Section 5.4

2. We see that $P(x) = 0$ when $x = 0$ and 1. Since the three coefficients have no factors
in common, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x \frac{2x}{x^2(1-x)^2} = 2\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{4}{x^2(1-x)^2} = 4\,.$$

The singular point $x = 0$ is *regular*. Considering $x = 1$,

$$\lim_{x\to 1} (x-1) p(x) = \lim_{x\to 1} (x-1) \frac{2x}{x^2(1-x)^2}\,.$$

The latter limit *does not exist*. Hence $x = 1$ is an *irregular* singular point.

3. $P(x) = 0$ when $x = 0$ and 1. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x \frac{x-2}{x^2(1-x)}\,.$$

The limit *does not exist*, and so $x = 0$ is an *irregular* singular point. Considering $x = 1$,

$$\lim_{x\to 1} (x-1) p(x) = \lim_{x\to 1} (x-1) \frac{x-2}{x^2(1-x)} = 1\,.$$

$$\lim_{x\to 1} (x-1)^2 q(x) = \lim_{x\to 1} (x-1)^2 \frac{-3x}{x^2(1-x)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

4. $P(x) = 0$ when $x = 0$ and ± 1. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x \frac{2}{x^3(1-x^2)}\,.$$

The limit *does not exist*, and so $x = 0$ is an *irregular* singular point. Near $x = -1$,

$$\lim_{x\to -1} (x+1) p(x) = \lim_{x\to -1} (x+1) \frac{2}{x^3(1-x^2)} = -1\,.$$

$$\lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}(x+1)^2 \frac{2}{x^3(1-x^2)} = 0\,.$$

Hence $x = -1$ is a *regular* singular point. At $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{2}{x^3(1-x^2)} = -1\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2 \frac{2}{x^3(1-x^2)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

6. The only singular point is at $x = 0$. We find that

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x\frac{x}{x^2} = 1\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\, \frac{x^2-\nu^2}{x^2} = -\nu^2.$$

Hence $x = 0$ is a *regular* singular point.

7. The only singular point is at $x = -3$. We find that

$$\lim_{x\to -3}(x+3)p(x) = \lim_{x\to -3}(x+3)\frac{-2x}{x+3} = 6\,.$$

$$\lim_{x\to -3}(x+3)^2 q(x) = \lim_{x\to -3}(x+3)^2\, \frac{1-x^2}{x+3} = 0.$$

Hence $x = -3$ is a *regular* singular point.

8. Dividing the ODE by $x(1-x^2)^3$, we find that

$$p(x) = \frac{1}{x(1-x^2)} \quad \text{and} \quad q(x) = \frac{2}{x(1+x)^2(1-x)^3}\,.$$

The singular points are at $x = 0$ and ± 1. For $x = 0$,

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x\frac{1}{x(1-x^2)} = 1\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\, \frac{2}{x(1+x)^2(1-x)^3} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. For $x = -1$,

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x+1)\frac{1}{x(1-x^2)} = -\frac{1}{2}\,.$$

$$\lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}(x+1)^2\,\frac{2}{x(1+x)^2(1-x)^3} = -\frac{1}{4}\,.$$

Hence $x = -1$ is a *regular* singular point. For $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{1}{x(1-x^2)} = -\frac{1}{2}\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\,\frac{2}{x(1+x)^2(1-x)^3}\,.$$

The latter limit *does not exist*. Hence $x = 1$ is an *irregular* singular point.

9. Dividing the ODE by $(x+2)^2(x-1)$, we find that

$$p(x) = \frac{3}{(x+2)^2} \quad\text{and}\quad q(x) = \frac{-2}{(x+2)(x-1)}\,.$$

The singular points are at $x = -2$ and 1. For $x = -2$,

$$\lim_{x\to -2}(x+2)p(x) = \lim_{x\to -2}(x+2)\frac{3}{(x+2)^2}\,.$$

The limit *does not exist*. Hence $x = -2$ is an *irregular* singular point. For $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{3}{(x+2)^2} = 0\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\,\frac{-2}{(x+2)(x-1)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

10. $P(x) = 0$ when $x = 0$ and 3. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0}x\,p(x) = \lim_{x\to 0}x\frac{x+1}{x(3-x)} = \frac{1}{3}\,.$$

$$\lim_{x\to 0}x^2 q(x) = \lim_{x\to 0}x^2\frac{-2}{x(3-x)} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. For $x = 3$,

$$\lim_{x\to 3}(x-3)p(x) = \lim_{x\to 3}(x-3)\frac{x+1}{x(3-x)} = -\frac{4}{3}.$$

$$\lim_{x\to 3}(x-3)^2 q(x) = \lim_{x\to 3}(x-3)^2\frac{-2}{x(3-x)} = 0.$$

Hence $x=3$ is a *regular* singular point.

