Chapter Four

Section 4.1

1. The differential equation is in standard form. Its coefficients, as well as the function $g(t) = t$, are continuous *everywhere*. Hence solutions are valid on the entire real line.

3. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $t = 0$ and $t = 1$. Hence the solutions are valid on the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.

4. The coefficients are continuous everywhere, but the function $g(t) = ln\, t$ is defined and
continuous only on the interval $(0, \infty)$. Hence solutions are defined for positive reals.

5. Writing the equation in standard form, the coefficients are *rational* functions with a singularity at $x_0 = 1$. Furthermore, $p_4(x) = tan\, x/(x-1)$ is *undefined*, and hence not continuous, at $x_k = \pm(2k+1)\pi/2$, $k = 0, 1, 2, \cdots$. Hence solutions are defined on any *interval* that *does not* contain x_0 or x_k.

6. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $x = \pm 2$. Hence the solutions are valid on the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

7. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = -14$. Hence the functions are linearly *independent*.

9. Evaluating the Wronskian of the four functions, $W(f_1, f_2, f_3, f_4) = 0$. Hence the functions are linearly *dependent*. To find a linear relation among the functions, we need to find constants c_1, c_2, c_3, c_4, not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) + c_4 f_4(t) = 0\,.$$

Collecting the common terms, we obtain

$$(c_2 + 2c_3 + c_4)t^2 + (2c_1 - c_3 + c_4)t + (-3c_1 + c_2 + c_4) = 0\,,$$

which results in *three* equations in *four* unknowns. Arbitrarily setting $c_4 = -1$, we can solve the equations $c_2 + 2c_3 = 1$, $2c_1 - c_3 = 1$, $-3c_1 + c_2 = 1$, to find that $c_1 = 2/7$, $c_2 = 13/7$, $c_3 = -3/7$. Hence

$$2f_1(t) + 13f_2(t) - 3f_3(t) - 7f_4(t) = 0\,.$$

10. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = 156$. Hence the functions are linearly *independent*.

11. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have
$W(1, cos\,t, sin\,t) = 1$.

12. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, cos\,t, sin\,t) = 1$.

14. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, e^{-t}, t\,e^{-t}) = e^{-2t}$.

15. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, x, x^3) = 6x$.

16. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(x, x^2, 1/x) = 6/x$.

18. The operation of taking a derivative is linear, and hence

$$(c_1y_1 + c_2y_2)^{(k)} = c_1y_1^{(k)} + c_2y_2^{(k)}.$$

It follows that

$$L[c_1y_1 + c_2y_2] = c_1y_1^{(n)} + c_2y_2^{(n)} + p_1\left[c_1y_1^{(n-1)} + c_2y_2^{(n-1)}\right] + \cdots + p_n[c_1y_1 + c_2y_2].$$

Rearranging the terms, we obtain $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$. Since y_1 and y_2 are solutions, $L[c_1y_1 + c_2y_2] = 0$. The rest follows by induction.

19(a). Note that $d^k(t^n)/dt^k = n(n-1)\cdots(n-k+1)t^{n-k}$, for $k = 1, 2, \cdots, n$.
Hence

$$L[t^n] = a_0\,n! + a_1[n(n-1)\cdots 2]t + \cdots a_{n-1}\,n\,t^{n-1} + a_n\,t^n.$$

(b). We have $d^k(e^{rt})/dt^k = r^ke^{rt}$, for $k = 0, 1, 2, \cdots$. Hence

$$\begin{aligned} L\left[e^{rt}\right] &= a_0\,r^ne^{rt} + a_1r^{n-1}e^{rt} + \cdots a_{n-1}r\,e^{rt} + a_n\,e^{rt} \\ &= \left[a_0\,r^n + a_1r^{n-1} + \cdots a_{n-1}\,r\, + a_n\right]e^{rt}. \end{aligned}$$

(c). Set $y = e^{rt}$, and substitute into the ODE. It follows that $r^4 - 5r^2 + 4 = 0$, with $r = \pm 1, \pm 2$. Furthermore, $W(e^t, e^{-t}, e^{2t},\, e^{-2t}) = 72$.

20(a). Let $f(t)$ and $g(t)$ be arbitrary functions. Then $W(f, g) = fg' - f'g$. Hence $W'(f, g) = f'g' + fg'' - f''g - f'g' = fg'' - f''g$. That is,

$$W'(f, g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}.$$

Now expand the 3-*by*-3 determinant as

$$W(y_1, y_2, y_3) = y_1 \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}.$$

Differentiating, we obtain

$$\begin{aligned} W'(y_1, y_2, y_3) &= y_1' \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2' \begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3' \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} + \\ &+ y_1 \begin{vmatrix} y_2' & y_3' \\ y_2''' & y_3''' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1''' & y_3''' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1''' & y_2''' \end{vmatrix}. \end{aligned}$$

The *second* line follows from the observation above. Now we find that

$$W'(y_1, y_2, y_3) = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hence the assertion is true, since the first determinant is equal to *zero*.

