$$e^t v'' + \left[2e^t + t(1-t)^{-1}e^t\right]v' = 2(1-t)\,e^{-t},$$

that is, $v'' + [(2-t)/(1-t)]v' = 2(1-t)\,e^{-2t}$. This equation is first order linear in v', with integrating factor $\mu = e^t/(t-1)$. The solution is

$$v' = (t-1)\left(2e^{-2t} + c_1 e^{-t}\right).$$

Integrating, we obtain $v(t) = (1/2 - t)e^{-2t} - c_1 te^{-t} + c_2$. Hence the solution of the original ODE is $y(t) = (1/2 - t)e^{-t} - c_1 t + c_2 e^t$.

Section 3.8

1. $R\cos\delta = 3$ and $R\sin\delta = 4 \Rightarrow R = \sqrt{25} = 5$ and $\delta = arctan(4/3)$. Hence

$$u = 5\,cos(2t - 0.9273).$$

3. $R\cos\delta = 4$ and $R\sin\delta = -2 \Rightarrow R = \sqrt{20} = 2\sqrt{5}$ and $\delta = -arctan(1/2)$. Hence

$$u = 2\sqrt{5}\,cos(3t + 0.4636).$$

4. $R\cos\delta = -2$ and $R\sin\delta = -3 \Rightarrow R = \sqrt{13}$ and $\delta = \pi + arctan(3/2)$. Hence

$$u = \sqrt{13}\,cos(\pi t - 4.1244).$$

5. The spring constant is $k = 2/(1/2) = 4$ *lb/ft*. Mass $m = 2/32 = 1/16$ *lb-s*^{2}*/ft*. Since there is no damping, the equation of motion is

$$\frac{1}{16}u'' + 4u = 0\,,$$

that is, $u'' + 64u = 0$. The initial conditions are $u(0) = 1/4\,ft$, $u'(0) = 0\,fps$. The general solution is $u(t) = A\,cos\,8t + B\,sin\,8t$. Invoking the initial conditions, we have $u(t) = \frac{1}{4}\,cos\,8t$. $R = 3$ *inches*, $\delta = 0$ *rad*, $\omega_0 = 8$ *rad/s*, and $T = \pi/4$ *sec*.

7. The spring constant is $k = 3/(1/4) = 12$ *lb/ft*. Mass $m = 3/32$ *lb-s*^{2}*/ft*. Since there is no damping, the equation of motion is

$$\frac{3}{32}u'' + 12u = 0\,,$$

that is, $u'' + 128u = 0$. The initial conditions are $u(0) = -1/12\,ft$, $u'(0) = 2\,fps$. The general solution is $u(t) = A\,cos\,8\sqrt{2}\,t + B\,sin\,8\sqrt{2}\,t$. Invoking the initial conditions, we have

$$u(t) = -\frac{1}{12}\,cos\,8\sqrt{2}\,t + \frac{1}{4\sqrt{2}}\,sin\,8\sqrt{2}\,t\,.$$

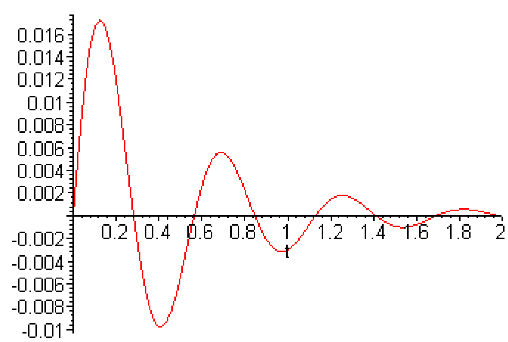
$R = \sqrt{11}/12\,ft$, $\delta = \pi - atan\left(3/\sqrt{2}\right)\,rad$, $\omega_0 = 8\sqrt{2}\;rad/s$, and $T = \pi/\left(4\sqrt{2}\right)\,sec$.

