Section 9.3

1. Write the system in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$. In this case, it is evident that

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}.$$

That is, $\mathbf{g}(\mathbf{x}) = (-y^2, x^2)^T$. Using polar coordinates, $\|\mathbf{g}(\mathbf{x})\| = r^2\sqrt{sin^4\theta + cos^4\theta}$ and $\|\mathbf{x}\| = r$. Hence

$$\lim_{r \to 0} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = \lim_{r \to 0} r\sqrt{sin^4\theta + cos^4\theta} = 0\,,$$

and the system is *almost linear*. The origin is an isolated critical point of the linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^2 + r - 2 = 0$, with roots $r_1 = 1$ and $r_2 = -2$. Hence the critical point is a *saddle*, which is *unstable*.

2. The system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}.$$

Following the discussion in Example 3, we note that $F(x\,,y) = -x + y + 2xy$ and $G(x\,,y) = -4x - y + x^2 - y^2$. Both of the functions F and G are *twice differentiable*, hence the system is *almost linear*. Furthermore,

$$F_x = -1 + 2y\,,\ F_y = 1 + 2x\,,\ G_x = -4 + 2x\,,\ G_y = -1 - 2y\,.$$

The origin is an isolated critical point, with

$$\begin{pmatrix} F_x(0\,,0) & F_y(0\,,0) \\ G_x(0\,,0) & G_y(0\,,0) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}.$$

The characteristic equation of the associated linear system is $r^2 + 2\,r + 5 = 0$, with complex conjugate roots $r_{1,2} = -1 \pm 2i$. The origin is a *stable spiral*, which is *asymptotically stable*.

$5(a)$. The critical points consist of the solution set of the equations

$$\begin{aligned}(2 + x)(y - x) &= 0 \\ (4 - x)(y + x) &= 0\,.\end{aligned}$$

As shown in Prob. 13 of Section 9.2, the only critical points are at $(0\,,0)$, $(4\,,4)$ and $(-2\,,2)$.

(b, c). First note that $F(x\,, y) = (2 + x)(y - x)$ and $G(x\,, y) = (4 - x)(y + x)$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} -2 - 2x + y & 2 + x \\ 4 - y - 2x & 4 - x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix},$$

with eigenvalues $r_1 = 1 - \sqrt{17}$ and $r_2 = 1 + \sqrt{17}$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(-2\,,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-2\,,2) = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix},$$

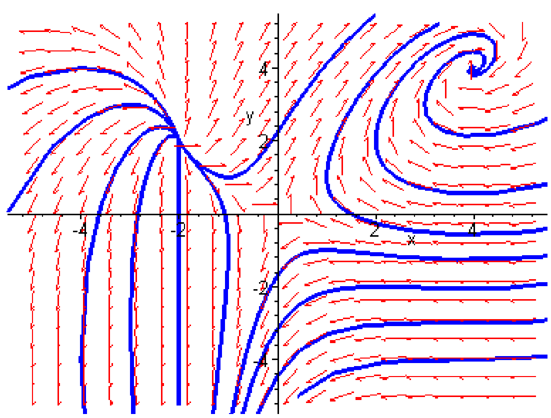
with eigenvalues $r_1 = 4$ and $r_2 = 6$. The eigenvalues are real, unequal and positive, hence the critical point is an *unstable node*. At the point $(4\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(4\,,4) = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -3 \pm i\sqrt{39}$. The critical point is a *stable spiral*, which is *asymptotically stable*.

Based on Table 9.3.1 , the nonlinear terms do not affect the stability and type of each critical point.

(d).



$7(a)$. The critical points are solutions of the equations

$$1 - y = 0$$
$$(x - y)(x + y) = 0\,.$$

The first equation requires that $y = 1$. Based on the second equation, $x = \pm 1$. Hence the critical points are $(-1, 1)$ and $(1, 1)$.

(b, c). $F(x, y) = 1 - y$ and $G(x, y) = x^2 - y^2$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x, y) & F_y(x, y) \\ G_x(x, y) & G_y(x, y) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2x & -2y \end{pmatrix}.$$

At the critical point $(-1, 1)$, the coefficient matrix of the linearized system is

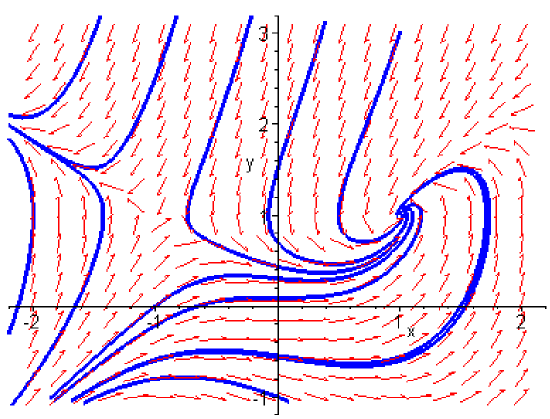
$$\mathbf{J}(-1, 1) = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = -1 - \sqrt{3}$ and $r_2 = -1 + \sqrt{3}$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(1, 1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1, 1) = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1 \pm i$. The critical point is a *stable spiral*, which is *asymptotically stable*.

(d).



Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

$8(a)$. The critical points are given by the solution set of the equations

$$x(1 - x - y) = 0$$
$$y(2 - y - 3x) = 0\,.$$

If $x = 0$, then either $y = 0$ or $y = 2$. If $y = 0$, then $x = 0$ or $x = 1$. If $y = 1 - x$, then either $x = 1/2$ or $x = 1$. If $y = 2 - 3x$, then $x = 0$ or $x = 1/2$. Hence the critical points are at $(0\,,0)$, $(0\,,2)$, $(1\,,0)$ and $(1/2\,,1/2)$.

