Section 11.5

3. The equations relating to this problem are given by Eqs. (2) to (17) in the text. Based on the boundary conditions, the eigenfunctions are $\phi_n(x) = J_0(\lambda_n r)$ and the associated eigenvalues $\lambda_1, \lambda_2, \cdots$ are the positive zeroes of $J_0(\lambda)$. The general solution has the form

$$u(r\,,t) = \sum_{n=1}^{\infty}[c_n J_0(\lambda_n r)\, cos\,\lambda_n at + k_n J_0(\lambda_n r)\, sin\,\lambda_n at]\,.$$

The initial conditions require that

$$u(r\,,0) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = f(r)$$

and

$$u_t(r\,,0) = \sum_{n=1}^{\infty} a\lambda_n k_n J_0(\lambda_n r) = g(r)\,.$$

The coefficients c_n and k_n are obtained from the respective eigenfunction expansions. That is,

$$c_n = \frac{1}{\|J_0(\lambda_n r)\|^2}\int_0^1 rf(r)J_0(\lambda_n r)dr$$

and

$$k_n = \frac{1}{a\lambda_n\,\|J_0(\lambda_n r)\|^2}\int_0^1 rg(r)J_0(\lambda_n r)dr\,,$$

in which

$$\|J_0(\lambda_n r)\|^2 = \int_0^1 r[J_0(\lambda_n r)]^2 dr$$

for $n = 1, 2, \cdots$.

8. A more general equation was considered in Prob. 23 of Section 10.5. Assuming a solution of the form $u(r\,,t) = R(r)T(t)$, substitution into the PDE results in

$$\alpha^2\left[R''T + \frac{1}{r}\,R'T\right] = RT'\,.$$

Dividing both sides of the equation by the factor RT, we obtain

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} = \frac{T'}{\alpha^2 T}\,.$$

Since both sides of the resulting differential equation depend on *different* variables, each side must be equal to a constant, say $-\lambda^2$. That is,

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} = \frac{T'}{\alpha^2 T} = -\lambda^2\,.$$

It follows that $T' + \alpha^2\lambda^2 T = 0$, and

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} = -\lambda^2\,,$$

which can be written as $r^2 R'' + r\,R' + \lambda^2 r^2 R = 0$. Introducing the variable $\xi = \lambda r$, the last equation can be expressed as $\xi^2 R'' + \xi\,R' + \xi^2 R = 0$, which is the Bessel equation of order zero.

The temporal equation has solutions which are multiples of $T(t) = exp(-\alpha^2\lambda^2 t)$. The general solution of the Bessel equation is

$$R(r) = b_1 J_0(\lambda_n r) + b_2 Y_0(\lambda_n r)\,.$$

Since the steady state temperature will be *zero*, all solutions must be bounded, and hence we set $b_2 = 0$. Furthermore, the boundary condition $u(1\,,t) = 0$ requires that $R(1) = 0$ and hence $J_0(\lambda) = 0$. It follows that the eigenfunctions are $\phi_n(x) = J_0(\lambda_n r)$, with the associated eigenvalues $\lambda_1, \lambda_2, \cdots$, which are the positive zeroes of $J_0(\lambda)$. Therefore the fundamental solutions of the PDE are $u_n(r\,,t) = J_0(\lambda_n r)exp(-\alpha^2\lambda_n^2 t)$, and the general solution has the form

$$u(r\,,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) exp\big(-\alpha^2\lambda_n^2 t\big)\,.$$

The initial condition requires that

$$u(r\,,0) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = f(r).$$

The coefficients in the general solution are obtained from the eigenfunction expansion of $f(r)$. That is,

$$c_n = \frac{1}{\|J_0(\lambda_n r)\|^2}\int_0^1 r f(r) J_0(\lambda_n r)dr\,,$$

in which

$$\|J_0(\lambda_n r)\|^2 = \int_0^1 r[J_0(\lambda_n r)]^2 dr \quad (n = 1, 2, \cdots).$$