Section 8.2

1. The improved Euler formula for this problem is

$$y_{n+1} = y_n + h\left(3 + \frac{1}{2}t_n + \frac{1}{2}t_{n+1} - y_n\right) - \frac{h^2}{2}(3 + t_n - y_n).$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h(3 - y_n) + \frac{h^2}{2}(y_n - 2 + 2n) - \frac{nh^3}{2} \ ,$$

with $y_0 = 1$.

(a). $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	1.19512	1.38120	1.55909	1.72956

(b). $h = 0.025$:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
t_n	0.1	0.2	0.3	0.4
y_n	1.19515	1.38125	1.55916	1.72965

(c). $h = 0.0125$:

	$n = 8$	$n = 16$	$n = 24$	$n = 32$
t_n	0.1	0.2	0.3	0.4
y_n	1.19516	1.38126	1.55918	1.72967

2. The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(5\,t_n - 3\sqrt{y_n}\,\right) + \frac{h}{2}\left(5\,t_{n+1} - 3\sqrt{K_n}\,\right),$$

in which $K_n = y_n + h\left(5\,t_n - 3\sqrt{y_n}\,\right)$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \frac{h}{2}\left(5\,nh - 3\sqrt{y_n}\,\right) + \frac{h}{2}\left[5\,(n+1)h - 3\sqrt{K_n}\,\right],$$

with $y_0 = 2$.

(a). $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.62283	1.33460	1.12820	0.995445

(b). $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.62243	1.33386	1.12718	0.994215

(c). $h = 0.0125$:

	$n=8$	$n=16$	$n=24$	$n=32$
t_n	0.1	0.2	0.3	0.4
y_n	1.62234	1.33368	1.12693	0.993921

3. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}(4\,y_n - 3\,t_n - 3\,t_{n+1}) + h^2(2\,y_n - 3\,t_n).$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + 2h\,y_n + \frac{h^2}{2}(4\,y_n - 3 - 6n) - 3nh^3 \ ,$$

with $y_0 = 1$.

(a). $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	1.20526	1.42273	1.65511	1.90570

(b). $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	1.20533	1.42290	1.65542	1.90621

(c). $h = 0.0125$:

	$n=8$	$n=16$	$n=24$	$n=32$
t_n	0.1	0.2	0.3	0.4
y_n	1.20534	1.42294	1.65550	1.90634

5. The improved Euler formula is

$$y_{n+1} = y_n + h\,\frac{y_n^2 + 2\,t_n\,y_n}{2(3+t_n^2)} + h\,\frac{K_n^2 + 2\,t_{n+1}K_n}{2\big(3+t_{n+1}^2\big)},$$

in which

$$K_n = y_n + h\,\frac{y_n^2 + 2\,t_n\,y_n}{3+t_n^2}.$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h\,\frac{y_n^2 + 2nh\,y_n}{2(3+n^2h^2)} + h\,\frac{K_n^2 + 2(n+1)hK_n}{2\big[3+(n+1)^2h^2\big]},$$

with $y_0 = 0.5$.

(a). $h = 0.05$:

	$n=2$	$n=4$	$n=6$	$n=8$
t_n	0.1	0.2	0.3	0.4
y_n	0.510164	0.524126	0.54083	0.564251

(b). $h = 0.025$:

	$n=4$	$n=8$	$n=12$	$n=16$
t_n	0.1	0.2	0.3	0.4
y_n	0.510168	0.524135	0.542100	0.564277

(c). $h = 0.0125$:

	$n=8$	$n=16$	$n=24$	$n=32$
t_n	0.1	0.2	0.3	0.4
y_n	0.51069	0.524137	0.542104	0.564284

6. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}\big(t_n^2 - y_n^2\big)\sin y_n + \frac{h}{2}\big(t_{n+1}^2 - K_n^2\big)\sin K_n,$$

in which

$$K_n = y_n + h\left(t_n^2 - y_n^2\right) sin\, y_n \,.$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \frac{h}{2}\left(n^2h^2 - y_n^2\right) sin\, y_n + \frac{h}{2}\left[(n+1)^2h^2 - K_n^2\right] sin\, K_n,$$

with $y_0 = -1$.

(a). $h = 0.05$:

	$n = 2$	$n = 4$	$n = 6$	$n = 8$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924650	-0.864338	-0.816642	-0.780008

(b). $h = 0.025$:

	$n = 4$	$n = 8$	$n = 12$	$n = 16$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924550	-0.864177	-0.816442	-0.779781

(c). $h = 0.0125$:

	$n = 8$	$n = 16$	$n = 24$	$n = 32$
t_n	0.1	0.2	0.3	0.4
y_n	-0.924525	-0.864138	-0.816393	-0.779725

7. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}(4\,y_n - t_n - t_{n+1} + 1) + h^2(2\,y_n - t_n + 0.5).$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h(2\,y_n + 0.5) + h^2(2\,y_n - n) - nh^3\,,$$

with $y_0 = 1$.