(b,c). Note that $F(x\,,y) = x - x^2 - xy$ and $G(x\,,y) = (2y - y^2 - 3xy)/4$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} 1 - 2x - y & -\,x \\ -\,3y/4 & 1/2 - y/2 - 3x/4 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = 1/2$. The eigenvalues are real and both positive. Hence the critical point is an *unstable node*. At the equilibrium point $(0\,,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,2) = \begin{pmatrix} -\,1 & 0 \\ -\,\frac{3}{2} & -\,\frac{1}{2} \end{pmatrix},$$

with eigenvalues $r_1 = -\,1$ and $r_2 = -\,1/2$. The eigenvalues are both negative, hence the critical point is a *stable node*. At the point $(1\,,0)$, the coefficient matrix of the linearized system is

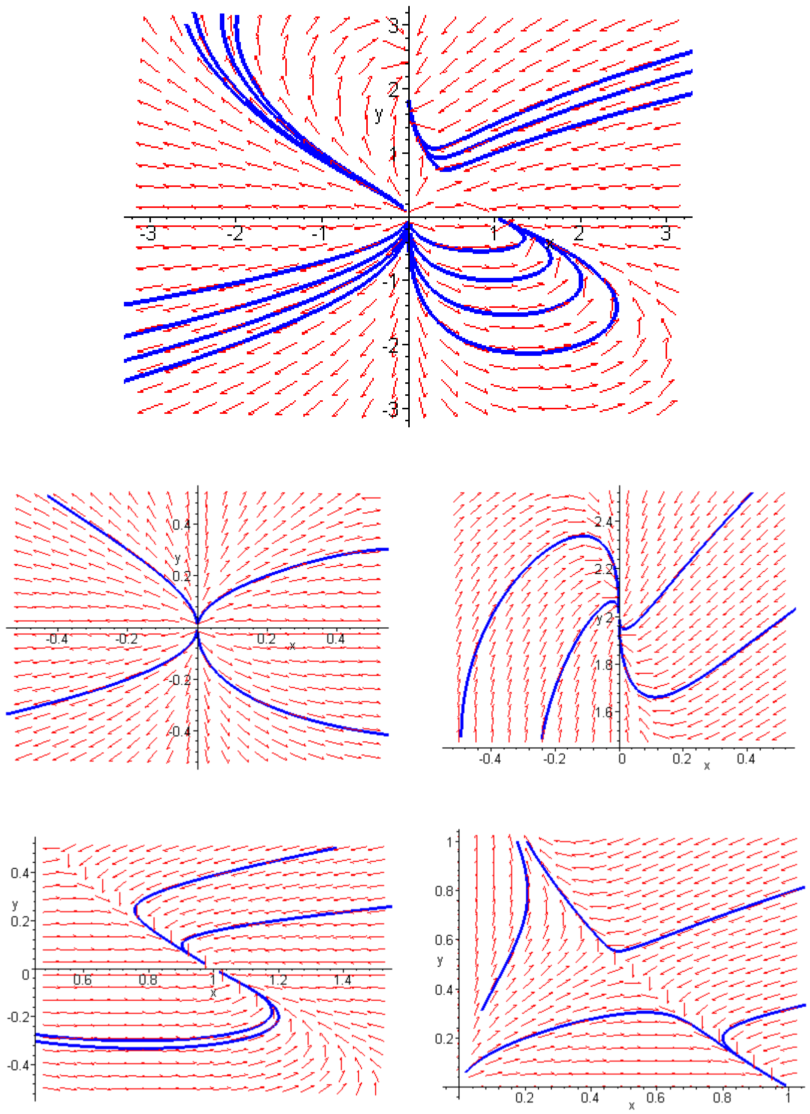
$$\mathbf{J}(1\,,0) = \begin{pmatrix} -\,1 & -\,1 \\ 0 & -\,\frac{1}{4} \end{pmatrix},$$

with eigenvalues $r_1 = -\,1$ and $r_2 = -\,1/4$. Both of the eigenvalues are negative, and hence the critical point is a *stable node*. At the critical point $(1/2\,,1/2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2\,,1/2) = \begin{pmatrix} -\,\frac{1}{2} & -\,\frac{1}{2} \\ -\,\frac{3}{8} & -\,\frac{1}{8} \end{pmatrix},$$

with eigenvalues $r_1 = -\,5/16 - \sqrt{57}/16$ and $r_2 = -\,5/16 + \sqrt{57}/16$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

(d).



Based on Table $9.3.1$, the nonlinear terms do not affect the stability and type of each critical point.

$9(a)$. Based on Prob. 8, in Section 9.2, the critical points are at $(0\,,0)$, $(-2\,,\,-2)$, $(0\,,1)$ and $(3\,,\,-2)$.

(b,c). First note that $F(x\,,y)=-(x-y)(1-x-y)$ and $G(x\,,y)=x(2+y)$. The *Jacobian* matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix} 2x-1 & 1-2y \\ 2+y & x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = -2$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the critical point $(0\,,1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,1) = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1/2 \pm i\sqrt{11}/2$. The critical point is a *stable spiral*, which is *asymptotically stable*. At the point $(-2\,,-2)$, the coefficient matrix of the linearized system is

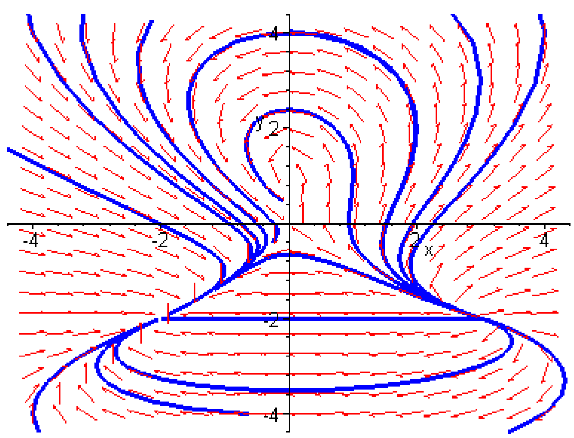
$$\mathbf{J}(-2\,,-2) = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = -2$ and $r_2 = -5$. The eigenvalues are unequal and negative, hence the critical point is a *stable node*. At the point $(3\,,-2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3\,,-2) = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix},$$

with eigenvalues $r_1 = 3$ and $r_2 = 5$. The eigenvalues are unequal and positive, hence the critical point is an *unstable node*.

