-3/2 + c_2)e^{2t/3} - \frac{3}{2}c_2 t\, e^{2t/3}$. The second initial condition requires that $-3/2 + c_2 = -4$, or $c_2 = -5/2$. Hence the specific solution is $y(t) = e^{-3t/2} - \frac{5}{2}t\, e^{-3t/2}$.



(b). The solution crosses the *x-axis* at $t = 0.4$.

(c). The solution has a minimum at the point $(16/15\,,\, -5e^{-8/5}/3)$.

(d). Given that $y'(0) = b$, we have $-3/2 + c_2 = b$, or $c_2 = b + 3/2$. Hence the solution is $y(t) = e^{-3t/2} + \left(b + \frac{3}{2}\right)t\, e^{-3t/2}$. Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $b + \frac{3}{2}$. The critical value is $b = -\frac{3}{2}$.

16. The characteristic roots are $r_1 = r_2 = 1/2$. Hence the general solution is given by $y(t) = c_1 e^{t/2} + c_2 t\, e^{t/2}$. Invoking the initial conditions, we require that $c_1 = 2$, and that $1 + c_2 = b$. The specific solution is $y(t) = 2e^{t/2} + (b - 1)t\, e^{t/2}$. Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $b - 1$. The critical value is $b = 1$.

$18(a)$. The characteristic roots are $r_1 = r_2 = -2/3$. Therefore the general solution is given by $y(t) = c_1 e^{-2t/3} + c_2 t\, e^{-2t/3}$. Invoking the initial conditions, we require that $c_1 = a$, and that $-2a/3 + c_2 = -1$. After solving for the coefficients, the specific solution is $y(t) = ae^{-2t/3} + \left(\frac{2a}{3} - 1\right) t\, e^{-2t/3}$.

(b). Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $\frac{2a}{3} - 1$. The critical value is $a = 3/2$.

$20(a)$. The characteristic equation is $r^2 + 2ar + a^2 = 0$, with *double* root $r = -a$. Hence one solution is $y_1(t) = c_1 e^{-at}$.

(b). Recall that the Wronskian satisfies the differential equation $W' + 2aW = 0$. The solution of this equation is $W(t) = c\, e^{-2at}$.

(c). By definition, $W = y_1 y_2' - y_1' y_2$. Hence $c_1 e^{-at} y_2' + ac_1 e^{-at} y_2 = c\, e^{-2at}$. That is, $y_2' + a\, y_2 = c_2 e^{-at}$. This equation is first order *linear*, with general solution $y_2(t) = c_2 te^{-at} + c_3 e^{-at}$. Setting $c_2 = 1$ and $c_3 = 0$, we obtain $y_2(t) = te^{-at}$.

$22(a)$. Write $ar^2 + br + c = a\left(r^2 + \frac{b}{a} r + \frac{c}{a}\right)$. It follows that $\frac{b}{a} = -2r_1$ and $\frac{c}{a} = r_1^2$. Hence $ar^2 + br + c = ar^2 - 2ar_1 r + ar_1^2 = a(r^2 - 2r_1 r + r_1^2) = a(r - r_1)^2$. We find that $L[e^{rt}] = (ar^2 + br + c)e^{rt} = a(r - r_1)^2 e^{rt}$. Setting $r = r_1$, $L[e^{r_1 t}] = 0$.

(b). Differentiating Eq.(i) with respect to r,

$$\frac{\partial}{\partial r} L\left[e^{rt}\right] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1).$$

Now observe that

$$\begin{aligned}
\frac{\partial}{\partial r} L\left[e^{rt}\right] &= \frac{\partial}{\partial r}\left[a\frac{\partial^2}{\partial t^2}\left(e^{rt}\right) + b\frac{\partial}{\partial t}\left(e^{rt}\right) + c\left(e^{rt}\right)\right] \\
&= \left[a\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial r}e^{rt}\right) + b\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r}e^{rt}\right) + c\left(\frac{\partial}{\partial r}e^{rt}\right)\right] \\
&= a\frac{\partial^2}{\partial t^2}\left(te^{rt}\right) + b\frac{\partial}{\partial t}\left(te^{rt}\right) + c\left(te^{rt}\right).
\end{aligned}$$

Hence $L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1)$. Setting $r = r_1$, $L[te^{r_1 t}] = 0$.