10. The spring constant is $k = 16/(1/4) = 64\;lb/ft$. Mass $m = 1/2\;lb\text{-}s^2/ft$. The damping coefficient is $\gamma = 2\;lb\text{-}sec/ft$. Hence the equation of motion is

$$\frac{1}{2}u'' + 2u' + 64u = 0\,,$$

that is, $u'' + 4u' + 128u = 0$. The initial conditions are $u(0) = 0\,ft$, $u'(0) = 1/4\,fps$. The general solution is $u(t) = A\,cos\,2\sqrt{31}\,t + B\,sin\,2\sqrt{31}\,t$. Invoking the initial conditions, we have

$$u(t) = \frac{1}{8\sqrt{31}}e^{-2t}\,sin\,2\sqrt{31}\,t\,.$$



Solving $u(t) = 0$, on the interval $[0.2\,,\,0.4]$, we obtain $t = \pi/2\sqrt{31} = 0.2821\;sec$. Based on the graph, and the solution of $u(t) = 0.01$, we have $|u(t)| \leq 0.01$ for $t \geq \tau = 0.2145$.

11. The spring constant is $k = 3/(.1) = 30\;N/m$. The damping coefficient is given as $\gamma = 3/5\;N\text{-}sec/m$. Hence the equation of motion is

$$2u'' + \frac{3}{5}u' + 30u = 0\,,$$

that is, $u'' + 0.3u' + 15u = 0$. The initial conditions are $u(0) = 0.05\;m$ and $u'(0) = 0.01\,m/s$. The general solution is $u(t) = A\,cos\,\mu t + B\,sin\,\mu t$, in which $\mu = 3.87008\;rad/s$. Invoking the initial conditions, we have

$$u(t) = e^{-0.15t}(0.05cos\,\mu t + 0.00452sin\,\mu t)\,.$$

Also, $\mu/\omega_0 = 3.87008/\sqrt{15} \approx 0.99925$.

13. The frequency of the *undamped* motion is $\omega_0 = 1$. The quasi frequency of the damped
motion is $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$. Setting $\mu = \frac{2}{3}\omega_0$, we obtain $\gamma = \frac{2}{3}\sqrt{5}$.

14. The spring constant is $k = mg/L$. The equation of motion for an undamped system is

$$mu'' + \frac{mg}{L}u = 0\,.$$

Hence the natural frequency of the system is $\omega_0 = \sqrt{\frac{g}{L}}$. The period is $T = 2\pi/\omega_0$.

15. The general solution of the system is $u(t) = A\,cos\,\gamma(t-t_0) + B\,sin\,\gamma(t-t_0)$. Invoking the initial conditions, we have $u(t) = u_0 cos\gamma(t-t_0) + (u_0'/\gamma)sin\gamma(t-t_0)$. Clearly, the functions $v = u_0 cos\gamma(t-t_0)$ and $w = (u_0'/\gamma)sin\gamma(t-t_0)$ satisfy the given criteria.

16. Note that $r\,sin(\,\omega_0 t - \theta) = r\,sin\omega_0 t\,cos\,\theta - r\,cos\omega_0 t\,sin\theta$. Comparing the given expressions, we have $A = -r\,sin\,\theta$ and $B = r\,cos\,\theta$. That is, $r = R = \sqrt{A^2+B^2}$, and $tan\,\theta = -A/B = -1/tan\,\delta$. The latter relation is also $tan\,\theta + cot\,\delta = 1$.

18. The system is *critically damped*, when $R = 2\sqrt{L/C}$. Here $R = 1000$ *ohms*.

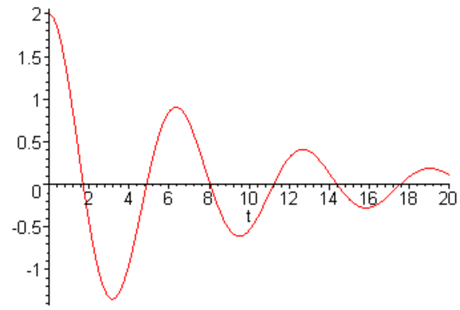
$21(a)$. Let $u = Re^{-\gamma t/2m}cos(\mu t - \delta)$. Then attains a *maximum* when $\mu t_k - \delta = 2k\pi$. Hence $T_d = t_{k+1} - t_k = 2\pi/\mu$.