(d).



Based on Table 9.3.1 , the nonlinear terms do not affect the stability and type of each critical point.

$11(a)$. The critical points are solutions of the equations

$$\begin{aligned} 2x + y + xy^3 &= 0 \\ x - 2y - xy &= 0\,. \end{aligned}$$

Substitution of $y = x/(x+2)$ into the first equation results in

$$3x^4 + 13x^3 + 28x^2 + 20x = 0\,.$$

One root of the resulting equation is $x = 0$. The only other real root of the equation is

$$x = \frac{1}{9}\left[\left(287 + 18\sqrt{2019}\right)^{1/3} - 83\left(287 + 18\sqrt{2019}\right)^{-1/3} - 13\right].$$

Hence the critical points are $(0\,,0)$ and $(\,-\,1.19345...\,,1.4797...)$.

(b, c). $F(x\,,y) = x - x^2 - xy$ and $G(x\,,y) = (2y - y^2 - 3xy)/4$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} 2 + y^3 & 1 + 3xy^2 \\ 1 - y & -2 - x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

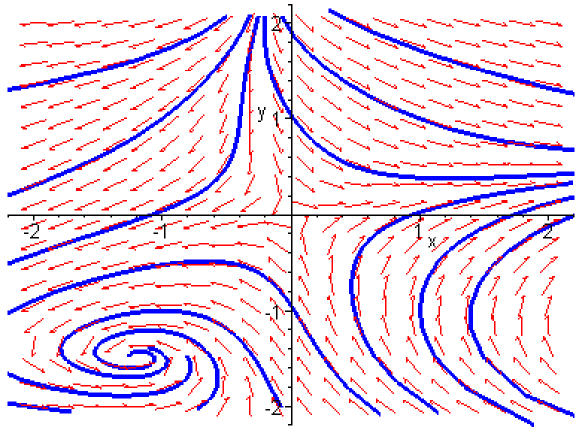
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{5}$ and $r_2 = -\sqrt{5}$. The eigenvalues are real and of opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(\,-\,1.19345...\,,1.4797...)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\,-\,1.19345\,,1.4797) = \begin{pmatrix} -\,1.2399 & -\,6.8393 \\ -\,2.4797 & -\,0.8065 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -\,1.0232 \pm 4.1125\,i$. The critical point is a *stable spiral*, which is *asymptotically stable*.

(d).



In both cases, the nonlinear terms do not affect the stability and type of the critical point.

$12(a)$. The critical points are given by the solution set of the equations

$$(1+x)sin\,y = 0$$
$$1 - x - cos\,y = 0\,.$$

If $x = -1$, then we must have $cos\,y = 2$, which is impossible. Therefore $sin\,y = 0$, which implies that $y = n\pi\,,\, n = 0\,,\pm 1\,,2\,,\ldots$. Based on the second equation,

$$x = 1 - cos\,n\pi\,.$$

It follows that the critical points are located at $(0\,,2k\pi)$ and $(2\,,(2k+1)\pi)$, where $k = 0\,,\pm 1\,,2\,,\ldots$.

(b,c). Given that $F(x\,,y) = (1+x)sin\,y$ and $G(x\,,y) = 1 - x - cos\,y$, the *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} sin\,y & (1+x)cos\,y \\ -1 & sin\,y \end{pmatrix}.$$

At the critical points $(0\,,2k\pi)$, the coefficient matrix of the linearized system is

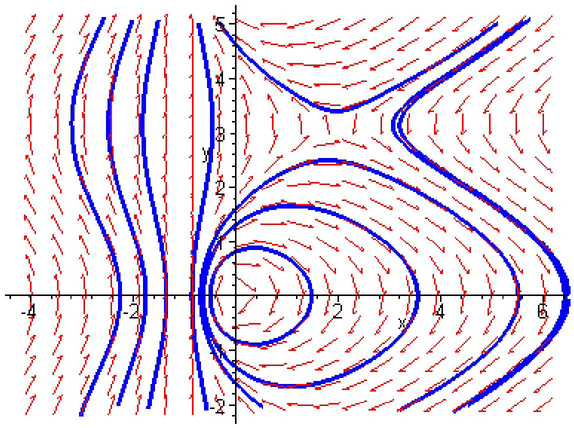
$$\mathbf{J}(0\,,2k\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with purely complex eigenvalues $r_{1,2} = \pm\, i$. The critical points of the associated linear systems are *centers*, which are *stable*. Note that Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical points of the nonlinear systems and their corresponding linearizations. At the points $(2\,,(2k+1)\pi)$, the coefficient matrix of the linearized system is

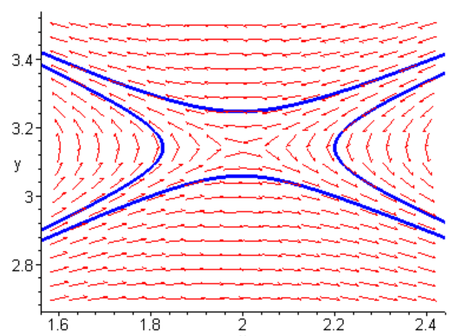
$$\mathbf{J}[2\,,(2k+1)\pi] = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{3}$ and $r_2 = -\sqrt{3}$. The eigenvalues are real, with opposite sign. Hence the critical points of the associated linear systems are *saddles*, which are *unstable*.