11. Dividing the ODE by (x^2+x-2), we find that

$$p(x) = \frac{x+1}{(x+2)(x-1)} \quad \text{and} \quad q(x) = \frac{2}{(x+2)(x-1)}.$$

The singular points are at $x = -2$ and 1. For $x = -2$,

$$\lim_{x\to -2}(x+2)p(x) = \lim_{x\to -2}\frac{x+1}{x-1} = \frac{1}{3}.$$

$$\lim_{x\to -2}(x+2)^2 q(x) = \lim_{x\to -2}\frac{2(x+2)}{x-1} = 0.$$

Hence $x = -2$ is a *regular* singular point. For $x=1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{x+1}{x+2} = \frac{2}{3}.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}\frac{2(x-1)}{(x+2)} = 0.$$

Hence $x=1$ is a *regular* singular point.

13. Note that $p(x) = ln|x|$ and $q(x) = 3x$. Evidently, $p(x)$ is *not* analytic at $x_0 = 0$. Furthermore, the function $x\,p(x) = x\,ln|x|$ does *not* have a Taylor series about $x_0 = 0$. Hence $x=0$ is an *irregular* singular point.

14. $P(x) = 0$ when $x=0$. Since the three coefficients have no common factors, $x=0$ is a singular point. The Taylor series of $e^x - 1$, about $x=0$, is

$$e^x - 1 = x + x^2/2 + x^3/6 + \cdots.$$

Hence the function $x\,p(x) = 2(e^x-1)/x$ is analytic at $x=0$. Similarly, the Taylor series of $e^{-x}cos\,x$, about $x=0$, is

$$e^{-x}cos\,x = 1 - x + x^3/3 - x^4/6 + \cdots.$$

The function $x^2 q(x) = e^{-x}cos\,x$ is also analytic at $x=0$. Hence $x=0$ is a *regular* singular point.

15. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. The Taylor series of $sin\,x$, about $x = 0$, is

$$sin\,x = x - x^3/3! + x^5/5! - \cdots .$$

Hence the function $x\,p(x) = -3sin\,x/x$ is analytic at $x = 0$. On the other hand, $q(x)$ is a rational function, with

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1+x^2}{x^2} = 1 .$$

Hence $x = 0$ is a *regular* singular point.

16. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. We find that

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{1}{x} = 1 .$$

Although the function $R(x) = cot\,x$ does not have a Taylor series about $x = 0$, note that $x^2 q(x) = x\,cot\,x = 1 - x^2/3 - x^4/45 - 2x^6/945 - \cdots$. Hence $x = 0$ is a *regular* singular point. Furthermore, $q(x) = cot\,x/x^2$ is undefined at $x = \pm\, n\pi$. Therefore the points $x = \pm\, n\pi$ are *also* singular points. First note that

$$\lim_{x\to \pm n\pi} (x \mp n\pi) p(x) = \lim_{x\to \pm n\pi} (x \mp n\pi)\frac{1}{x} = 0 .$$

Furthermore, since $cot\,x$ has period π,

$$\begin{aligned} q(x) &= cot\,x/x = cot(x \mp n\pi)/x \\ &= cot(x \mp n\pi)\frac{1}{(x \mp n\pi) \pm n\pi} . \end{aligned}$$

Therefore

$$(x \mp n\pi)^2 q(x) = (x \mp n\pi) cot(x \mp n\pi)\left[\frac{(x \mp n\pi)}{(x \mp n\pi) \pm n\pi}\right] .$$

From above,

$$(x \mp n\pi) cot(x \mp n\pi) = 1 - (x \mp n\pi)^2/3 - (x \mp n\pi)^4/45 - \cdots .$$

Note that the function in *brackets* is analytic *near* $x = \pm\, n\pi$. It follows that the function $(x \mp n\pi)^2 q(x)$ is also analytic near $x = \pm\, n\pi$. Hence all the singular points are *regular*.