(b). $u(t_k)/u(t_{k+1}) = exp(-\gamma t_k/2m)/exp(-\gamma t_{k+1}/2m) = exp[(\gamma t_{k+1} - \gamma t_k)/2m]$. Hence $u(t_k)/u(t_{k+1}) = exp[\gamma(2\pi/\mu)/2m] = exp(\gamma T_d/2m)$.

(c). $\Delta = ln[u(t_k)/u(t_{k+1})] = \gamma(2\pi/\mu)/2m = \pi\gamma/\mu m$.

22. The spring constant is $k = 16/(1/4) = 64$ *lb/ft*. Mass $m = 1/2$ *lb-s²/ft*. The damping coefficient is $\gamma = 2$ *lb-sec/ft*. The quasi frequency is $\mu = 2\sqrt{31}$ *rad/s*. Hence $\Delta = \frac{2\pi}{\sqrt{31}} \approx 1.1285$.

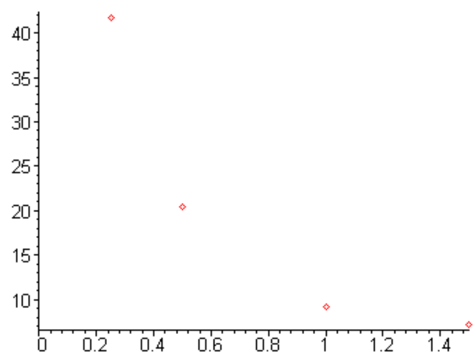
$25(a)$. The solution of the IVP is $u(t) = e^{-t/8}\left(2\,cos\,\frac{3}{8}\sqrt{7}\,t + 0.252 sin\,\frac{3}{8}\sqrt{7}\,t\right)$.



Using the plot, and numerical analysis, $\tau \approx 41.715$.

(b). For $\gamma = 0.5$, $\tau \approx 20.402$; for $\gamma = 1.0$, $\tau \approx 9.168$; for $\gamma = 1.5$, $\tau \approx 7.184$.

(c).



(d). For $\gamma = 1.6$, $\tau \approx 7.218$; for $\gamma = 1.7$, $\tau \approx 6.767$; for $\gamma = 1.8$, $\tau \approx 5.473$; for $\gamma = 1.9$, $\tau \approx 6.460$. τ steadily decreases to about $\tau_{min} \approx 4.873$, corresponding to the critical value $\gamma_0 \approx 1.73$.

(e). We have $u(t) = \frac{4e^{-\gamma t/2}}{\sqrt{4-\gamma^2}} cos(\mu t - \delta)$, in which $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$, and $\delta = tan^{-1}\frac{\gamma}{\sqrt{4-\gamma^2}}$. Hence $|u(t)| \leq \frac{4e^{-\gamma t/2}}{\sqrt{4-\gamma^2}}$.

26(a). The characteristic equation is $mr^2 + \gamma r + k = 0$. Since $\gamma^2 < 4km$, the roots are $r_{1,2} = -\frac{\gamma}{2m} \pm i\frac{\sqrt{4mk-\gamma^2}}{2m}$. The general solution is

$$u(t) = e^{-\gamma t/2m}\left[A\,cos\frac{\sqrt{4mk-\gamma^2}}{2m}t + B\,sin\frac{\sqrt{4mk-\gamma^2}}{2m}t\right].$$

Invoking the initial conditions, $A = u_0$ and

$$B = \frac{(2mv_0 - \gamma u_0)}{\sqrt{4mk-\gamma^2}}\,.$$

(b). We can write $u(t) = R\,e^{-\gamma t/2m} cos(\mu t - \delta)$, in which

$$R = \sqrt{u_0^2 + \frac{(2mv_0 - \gamma u_0)^2}{4mk - \gamma^2}}\,,$$

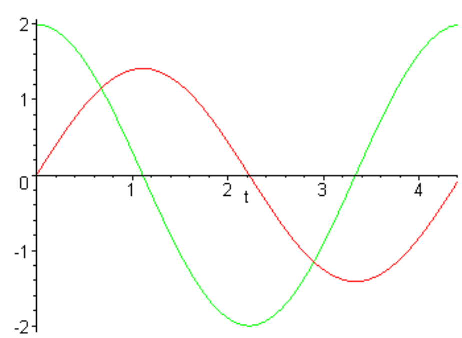
and

$$\delta = arctan\left[\frac{(2mv_0 - \gamma u_0)}{u_0\sqrt{4mk - \gamma^2}}\right].$$