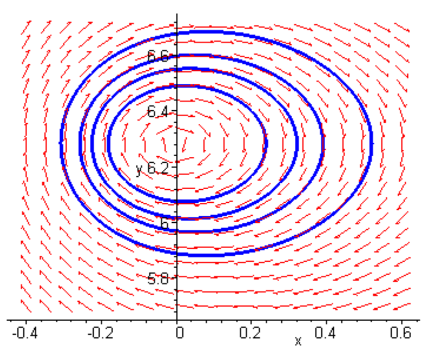
(d).



As asserted in Theorem $9.3.2$, the trajectories near the critical points $(2\,,(2k+1)\pi)$ resemble those near a saddle.



Upon closer examination, the critical points $(0\,,2k\pi)$ are indeed centers.



$13(a)$. The critical points are solutions of the equations

$$x - y^2 = 0$$
$$y - x^2 = 0\,.$$

Substitution of $y = x^2$ into the first equation results in

$$x - x^4 = 0\,,$$

with real roots $x = 0\,,1\,.$ Hence the critical points are at $(0\,,0)$ and $(1\,,1)$.

(b, c). In this problem, $F(x\,,y) = x - y^2$ and $G(x\,,y) = y - x^2\,.$ The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 & -\,2y \\ -\,2x & 1 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

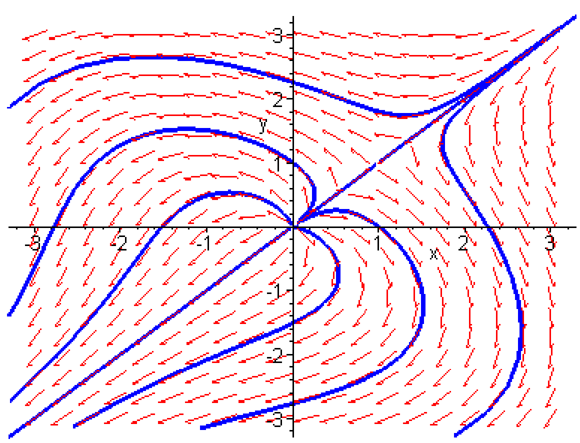
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with *repeated* eigenvalues $r_1 = 1$ and $r_2 = 1\,.$ It is easy to see that the corresponding eigenvectors are linearly independent. Hence the critical point is an *unstable proper node*. Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical point of the nonlinear system and the corresponding linearization. At the critical point $(1\,,1)$, the coefficient matrix of the linearized system is

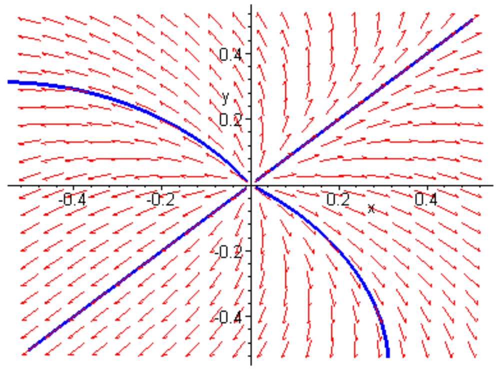
$$\mathbf{J}(1\,,1) = \begin{pmatrix} 1 & -\,2 \\ -\,2 & 1 \end{pmatrix},$$

with eigenvalues $r_1 = 3$ and $r_2 = -\,1\,.$ The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

(d).



Closer examination reveals that the critical point at the origin is indeed a proper node.



$14(a)$. The critical points are given by the solution set of the equations

$$1 - xy = 0$$
$$x - y^3 = 0\,.$$

After multiplying the second equation by y, it follows that $y = \pm 1$. Hence the critical points of the system are at $(1\,,1)$ and $(\,-1\,,\,-1)$.

(b, c). Note that $F(x\,,y) = 1 - xy$ and $G(x\,,y) = x - y^3$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -\,y & -\,x \\ 1 & -\,3y^2 \end{pmatrix}.$$

At the critical point $(1\,,1)$, the coefficient matrix of the linearized system is

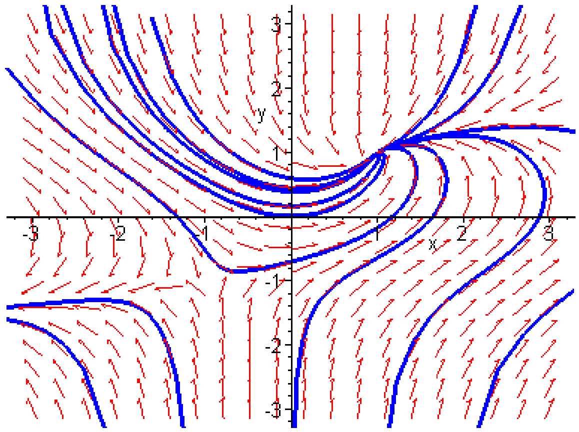
$$\mathbf{J}(1\,,1) = \begin{pmatrix} -\,1 & -\,1 \\ 1 & -\,3 \end{pmatrix},$$

with eigenvalues $r_1 = \,-\,2$ and $r_2 = \,-\,2$. The eigenvalues are real and *equal*. It is easy to show that there is only *one* linearly independent eigenvector. Hence the critical point is a *stable improper node*. Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical point of the nonlinear system and the corresponding linearization. At the point $(\,-1\,,\,-1)$, the coefficient matrix of the linearized system is

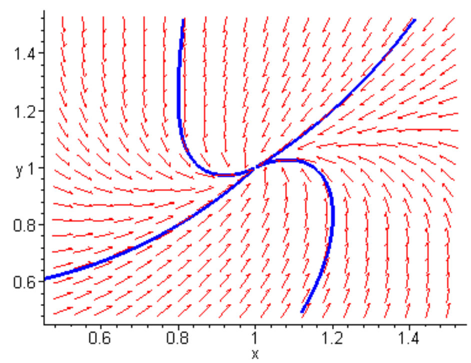
$$\mathbf{J}(\,-1\,,\,-1) = \begin{pmatrix} 1 & 1 \\ 1 & -\,3 \end{pmatrix},$$

with eigenvalues $r_1 = \,-\,1 + \sqrt{5}$ and $r_2 = \,-\,1 - \sqrt{5}$. The eigenvalues are real, with opposite sign. Hence the critical point of the associated linear system is a *saddle*, which is *unstable*.