18. The singular points are located at $x = \pm\, n\pi$, $n = 0, 1, \cdots$. Dividing the ODE by $x\,sin\,x$, we find that $x\,p(x) = 3\,csc\,x$ and $x^2 q(x) = x^2 csc\,x$. Evidently, $x\,p(x)$ is not even defined at $x = 0$. Hence $x = 0$ is an *irregular* singular point. On the other hand, the Taylor series of $x\,csc\,x$, about $x = 0$, is

$$x\,csc\,x = 1 + x^2/6 + 7x^4 360 + \cdots .$$

Noting that $csc(x \mp n\pi) = (-1)^n csc\,x$,

$$\begin{aligned}(x \mp n\pi)p(x) &= 3(-1)^n (x \mp n\pi) csc(x \mp n\pi)/x \\ &= 3(-1)^n (x \mp n\pi) csc(x \mp n\pi)\left[\frac{1}{(x \mp n\pi) \pm n\pi}\right].\end{aligned}$$

It is apparent that $(x \mp n\pi)p(x)$ is analytic at $x = \pm n\pi$. Similarly,

$$\begin{aligned}(x \mp n\pi)^2 q(x) &= (x \mp n\pi)^2 csc\,x \\ &= (-1)^n (x \mp n\pi)^2 csc(x \mp n\pi),\end{aligned}$$

which is also analytic at $x = \pm n\pi$. Hence all other singular points are *regular*.

20. $x = 0$ is the only singular point. Dividing the ODE by $2x^2$, we have $p(x) = 3/(2x)$ and $q(x) = -x^{-2}(1+x)/2$. It follows that

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{3}{2x} = \frac{3}{2},$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{-(1+x)}{2x^2} = -\frac{1}{2}.$$

Hence $x = 0$ is a *regular* singular point. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substitution into the ODE results in

$$2x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + 3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - (1+x)\sum_{n=0}^{\infty}a_n x^n = 0.$$

That is,

$$2\sum_{n=2}^{\infty}n(n-1)a_n\,x^n + 3\sum_{n=1}^{\infty}n\,a_n x^n - \sum_{n=0}^{\infty}a_n x^n - \sum_{n=1}^{\infty}a_{n-1}x^n = 0.$$

It follows that

$$-a_0 + (2a_1 - a_0)x + \sum_{n=2}^{\infty}[2n(n-1)a_n + 3n\,a_n - a_n - a_{n-1}]x^n = 0.$$

Equating the coefficients to *zero*, we find that $a_0 = 0$, $2a_1 - a_0 = 0$, and

$$(2n-1)(n+1)a_n = a_{n-1}, \quad n = 2, 3, \cdots .$$

We conclude that *all* the a_n are *equal to zero*. Hence $y(x) = 0$ is the only solution that can be obtained.

22. Based on Prob. 21, the change of variable, $x = 1/\xi$, transforms the ODE into the

form

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + y = 0\,.$$

Evidently, $\xi = 0$ is a singular point. Now $p(\xi) = 2/\xi$ and $q(\xi) = 1/\xi^4$. Since the value of $\lim\limits_{\xi \to 0} \xi^2 q(\xi)$ does not exist, $\xi = 0$, that is, $x = \infty$, is an *irregular* singular point.

24. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \left(1 - \frac{1}{\xi^2}\right) \frac{d^2y}{d\xi^2} + \left[2\xi^3 \left(1 - \frac{1}{\xi^2}\right) + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0\,,$$

that is,

$$\left(\xi^4 - \xi^2\right) \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2\xi}{\xi^2 - 1} \text{ and } q(\xi) = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)}\,.$$

It follows that

$$\lim_{\xi \to 0} \xi\, p(\xi) = \lim_{\xi \to 0} \xi \frac{2\xi}{\xi^2 - 1} = 0\,,$$

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)} = -\alpha(\alpha+1)\,.$$

Hence $\xi = 0$ $(x = \infty)$ is a *regular* singular point.

26. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + \left[2\xi^3 + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \lambda\, y = 0\,,$$

that is,

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\left(\xi^3 + \xi\right) \frac{dy}{d\xi} + \lambda\, y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2(\xi^2+1)}{\xi^3} \text{ and } q(\xi) = \frac{\lambda}{\xi^4}\,.$$

It immediately follows that the limit $\lim\limits_{\xi \to 0} \xi\, p(\xi)$ *does not exist*. Hence $\xi = 0$ $(x = \infty)$

is an *irregular* singular point.

27. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} - \frac{1}{\xi} y = 0 \,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2}{\xi} \text{ and } q(\xi) = \frac{-1}{\xi^5}.$$

We find that

$$\lim_{\xi \to 0} \xi \, p(\xi) = \lim_{\xi \to 0} \xi \frac{2}{\xi} = 2 \,,$$

but

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \, \frac{(-1)}{\xi^5} \,.$$

The latter limit *does not exist*. Hence $\xi = 0$ $(x = \infty)$ is an *irregular* singular point.