(a). $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	2.96719	7.88313	20.8114	55.5106

(b). $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	2.96800	7.88755	20.8294	55.5758

8. The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}\left(5\,t_n - 3\sqrt{y_n}\,\right) + \frac{h}{2}\left(5\,t_{n+1} - 3\sqrt{K_n}\,\right),$$

in which $K_n = y_n + h\left(5\,t_n - 3\sqrt{y_n}\,\right)$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \frac{h}{2}\left(5\,nh - 3\sqrt{y_n}\,\right) + \frac{h}{2}\left[5\,(n+1)h - 3\sqrt{K_n}\right],$$

with $y_0 = 2$.

(a). $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.926139	1.28558	2.40898	4.10386

(b). $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.925815	1.28525	2.40869	4.10359

9. The improved Euler formula for this problem is

$$y_{n+1} = y_n + \frac{h}{2}\sqrt{t_n + y_n} + \frac{h}{2}\sqrt{t_{n+1} + K_n}\,,$$

in which $K_n = y_n + h\sqrt{t_n + y_n}$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \frac{h}{2}\sqrt{nh + y_n} + \frac{h}{2}\sqrt{(n+1)h + K_n}\,,$$

with $y_0 = 3$.

(a). $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	3.96217	5.10887	6.43134	7.92332

$(b).\ h = 0.0125\ :$

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	3.96218	5.10889	6.43138	7.92337

10. The improved Euler formula is

$$y_{n+1} = y_n + \frac{h}{2}[2\,t_n + exp(-\,t_n\,y_n)] + \frac{h}{2}[2\,t_{n+1} + exp(-\,t_{n+1}K_n)],$$

in which $K_n = y_n + h[2\,t_n + exp(-\,t_n\,y_n)]$. Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + \frac{h}{2}[2\,nh + exp(-\,nh\,y_n)] + \frac{h}{2}\{2(n+1)h + exp[-\,(n+1)hK_n]\},$$

with $y_0 = 1$.

$(a).\ h = 0.025\ :$

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	1.61263	2.48097	3.74556	5.49595

$(b).\ h = 0.0125\ :$

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	1.61263	2.48092	3.74550	5.49589

12. The improved Euler formula is

$$y_{n+1} = y_n + h\,\frac{y_n^2 + 2\,t_n\,y_n}{2(3 + t_n^2)} + h\,\frac{K_n^2 + 2\,t_{n+1}K_n}{2\big(3 + t_{n+1}^2\big)},$$

in which

$$K_n = y_n + h\,\frac{y_n^2 + 2\,t_n\,y_n}{3 + t_n^2}.$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h\,\frac{y_n^2 + 2nh\,y_n}{2(3+n^2h^2)} + h\,\frac{K_n^2 + 2(n+1)hK_n}{2\left[3+(n+1)^2h^2\right]},$$

with $y_0 = 0.5$.

(a). $h = 0.025$:

	$n = 20$	$n = 40$	$n = 60$	$n = 80$
t_n	0.5	1.0	1.5	2.0
y_n	0.590897	0.799950	1.16653	1.74969

(b). $h = 0.0125$:

	$n = 40$	$n = 80$	$n = 120$	$n = 160$
t_n	0.5	1.0	1.5	2.0
y_n	0.590906	0.799988	1.16663	1.74992

16. The exact solution of the initial value problem is $\phi(t) = \frac{1}{2} + \frac{1}{2}e^{2t}$. Based on the result in Prob. $14(c)$, the local truncation error for a *linear* differential equation is

$$e_{n+1} = \frac{1}{6}\phi'''(\bar{t}_n)h^3,$$

where $t_n < \bar{t}_n < t_{n+1}$. Since $\phi'''(t) = 4\,e^{2t}$, the local truncation error is

$$e_{n+1} = \frac{2}{3}exp(2\,\bar{t}_n)h^3.$$

Furthermore, with $0 \le \bar{t}_n \le 1$,

$$|e_{n+1}| \le \frac{2}{3}\,e^2h^3.$$

It also follows that for $h = 0.1$,

$$|e_1| \le \frac{2}{3}\,e^{0.2}(0.1)^3 = \frac{1}{1500}\,e^{0.2}.$$

Using the improved Euler method, with $h = 0.1$, we have $y_1 \approx 1.11000$. The exact value is given by $\phi(0.1) = 1.1107014$.