(b). Based on the properties of determinants,

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3\, y_1 & p_3\, y_2 & p_3\, y_3 \\ p_2\, y_1' & p_2\, y_2' & p_2\, y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

Adding the *first two* rows to the *third* row does not change the value of the determinant. Since the functions are assumed to be solutions of the given ODE, addition of the rows results in

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3\, y_1 & p_3\, y_2 & p_3\, y_3 \\ p_2\, y_1' & p_2\, y_2' & p_2\, y_3' \\ -p_1\, y_1'' & -p_1\, y_2'' & -p_1\, y_3'' \end{vmatrix}.$$

It follows that $p_2(t)p_3(t)W' = -p_1(t)p_2(t)p_3(t)W$. As long as the coefficients are not zero, we obtain $W' = -p_1(t)W$.

(c). The first order equation $W' = -p_1(t)W$ is linear, with integrating factor $\mu(t) = = exp\left(\int p_1(t)dt\right)$. Hence $W(t) = c\, exp\left(-\int p_1(t)dt\right)$. Furthermore, $W(t)$ is *zero* only if $c = 0$.

(d). It can be shown, by mathematical induction, that

$$W'(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y_1' & y_2' & \cdots & y_{n-1}' & y_n' \\ \vdots & & & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_{n-1}^{(n)} & y_n^{(n)} \end{vmatrix}.$$

Based on the reasoning in Part(b), it follows that

$$p_2(t)p_3(t)\cdots p_n(t)W' = -p_1(t)p_2(t)p_3(t)\cdots p_n(t)W,$$

and hence $W' = -p_1(t)W$.

22. Inspection of the coefficients reveals that $p_1(t) = 0$. Based on Prob. 20, we find that $W' = 0$, and hence $W = c$.

23. After writing the equation in standard form, observe that $p_1(t) = 2/t$. Based on the results in Prob. 20, we find that $W' = (-2/t)W$, and hence $W = c/t^2$.

24. Writing the equation in standard form, we find that $p_1(t) = 1/t$. Using *Abel's formula*, the Wronskian has the form $W(t) = c\,exp\big(-\int \frac{1}{t}dt\big) = c/t$.

25(a). Assuming that $c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0$, then taking the first $n-1$ derivatives of this equation results in

$$c_1y_1^{(k)}(t) + c_2y_2^{(k)}(t) + \cdots + c_ny_n^{(k)}(t) = 0$$

for $k = 0, 1, \cdots, n-1$. Setting $t = t_0$, we obtain a system of n algebraic equations with unknowns $c_1, c_2, \cdots, c_n$. The Wronskian, $W(y_1, y_2, \cdots, y_n)(t_0)$, is the determinant of the coefficient matrix. Since system of equations is homogeneous, $W(y_1, y_2, \cdots, y_n)(t_0) \neq 0$ implies that the only solution is the *trivial* solution, $c_1 = c_2 = \cdots = c_n = 0$.

(b). Suppose that $W(y_1, y_2, \cdots, y_n)(t_0) = 0$ for some t_0. Consider the system of algebraic
equations

$$c_1y_1^{(k)}(t_0) + c_2y_2^{(k)}(t_0) + \cdots + c_ny_n^{(k)}(t_0) = 0,$$

$k = 0, 1, \cdots, n-1$, with unknowns $c_1, c_2, \cdots, c_n$. Vanishing of the Wronskian, which is the determinant of the coefficient matrix, implies that there is a *nontrivial* solution of the system of homogeneous equations. That is, there exist constants $c_1, c_2, \cdots, c_n$, not all zero, which satisfy the above equations. Now let

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t).$$

Since the ODE is linear, $y(t)$ is also a *nonzero* solution. Based on the system of algebraic equations above, $y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$. This contradicts the uniqueness of the *identically zero* solution.

26. Let $y(t) = y_1(t)v(t)$. Then $y' = y_1'v + y_1v'$, $y'' = y_1''v + 2y_1'v' + y_1v''$, and $y''' = y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$. Substitution into the ODE results in

$$y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v''' + p_1[y_1''v + 2y_1'v' + y_1v''] + p_2[y_1'v + y_1v'] + p_3y_1v = 0.$$

Since y_1 is assumed to be a solution, all terms containing the factor $v(t)$ vanish. Hence

$$y_1 v''' + [p_1 y_1 + 3y_1']v'' + [3y_1'' + 2p_1 y_1' + p_2 y_1]v' = 0\,,$$

which is a *second order* ODE in the variable $u = v'$.

28. First write the equation in standard form:

$$y''' - 3\frac{t+2}{t(t+3)}y'' + 6\frac{t+1}{t^2(t+3)}y' - \frac{6}{t^2(t+3)}y = 0\,.$$

Let $y(t) = t^2 v(t)$. Substitution into the given ODE results in

$$t^2 v''' + 3\frac{t(t+4)}{t+3}v'' = 0\,.$$

Set $w = v''$. Then w is a solution of the first order differential equation

$$w' + 3\frac{t+4}{t(t+3)}w = 0\,.$$

This equation is *linear*, with integrating factor $\mu(t) = t^4/(t+3)$. The general solution is $w = c(t+3)/t^4$. Integrating twice, it follows that $v(t) = c_1 t^{-1} + c_1 t^{-2} + c_2 t + c_3$. Hence $y(t) = c_1 t + c_1 + c_2 t^3 + c_3 t^2$. Finally, since $y_1(t) = t^2$ and $y_2(t) = t^3$are given solutions, the *third* independent solution is $y_3(t) = c_1 t + c_1$.