(d).



Closer examination reveals that the critical point at $(1\,,1)$ is indeed a *stable* improper node, which is asymptotically stable.



$15(a)$. The critical points are given by the solution set of the equations

$$-\,2x - y - x\left(x^2 + y^2\right) = 0$$
$$x - y + y\left(x^2 + y^2\right) = 0\,.$$

It is clear that the origin is a critical point. Solving the *first* equation for y, we find that

$$y = \frac{-\,1 \pm \sqrt{1 - 8x^2 - 4x^4}}{2x}\,.$$

Substitution of these relations into the *second* equation results in two equations of the form $f_1(x) = 0$ and $f_2(x) = 0$. Plotting these functions, we note that only $f_1(x) = 0$ has real roots given by $x \approx \pm\,0.33076$. It follows that the additional critical points are at $(-\,0.33076\,, 1.0924)$ and $(0.33076\,, -\,1.0924)$.

(b, c). Given that

$$F(x\,, y) = \,-\,2x - y - x\big(x^2 + y^2\big)$$
$$G(x\,, y) = x - y + y\big(x^2 + y^2\big),$$

the *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -\,2 - 3x^2 - y^2 & -\,1 - 2xy \\ 1 + 2xy & -\,1 + x^2 + 3y^2 \end{pmatrix}.$$

At the critical point $(0\,, 0)$, the coefficient matrix of the linearized system is

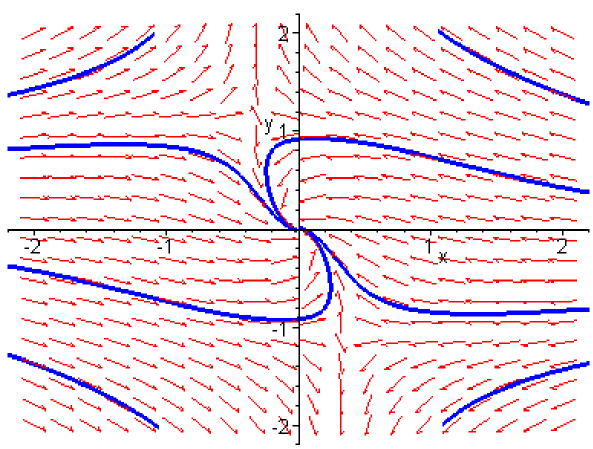
$$\mathbf{J}(0\,, 0) = \begin{pmatrix} -\,2 & -\,1 \\ 1 & -\,1 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = \left(-\,3 \pm i\sqrt{3}\right)/2$. Hence the critical point is a *stable spiral*, which is *asymptotically stable*. At the point $(-\,0.33076\,, 1.0924)$, the coefficient matrix of the linearized system is

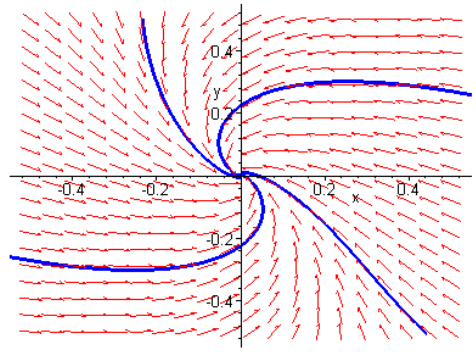
$$\mathbf{J}(-\,0.33076\,, 1.0924) = \begin{pmatrix} -\,3.5216 & -\,0.27735 \\ 0.27735 & 2.6895 \end{pmatrix},$$

with eigenvalues $r_1 = \,-\,3.5092$ and $r_2 = 2.6771$. The eigenvalues are real, with opposite sign. Hence the critical point of the associated linear system is a *saddle*, which is *unstable*. Identical results hold for the point at $(\,0.33076\,, \,-\,1.0924)$.

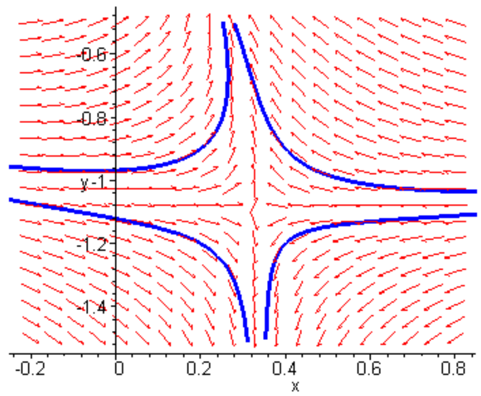
(d).



A closer look at the origin reveals a spiral:



Near the point $(0.33076\,,\,-1.0924)$ the nature of the critical point is evident:



Based on Table 9.3.1 , the nonlinear terms do not affect the stability and type of each critical point.

$16(a)$. The critical points are solutions of the equations

$$y + x\left(1 - x^2 - y^2\right) = 0$$
$$-x + y\left(1 - x^2 - y^2\right) = 0\,.$$

Multiply the *first* equation by y and the *second* equation by x. The difference of the two equations gives $x^2 + y^2 = 0$. Hence the only critical point is at the origin.