23. Set $y_2(t) = t^2\, v(t)$. Substitution into the ODE results in

$$t^2\left(t^2 v'' + 4tv' + 2v\right) - 4t\left(t^2 v' + 2tv\right) + 6t^2 v = 0.$$

After collecting terms, we end up with $t^4 v'' = 0$. Hence $v(t) = c_1 + c_2 t$, and thus $y_2(t) = c_1 t^2 + c_2 t^3$. Setting $c_1 = 0$ and $c_2 = 1$, we obtain $y_2(t) = t^3$.

24. Set $y_2(t) = t\, v(t)$. Substitution into the ODE results in

$$t^2(tv'' + 2v') + 2t(tv' + v) - 2tv = 0\,.$$

After collecting terms, we end up with $t^3v'' + 4t^2v' = 0$. This equation is *linear* in the variable $w = v'$. It follows that $v'(t) = c\,t^{-4}$, and $v(t) = c_1t^{-3} + c_2$. Thus $y_2(t) = c_1t^{-2} + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = t^{-2}$.

26. Set $y_2(t) = t\,v(t)$. Substitution into the ODE results in $v'' - v' = 0$. This ODE is *linear* in the variable $w = v'$. It follows that $v'(t) = c_1e^t$, and $v(t) = c_1e^t + c_2$. Thus $y_2(t) = c_1te^t + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = te^t$.

28. Set $y_2(x) = e^xv(x)$. Substitution into the ODE results in

$$v'' + \frac{x-2}{x-1}v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is

$$\begin{aligned}\mu &= exp\left(\int \frac{x-2}{x-1}dx\right)\\ &= \frac{e^x}{x-1}\,.\end{aligned}$$

Rewrite the equation as $\left[\frac{e^xv'}{x-1}\right]' = 0$, from which it follows that $v'(x) = c(x-1)e^{-x}$. Hence $v(x) = c_1xe^{-x} + c_2$ and $y_2(x) = c_1x + c_2e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

29. Set $y_2(x) = y_1(x)\,v(x)$, in which $y_1(x) = x^{1/4}exp\big(2\sqrt{x}\big)$. It can be verified that y_1 is a solution of the ODE, that is, $x^2y_1'' - (x - 0.1875)y_1 = 0$. Substitution of the given form of y_2 results in the differential equation

$$2x^{9/4}v'' + \big(4x^{7/4} + x^{5/4}\big)v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is

$$\begin{aligned}\mu &= exp\left(\int \left[2x^{-1/2} + \frac{1}{2x}\right]dx\right)\\ &= \sqrt{x}\;exp\big(4\sqrt{x}\big).\end{aligned}$$

Rewrite the equation as $\left[\sqrt{x}\;exp\big(4\sqrt{x}\big)\;v'\right]' = 0$, from which it follows that

$$v'(x) = c\,exp\big(-4\sqrt{x}\big)/\sqrt{x}\,.$$

Integrating, $v(x) = c_1exp\big(-4\sqrt{x}\big) + c_2$ and as a result,

$$y_2(x) = c_1x^{1/4}exp\big(-2\sqrt{x}\big) + c_2x^{1/4}exp\big(2\sqrt{x}\big).$$

Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{1/4}exp\big(-2\sqrt{x}\big)$.