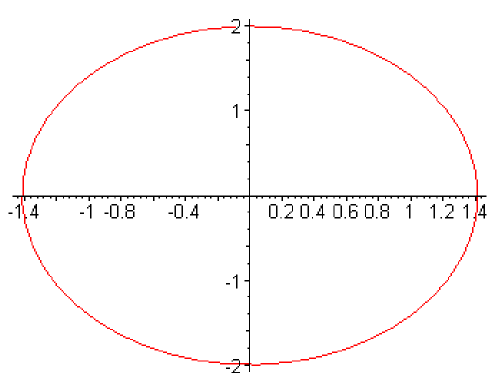
(c). $R = \sqrt{u_0^2 + \frac{(2mv_0-\gamma u_0)^2}{4mk-\gamma^2}} = 2\sqrt{\frac{m(ku_0^2+\gamma u_0 v_0+mv_0^2)}{4mk-\gamma^2}} = \sqrt{\frac{a+b\gamma}{4mk-\gamma^2}}$.
It is evident that R increases (*monotonically*) without bound as $\gamma \to \left(2\sqrt{mk}\right)^-$.

28(a). The general solution is $u(t) = Acos\sqrt{2}\,t + Bsin\sqrt{2}\,t$. Invoking the initial conditions, we have $u(t) = \sqrt{2}\ sin\sqrt{2}\ t$.

(b).

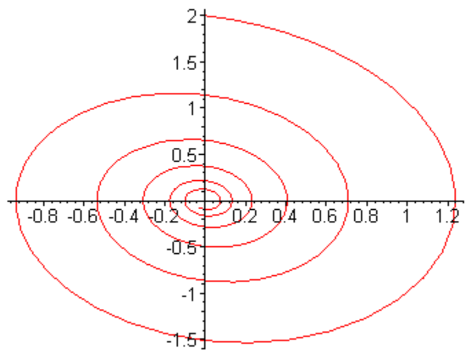


(c).



The condition $u'(0) = 2$ implies that $u(t)$ *initially* increases. Hence the phase point travels *clockwise*.

29. $u(t) = \frac{16}{\sqrt{127}} e^{-t/8} sin\, \frac{\sqrt{127}}{8} t$.



31. Based on *Newton's second law*, with the positive direction to the right,

$$\sum F = mu''$$

where

$$\sum F = -ku - \gamma u'.$$

Hence the equation of motion is $mu'' + \gamma u' + ku = 0$. The only difference in this problem is that the equilibrium position is located at the *unstretched* configuration of the spring.

$32(a)$. The *restoring* force exerted by the spring is $F_s = -(ku + \varepsilon u^3)$. The *opposing* viscous force is $F_d = -\gamma u'$. Based on *Newton's second law*, with the positive direction to the right,

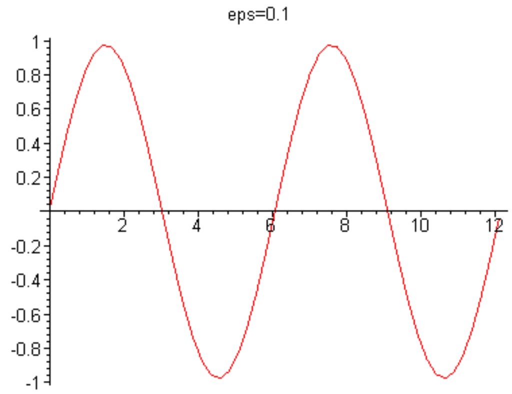
$$F_s + F_d = mu''.$$

Hence the equation of motion is $mu'' + \gamma u' + ku + \varepsilon u^3 = 0$.