(b, c). With $F(x\,,y) = y + x(1 - x^2 - y^2)$ and $G(x\,,y) = -x + y(1 - x^2 - y^2)$, the *Jacobian* matrix of the vector field is

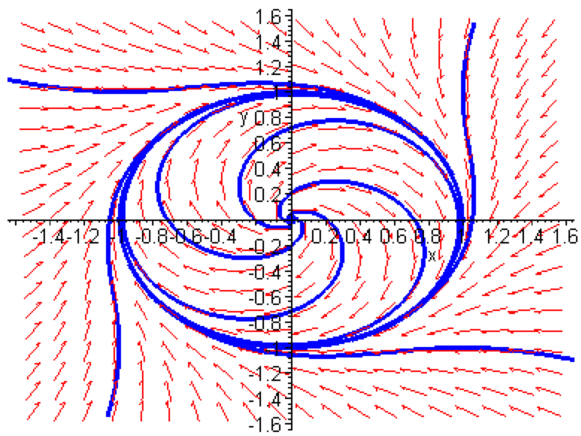
$$\mathbf{J} = \begin{pmatrix} 1 - 3x^2 - y^2 & 1 - 2xy \\ -1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = 1 \pm i$. Hence the origin is an *unstable* *spiral*.

(d).



$17(a)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 + 6x^2 & 0 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

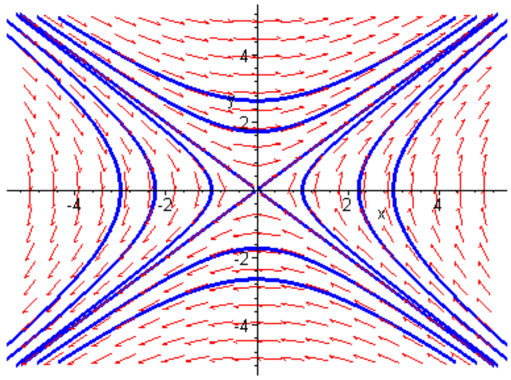
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = -1$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle point*.

(b). The trajectories of the *linearized* system are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x}{y},$$

which is separable. Integrating both sides of the equation $x\,dx - y\,dy = 0$, the solution is $x^2 - y^2 = C$. The trajectories consist of a family of hyperbolas.



It is easy to show that the general solution is given by $x(t) = c_1 e^t + c_2 e^{-t}$ and $y(t) = c_1 e^t - c_2 e^{-t}$. The only *bounded* solutions consist of those for which $c_1 = 0$. In that case, $x(t) = c_2 e^{-t} = -y(t)$.

(c). The trajectories of the given system are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x + 2x^3}{y},$$

which can also be written as $(x + 2x^3)dx - y\,dy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = x + 2x^3 \text{ and } \frac{\partial H}{\partial y} = -y\,.$$

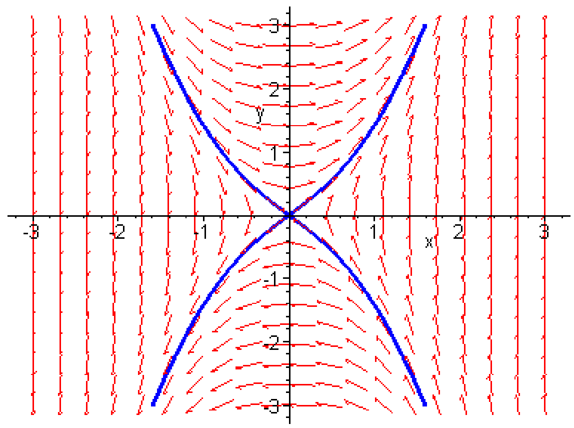
Integrating the first equation, we find that $H(x\,,y) = x^2/2 + x^4/2 + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f'(y).$$

Comparing the partial derivatives, we obtain $f(y) = -y^2/2 + c$. Hence the solutions are level curves of the function

$$H(x\,,y) = x^2/2 + x^4/2 - y^2/2\,.$$

The trajectories *approaching* to, or *diverging* from, the origin are no longer straight lines.



$19(a)$. The solutions of the system of equations

$$\begin{aligned} y &= 0 \\ -\,\omega^2 sin\, x &= 0 \end{aligned}$$

consist of the points $(\pm\, n\pi\,, 0)\,,\, n = 0\,,1\,,2\,,\cdots$. The functions $F(x\,,y) = y$ and $G(x\,,y) = \,-\,\omega^2 sin\, x$ are *analytic* on the entire plane. It follows that the system is almost linear near each of the critical points.

(b). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -\,\omega^2 cos\, x & 0 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & 1 \\ -\,\omega^2 & 0 \end{pmatrix},$$

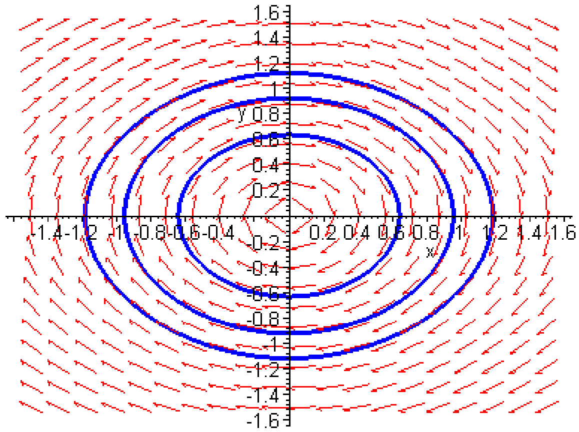
with purely complex eigenvalues $r_{1,2} = \,\pm\, i\omega$. Hence the origin is a *center*. Since the eigenvalues are purely complex, Theorem 9.3.2 gives no definite conclusion about the critical point of the nonlinear system. Physically, the critical point corresponds to the state $\theta = 0\,,\,\theta' = 0$. That is, the rest configuration of the pendulum.