17. The exact solution of the initial value problem is given by $\phi(t) = \frac{1}{2}t + e^{2t}$. Using the modified Euler method, the local truncation error for a *linear* differential equation is

$$e_{n+1} = \frac{1}{6}\phi'''(\bar{t}_n)h^3,$$

where $t_n < \bar{t}_n < t_{n+1}$. Since $\phi'''(t) = 8\,e^{2t}$, the local truncation error is

$$e_{n+1} = \frac{4}{3} exp(2\,\bar{t}_n)h^3.$$

Furthermore, with $0 \le \bar{t}_n \le 1$, the *local* error is bounded by

$$|e_{n+1}| \le \frac{4}{3}\,e^2 h^3.$$

It also follows that for $h = 0.1$,

$$|e_1| \le \frac{4}{3}\,e^{0.2}(0.1)^3 = \frac{1}{750}\,e^{0.2}.$$

Using the improved Euler method, with $h = 0.1$, we have $y_1 \approx 1.27000$. The exact value is given by $\phi(0.1) = 1.271403$.

18. Using the *Euler method*,

$$\begin{aligned} y_1 &= 1 + 0.1(0.5 - 0 + 2 \cdot 1) \\ &= 1.25\,. \end{aligned}$$

Using the *improved Euler method*,

$$\begin{aligned} y_1 &= 1 + 0.05(0.5 - 0 + 2 \cdot 1) + 0.05(0.5 - 0.1 + 2 \cdot 1.25) \\ &= 1.27\,. \end{aligned}$$

The estimated error is $e_1 \approx 1.27 - 1.25 = 0.02$. The step size should be adjusted by a factor of $\sqrt{0.0025/0.02} \approx 0.354$. Hence the required step size is estimated as

$$h \approx (0.1)(0.36) = 0.036\,.$$

20. Using the *Euler method*,

$$\begin{aligned} y_1 &= 3 + 0.1\sqrt{0+3} \\ &= 3.173205\,. \end{aligned}$$

Using the *improved Euler method*,

$$\begin{aligned} y_1 &= 3 + 0.05\sqrt{0+3} + 0.05\sqrt{0.1 + 3.173205} \\ &= 3.177063\,. \end{aligned}$$

The estimated error is $e_1 \approx 3.177063 - 3.173205 = 0.003858$. The step size should be adjusted by a factor of $\sqrt{0.0025/0.003858} \approx 0.805$. Hence the required step size is estimated as

$$h \approx (0.1)(0.805) = 0.0805\,.$$

21. Using the *Euler method*,

$$\begin{aligned} y_1 &= 0.5 + 0.1\frac{(0.5)^2 + 0}{3+0} \\ &= \; 0.508334 \end{aligned}$$

Using the *improved Euler method*,

$$\begin{aligned} y_1 &= 0.5 + 0.05\frac{(0.5)^2 + 0}{3+0} + 0.05\frac{(0.508334)^2 + 2(0.1)(0.508334)}{3 + (0.1)^2} \\ &= 0.510148\,. \end{aligned}$$

The estimated error is $e_1 \approx 0.510148 - 0.508334 = 0.0018$. The local truncation error is *less* than the given tolerance. The step size can be adjusted by a factor of $\sqrt{0.0025/0.0018} \approx 1.1785$. Hence it is possible to use a step size of

$$h \approx (0.1)(1.1785) \approx 0.117\,.$$

22. Assuming that the solution has continuous derivatives at least to the third order,

$$\phi(t_{n+1}) = \phi(t_n) + \phi'(t_n)h + \frac{\phi''(t_n)}{2!}h^2 + \frac{\phi'''(\bar{t}_n)}{3!}h^3,$$

where $t_n < \bar{t}_n < t_{n+1}$. Suppose that $y_n = \phi(t_n)$.

(a). The local truncation error is given by

$$e_{n+1} = \phi(t_{n+1}) - y_{n+1}.$$

The *modified Euler formula* is defined as

$$y_{n+1} = y_n + h\,f\left[t_n + \frac{1}{2}h\,,y_n + \frac{1}{2}h\,f(t_n, y_n)\right].$$

Observe that $\phi'(t_n) = f(t_n\,,\phi(t_n)) = f(t_n\,,y_n)$. It follows that

$$\begin{aligned} e_{n+1} &= \phi(t_{n+1}) - y_{n+1} \\ &= h\,f(t_n\,,y_n) + \frac{\phi''(t_n)}{2!}h^2 + \frac{\phi'''(\bar{t}_n)}{3!}h^3 - \\ &- h\,f\left[t_n + \frac{1}{2}h\,,y_n + \frac{1}{2}h\,f(t_n, y_n)\right]. \end{aligned}$$