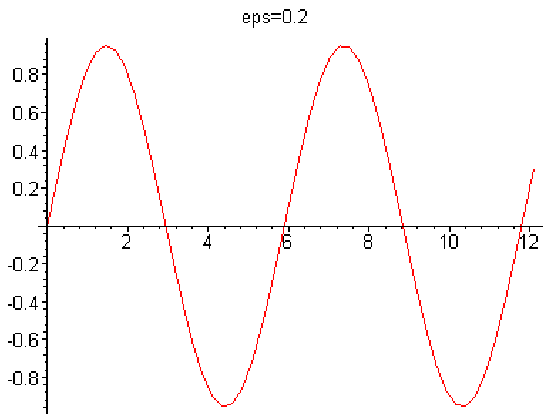
(b). With the specified parameter values, the equation of motion is $u'' + u = 0$. The general solution of this ODE is $u(t) = A\, cos\, t + B\, sin\, t$. Invoking the initial conditions,
the specific solution is $u(t) = sin\, t$. Clearly, the amplitude is $R = 1$, and the period of the motion is $T = 2\pi$.

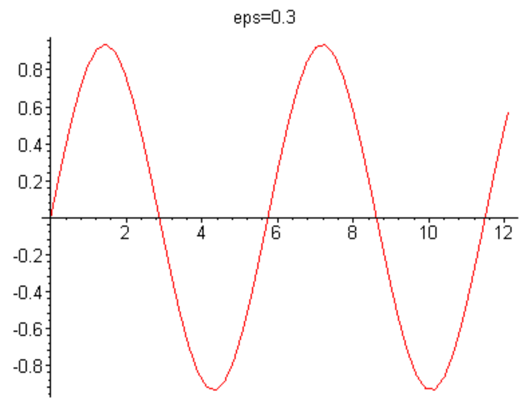
(c). Given $\varepsilon = 0.1$, the equation of motion is $u'' + u + 0.1\, u^3 = 0$. A solution of the

IVP can be generated numerically:



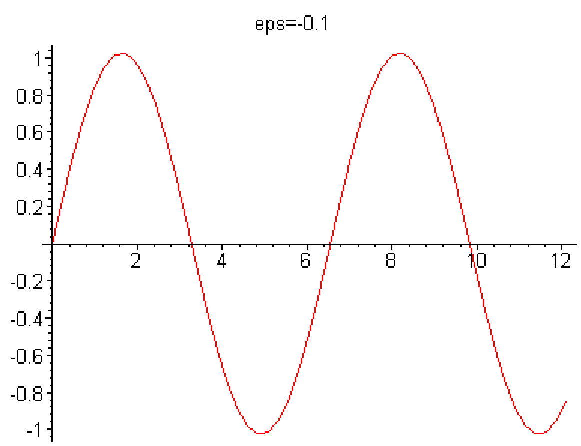
(d).

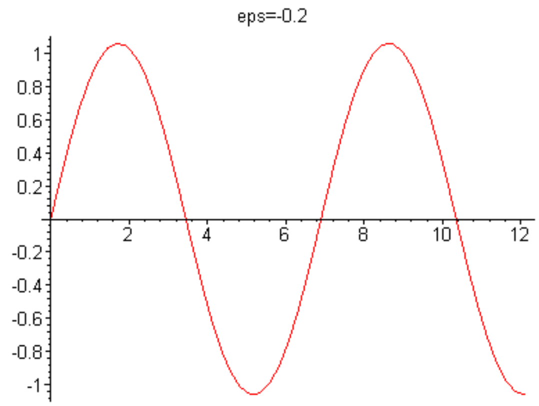




(e). The amplitude and period both seem to *decrease*.

(f).

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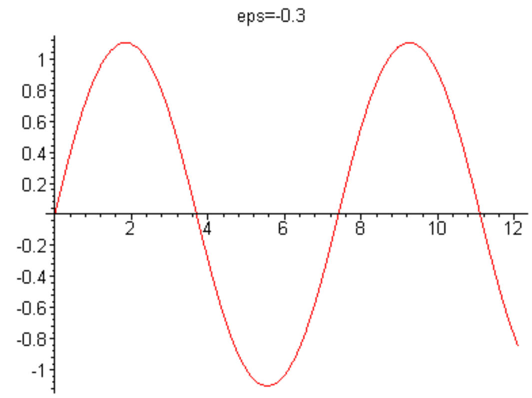
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