32. Direct substitution verifies that $y_1(t) = exp(-\delta x^2/2)$ is a solution of the ODE. Now set $y_2(x) = y_1(x)\,v(x)$. Substitution of y_2 into the ODE results in

$$v'' - \delta x v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is $\mu = exp(-\delta x^2/2)$. Rewrite the equation as $[\,exp(-\delta x^2/2)v']' = 0$, from which it follows that

$$v'(x) = c_1\,exp\big(\delta x^2/2\big)\,.$$

Integrating, we obtain

$$v(x) = c_1\int_{x_0}^{x} exp\big(\delta u^2/2\big)du + v(x_0).$$

Hence

$$y_2(x) = c_1 exp\big(-\delta x^2/2\big)\int_{x_0}^{x} exp\big(\delta u^2/2\big)du + c_2 exp\big(-\delta x^2/2\big).$$

Setting $c_2 = 0$, we obtain a second independent solution.

34. After writing the ODE in standard form, we have $p(t) = 3/t$. Based on *Abel's identity*, $W(y_1, y_2) = c_1 exp\big(-\int \frac{3}{t}dt\big) = c_1 t^{-3}$. As shown in Prob. 33, two solutions of a second order linear equation satisfy

$$(y_2/y_1)' = W(y_1, y_2)/y_1^2\,.$$

In the given problem, $y_1(t) = t^{-1}$. Hence $(t\,y_2)' = c_1 t^{-1}$. Integrating both sides of the equation, $y_2(t) = c_1 t^{-1} ln\,t + c_2 t^{-1}$.

36. After writing the ODE in standard form, we have $p(x) = -x/(x-1)$. Based on *Abel's identity*, $W(y_1, y_2) = c\ exp\big(\int \frac{x}{x-1}dx\big) = c\ e^x(x-1)$. Two solutions of a second order linear equation satisfy

$$(y_2/y_1)' = W(y_1, y_2)/y_1^2\,.$$

In the given problem, $y_1(x) = e^x$. Hence $(e^{-x}\,y_2)' = c\,e^{-x}(x-1)$. Integrating both sides of the equation, $y_2(x) = c_1 x + c_2 e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

37. Write the ODE in standard form to find $p(x) = 1/x$. Based on *Abel's identity*, $W(y_1, y_2) = c\,exp\big(-\int \frac{1}{x}dx\big) = c\,x^{-1}$. Two solutions of a second order linear ODE satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(x) = x^{-1/2} sin\,x$. Hence

$$\left(\frac{\sqrt{x}}{sin\,x}\,y_2\right)' = c\,\frac{1}{sin^2 x}\,.$$

Integrating both sides of the equation, $y_2(x) = c_1 x^{-1/2} cos\, x + c_2 x^{-1/2} sin\, x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{-1/2} cos\, x$.

39(a). The characteristic equation is $ar^2 + c = 0$. If $a\,, c > 0$, then the roots are $r_{1,2} = \pm\, i\sqrt{c/a}$. The general solution is

$$y(t) = c_1 cos\sqrt{\frac{c}{a}}\, t + c_2 sin\sqrt{\frac{c}{a}}\, t\,,$$

which is bounded.

(b). The characteristic equation is $ar^2 + br = 0$. The roots are $r_{1,2} = 0\,,\ - b/a\,,$ and hence the general solution is $y(t) = c_1 + c_2 exp(-bt/a)$. Clearly, $y(t) \to c_1$.

40. Note that $cos\, t\, sin\, t = \frac{1}{2} sin\, 2t$. So that $1 - k\, cos\, t\, sin\, t = 1 - \frac{k}{2} sin\, 2t$. If $0 < k < 2$, then $\frac{k}{2} sin\, 2t < |sin\, 2t|$ and $-\frac{k}{2} sin\, 2t > -|sin\, 2t|$. Hence

$$\begin{aligned} 1 - k\, cos\, t\, sin\, t &= 1 - \frac{k}{2} sin\, 2t \\ &> 1 - |sin\, 2t| \\ &\geq 0\,. \end{aligned}$$

41. $p(t) = -3/t$ and $q(t) = 4/t^2$. We have $x = 2\int t^{-1} dt = 2\, ln\, t$, and $t = e^{x/2}$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = -2\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0\,.$$

The general solution of this ODE is $y(x) = c_1 e^x + c_2 x e^x$. In terms of the original independent variable, $y(t) = c_1 t^2 + c_2 t^2\, ln\, t$.