(c). At the critical point $(\pi\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\pi\,,0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix},$$

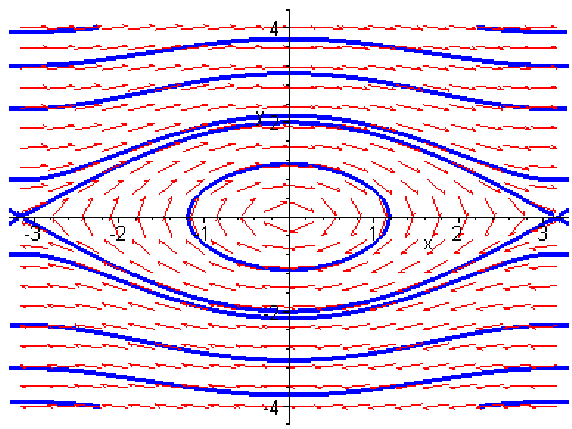
with eigenvalues $r_{1,2} = \,\pm\,\omega$. The eigenvalues are real and of opposite sign. Hence the critical point is a *saddle*. Theorem 9.3.2 asserts that the critical point for the nonlinear system is also a saddle, which is unstable. This critical point corresponds to the state $\theta = \pi\,,\,\theta' = 0$. That is, the *upright* rest configuration.

(d). Let $\omega^2 = 1$. The following is a plot of the phase curves near $(0\,,0)$.



The local phase portrait shows that the origin is indeed a center.

(e).



It should be noted that the phase portrait has a periodic pattern, since $\theta = x \; mod\, 2\pi$.

$20(a)$. The trajectories of the system in Problem 19 are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-\,\omega^2 sin\, x}{y},$$

which can also be written as $\omega^2 sin\, x\, dx + y\, dy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = \omega^2 sin\, x \;\text{ and }\; \frac{\partial H}{\partial y} = y\,.$$

Integrating the first equation, we find that $H(x\,,y) = -\,\omega^2 cos\, x + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f'(y).$$

Comparing the partial derivatives, we obtain $f(y) = y^2/2 + C$. Hence the solutions are level curves of the function

$$H(x\,,y) = -\omega^2 cos\,x + y^2/2\,.$$

Adding an arbitrary constant, say ω^2, to the function $H(x\,,y)$ does not change the nature of the level curves. Hence the trajectories are can be written as

$$\frac{1}{2}y^2 + \omega^2(1 - cos\,x) = c\,,$$

in which c is an arbitrary constant.

(b). Multiplying by mL^2 and reverting to the original physical variables, we obtain

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mL^2\omega^2(1 - cos\,\theta) = mL^2c\,.$$

Since $\omega^2 = g/L$, the equation can be written as

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1 - cos\,\theta) = E\,,$$

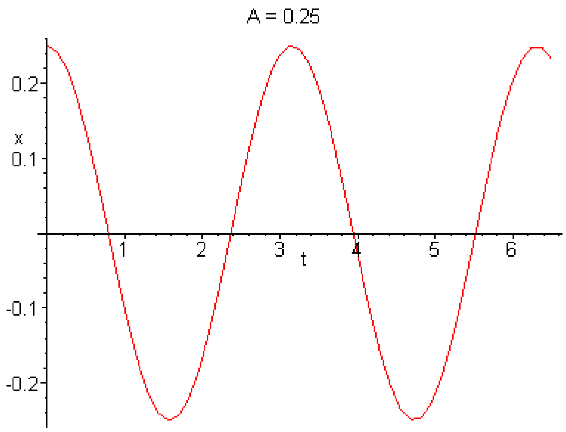
in which $E = mL^2c$.

(c). The *absolute velocity* of the point mass is given by $v = L\,d\theta/dt$. The kinetic energy of the mass is $T = mv^2/2$. Choosing the rest position as the *datum*, that is, the level of *zero potential energy*, the gravitational potential energy of the point mass is

$$V = mgL(1 - cos\,\theta).$$

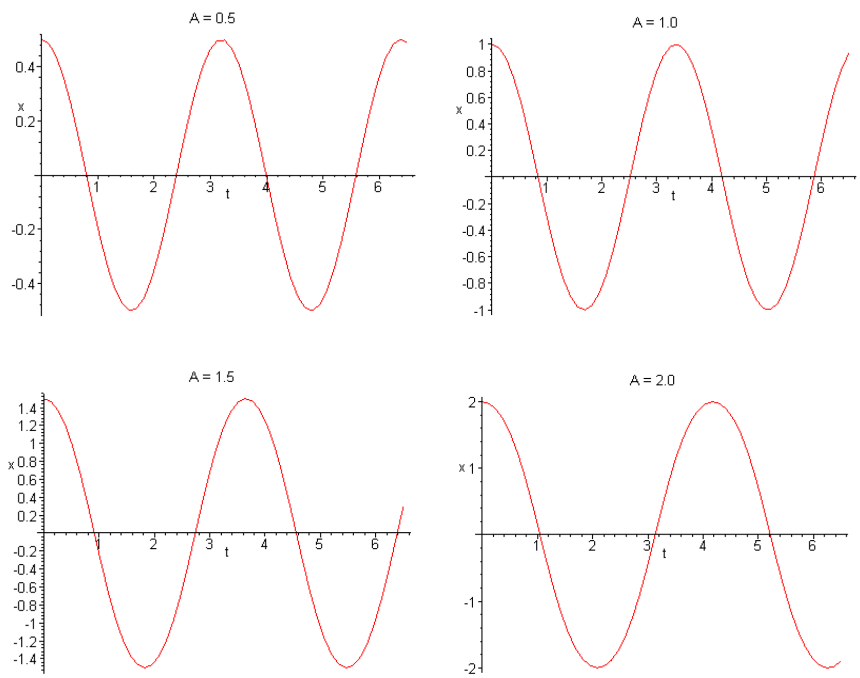
It follows that the total energy, $T + V$, is *constant* along the trajectories.