(b). As shown in Prob. $14(b)$,

$$\phi''(t_n) = f_t(t_n\,,y_n) + f_y(t_n\,,y_n)f(t_n\,,y_n)\,.$$

Furthermore,

$$\begin{aligned} f\left[t_n + \tfrac{1}{2}h\,,y_n + \tfrac{1}{2}h\,f(t_n, y_n)\right] &= f(t_n\,,y_n) + f_t(t_n\,,y_n)\frac{h}{2} + f_y(t_n\,,y_n)\,k + \\ &+ \frac{1}{2!}\left[\frac{h^2}{4}f_{tt} + hk\,f_{ty} + k^2 f_{yy}\right]_{t=\xi\,,\,y=\eta}, \end{aligned}$$

in which $k = \frac{1}{2}h\,f(t_n, y_n)$ and $t_n < \xi < t_n + h/2$, $y_n < \eta < y_n + k$. Therefore

$$e_{n+1} = \frac{\phi'''(\bar{t}_n)}{3!}h^3 - \frac{h}{2!}\left[\frac{h^2}{4}f_{tt} + hk\,f_{ty} + k^2 f_{yy}\right]_{t=\xi\,,\,y=\eta}.$$

Note that each term in the brackets has a factor of h^2 . Hence the local truncation error is *proportional* to h^3.

(c). If $f(t, y)$ is *linear*, then $f_{tt} = f_{ty} = f_{yy} = 0$, and

$$e_{n+1} = \frac{\phi'''(\bar{t}_n)}{3!}h^3.$$

23. The *modified* Euler formula for this problem is

$$\begin{aligned} y_{n+1} &= y_n + h\left\{3 + t_n + \frac{1}{2}h - \left[y_n + \frac{1}{2}h(3 + t_n - y_n)\right]\right\} \\ &= y_n + h(3 + t_n - y_n) + \frac{h^2}{2}(y_n - t_n - 2). \end{aligned}$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h(3 + nh - y_n) + \frac{h^2}{2}(y_n - nh - 2)\ ,$$

with $y_0 = 1$. Setting $h = 0.1$, we obtain the following values :

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	1.19500	1.38098	1.55878	1.72920

25. The *modified* Euler formula is

$$\begin{aligned} y_{n+1} &= y_n + h\left[2y_n - 3t_n - \frac{3}{2}h + h(2y_n - 3t_n)\right] \\ &= y_n + h(2y_n - 3t_n) + \frac{h^2}{2}(4y_n - 6t_n - 3). \end{aligned}$$

Since $t_n = t_0 + nh$ and $t_0 = 0$, we can also write

$$y_{n+1} = y_n + h(2y_n - 3nh) + \frac{h^2}{2}(4y_n - 6\,nh - 3)\ ,$$

with $y_0 = 1$. Setting $h = 0.1$, we obtain :

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	1.20500	1.42210	1.65396	1.90383

26. The *modified* Euler formula for this problem is

$$y_{n+1} = y_n + h\left\{2t_n + h + exp\left[-\left(t_n + \frac{h}{2}\right)K_n\right]\right\},$$

in which $K_n = y_n + \frac{h}{2}[2t_n + exp(-t_n y_n)]$. Now $t_n = t_0 + nh$, with $t_0 = 0$ and $y_0 = 1$. Setting $h = 0.1$, we obtain the following values :

	$n=1$	$n=2$	$n=3$	$n=4$
t_n	0.1	0.2	0.3	0.4
y_n	1.104885	1.21892	1.34157	1.472724

27. Let $f(t, y)$ be *linear* in both variables. The *improved* Euler formula is

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}h[f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))] \\ &= y_n + \frac{1}{2}hf(t_n, y_n) + \frac{1}{2}hf(t_n, y_n) + \frac{1}{2}hf[h, hf(t_n, y_n)] \\ &= hf(h, y_n) + \frac{1}{2}hf[h, hf(t_n, y_n)]. \end{aligned}$$

The *modified* Euler formula is

$$\begin{aligned} y_{n+1} &= y_n + hf\left[t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right] \\ &= y_n + hf(t_n, y_n) + hf\left[\frac{1}{2}h, \frac{1}{2}hf(t_n, y_n)\right]. \end{aligned}$$

Since $f(t, y)$ is *linear* in both variables,

$$f\left[\frac{1}{2}h, \frac{1}{2}hf(t_n, y_n)\right] = \frac{1}{2}f[h, hf(t_n, y_n)].$$