$21(a)$. $A = 0.25$



Since the system is *undamped*, and $y(0) = 0$, the amplitude is 0.25. The period is estimated at $\tau \approx 3.16$.

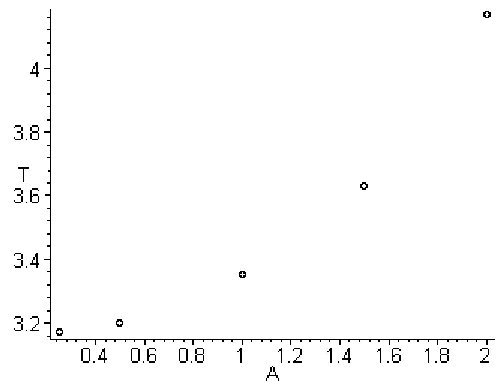
(b).



	R	τ
$A = 0.5$	0.5	3.20
$A = 1.0$	1.0	3.35
$A = 1.5$	1.5	3.63
$A = 2.0$	2.0	4.17

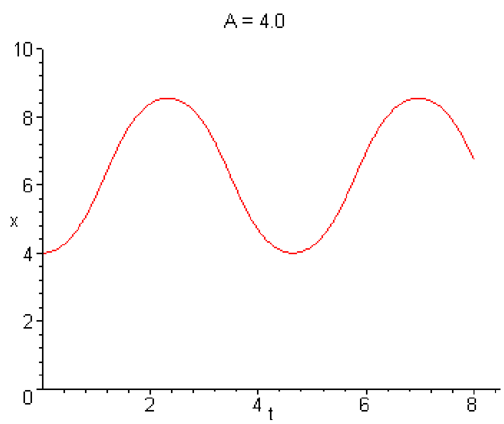
(c). Since the system is conservative, the amplitude is equal to the initial amplitude. On

the other hand, the period of the pendulum is a *monotone increasing* function of the initial
position A.



It appears that as $A \to 0$, the period approaches π, the period of the corresponding *linear* pendulum $(2\pi/\omega)$.

(d).



The pendulum is released from rest, at an inclination of $4-\pi$ *radians* from the vertical. Based on *conservation of energy*, the pendulum will swing past the lower equilibrium position $(\theta = 2\pi)$ and come to rest, momentarily, at a maximum rotational displacement of $\theta_{max} = 3\pi - (4-\pi) = 4\pi - 4$. The transition between the two dynamics occurs at $A = \pi$, that is, once the pendulum is released *beyond* the upright configuration.

24(a). It is evident that the origin is a critical point of each system. Furthermore, it is easy to see that the corresponding linear system, in each case, is given by

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x\,.$$

The eigenvalues of the coefficient matrix are $r_{1,2} = \pm i$. Hence the critical point of the linearized system is a *center*.

(b). Using polar coordinates, it is also easy to show that

$$\lim_{r\to 0}\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = 0\,.$$

Alternatively, the nonlinear terms are *analytic* in the entire plane. Hence both systems are
almost linear near the origin.

(c). For system (ii), note that

$$x\frac{dx}{dt} + y\frac{dy}{dt} = xy - x^2\left(x^2+y^2\right) - xy - y^2\left(x^2+y^2\right).$$

Converting to polar coordinates, and differentiating the equation $r^2 = x^2 + y^2$ with respect to t, we find that

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = -\left(x^2+y^2\right)^2 = -r^4.$$

That is, $r' = -r^3$. It follows that $r^2 = 1/(2t+c)$, where $c = 1/r_0^2$. Since $r \to 0$ as $t \to 0$, regardless of the value of r_0, the origin is an $asymptotically\ stable$ equilibrium point.

On the other hand, for system (i),

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = \left(x^2+y^2\right)^2 = r^4.$$

That is, $r' = r^3$. Solving the differential equation results in

$$r^2 = \frac{c-2t}{(2t-c)^2}\,.$$

Imposing the initial condition $r(0) = r_0$, we obtain a specific solution

$$r^2 = -\frac{r_0^2}{2\,r_0^2\,t-1}\,.$$

Since the solution becomes *unbounded* as $t \to 1/2r_0^2$, the critical point is *unstable*.

25. The characteristic equation of the coefficient matrix is $r^2+1=0$, with complex roots $r_{1,2} = \pm i$. Hence the critical point at the origin is a *center*. The characteristic equation of the perturbed matrix is $r^2 - 2\epsilon r + 1 + \epsilon^2 = 0$, with complex conjugate roots $r_{1,2} = \epsilon \pm i$. As long as $\epsilon \neq 0$, the critical point of the perturbed system is a *spiral point*. Its stability depends on the sign of ϵ.

26. The characteristic equation of the coefficient matrix is $(r+1)^2 = 0$, with roots

$r_1 = r_2 = -1$. Hence the critical point is an *asymptotically stable node*. On the other hand, the characteristic equation of the perturbed system is $r^2 + 2r + 1 + \epsilon = 0$, with roots $r_{1,2} = -1 \pm \sqrt{-\epsilon}$. If $\epsilon > 0$, then $r_{1,2} = -1 \pm i\sqrt{\epsilon}$ are complex roots. The critical point is a *stable spiral*. If $\epsilon < 0$, then $r_{1,2} = -1 \pm \sqrt{|\epsilon|}$ are real and both negative ($|\epsilon| \ll 1$). The critical point remains a *stable node*.

27(d). Set $k = sin(\alpha/2) = sin(A/2)$ and $g/L = 4$.