Section 3.7

1. The solution of the homogeneous equation is $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{5t}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{e^{3t}(2e^t)}{W(t)} dt \\ &= 2e^{-t} \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{2t}(2e^t)}{W(t)} dt \\ &= -\, e^{-2t} \end{aligned}$$

Hence the particular solution is $Y(t) = 2e^t - e^t = e^t$.

3. The solution of the homogeneous equation is $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$. The functions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-2t}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{-t}(3e^{-t})}{W(t)} dt \\ &= -\, 3t^2/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{-t}(3e^{-t})}{W(t)} dt \\ &= 3t \end{aligned}$$

Hence the particular solution is $Y(t) = -\, 3t^2 e^{-t}/2 + 3t^2 e^{-t} = 3t^2 e^{-t}/2$.

4. The functions $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^t$. First write the equation in standard form, so that $g(t) = 4e^{t/2}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{t/2}\left(4e^{t/2}\right)}{W(t)} dt \\ &= -\, 2t^2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{t/2}\left(4e^{t/2}\right)}{W(t)}dt \\ &= 4t \end{aligned}$$

Hence the particular solution is $Y(t) = -2t^2e^{t/2} + 4t^2e^{t/2} = 2t^2e^{t/2}$.

6. The solution of the homogeneous equation is $y_c(t) = c_1 cos\, 3t + c_2 sin\, 3t$. The two functions $y_1(t) = cos\, 3t$ and $y_2(t) = sin\, 3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{sin\, 3t(9\, sec^2 3t)}{W(t)}dt \\ &= -csc\, 3t \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{cos\, 3t(9\, sec^2 3t)}{W(t)}dt \\ &= ln|sec\, 3t + tan\, 3t| \end{aligned}$$

Hence the particular solution is $Y(t) = -1 + (sin\, 3t)ln|sec\, 3t + tan\, 3t|$. The general solution is given by $y(t) = c_1 cos\, 3t + c_2 sin\, 3t + (sin\, 3t)ln|sec\, 3t + tan\, 3t| - 1$.

7. The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-4t}$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)}dt \\ &= -ln\, t \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)}dt \\ &= -1/t \end{aligned}$$

Hence the particular solution is $Y(t) = -e^{-2t}ln\, t - e^{-2t}$. Since the *second term* is a solution of the homogeneous equation, the general solution is given by $y(t) = c_1 e^{-2t} +$ $+ c_2 te^{-2t} - e^{-2t}ln\, t$.

8. The solution of the homogeneous equation is $y_c(t) = c_1 cos\, 2t + c_2 sin\, 2t$. The two functions $y_1(t) = cos\, 2t$ and $y_2(t) = sin\, 2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{sin\,2t(3\,csc\,2t)}{W(t)}dt \\ &= -\,3t/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{cos\,2t(3\,csc\,2t)}{W(t)}dt \\ &= \frac{3}{4}ln|sin\,2t| \end{aligned}$$

Hence the particular solution is $Y(t) = -\frac{3}{2}tcos\,2t + \frac{3}{4}(sin\,3t)ln|sin\,2t|$. The general solution is given by $y(t) = c_1 cos\,2t + c_2 sin\,2t - \frac{3}{2}tcos\,2t + \frac{3}{4}(sin\,3t)ln|sin\,2t|$.

9. The functions $y_1(t) = cos\,(t/2)$ and $y_2(t) = sin(t/2)$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = 1/2$. First write the ODE in standard form, so that $g(t) = sec(t/2)/2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{cos\,(t/2)[sec(t/2)]}{2W(t)}dt \\ &= 2\,ln[cos\,(t/2)] \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{sin(t/2)[sec(t/2)]}{2W(t)}dt \\ &= t \end{aligned}$$

The particular solution is $Y(t) = 2cos(t/2)ln[cos\,(t/2)] + t\,sin(t/2)$. The general solution is given by

$$y(t) = c_1 cos\,(t/2) + c_2 sin(t/2) + 2\,cos(t/2)\,ln[cos\,(t/2)] + t\,sin(t/2).$$

10. The solution of the homogeneous equation is $y_c(t) = c_1 e^t + c_2 te^t$. The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions, with $W(y_1, y_2) = e^{2t}$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^t(e^t)}{W(t)(1+t^2)}dt \\ &= -\frac{1}{2}ln\big(1+t^2\big) \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^t(e^t)}{W(t)(1+t^2)}dt \\ &= arctan\,t \end{aligned}$$

The particular solution is $Y(t) = -\frac{1}{2}e^t\,ln(1+t^2) + te^t\,arctan(t)$. Hence the general solution is given by $y(t) = c_1 e^t + c_2 t e^t - \frac{1}{2} e^t \, ln(1+t^2) + t e^t \, arctan(t)$.

12. The functions $y_1(t) = cos\,2t$ and $y_2(t) = sin\,2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)$ $y_1(t) + u_2(t)\, y_2(t)$,
in which

$$u_1(t) = -\frac{1}{2}\int^t g(s)\, sin\,2s\, ds$$

$$u_2(t) = \frac{1}{2}\int^t g(s)\, cos\,2s\, ds$$

Hence the particular solution is

$$Y(t) = -\frac{1}{2} cos\,2t \int^t g(s)\, sin\,2s\, ds + \frac{1}{2} sin\,2t \int^t g(s)\, cos\,2s\, ds\,.$$

Note that $sin\,2t\, cos\,2s - cos\,2t\, sin\,2s = sin(2t - 2s)$. It follows that

$$Y(t) = \frac{1}{2}\int^t g(s) sin(2t-2s) ds\,.$$

The general solution of the differential equation is given by

$$y(t) = c_1 cos\,2t + c_2 sin\,2t + \frac{1}{2}\int^t g(s) sin(2t-2s) ds\,.$$

13. Note first that $p(t) = 0\,, q(t) = -2/t^2$ and $g(t) = (3t^2-1)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1, y_2) = -3$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{t^{-1}(3t^2-1)}{t^2\, W(t)} dt \\ &= t^{-2}/6 + ln\, t \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{t^2(3t^2-1)}{t^2\, W(t)} dt \\ &= -t^3/3 + t/3 \end{aligned}$$

Therefore $Y(t) = 1/6 + t^2 ln\, t - t^2/3 + 1/3$. Hence the general solution is

$$y(t) = c_1 t^2 + c_2 t^{-1} + t^2 ln\, t + 1/2\,.$$

15. Observe that $g(t) = t\,e^{2t}$. The functions $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions. The Wronskian of these two functions is $W(y_1,y_2) = t\,e^t$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{e^t(t\,e^{2t})}{W(t)}dt \\ &= -\,e^{2t}/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{(1+t)(t\,e^{2t})}{W(t)}dt \\ &= t\,e^t \end{aligned}$$

Therefore $Y(t) = -(1+t)e^{2t}/2 + t\,e^{2t} = -\,e^{2t}/2 + t\,e^{2t}/2\,.$

16. Observe that $g(t) = 2(1-t)\,e^{-t}$. Direct substitution of $y_1(t) = e^t$ and $y_2(t) = t$ verifies that they are solutions of the homogeneous equation. The Wronskian of the two solutions is $W(y_1,y_2) = (1-t)\,e^t$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{2t(1-t)e^{-t}}{W(t)}dt \\ &= te^{-2t} + e^{-2t}/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{2(1-t)}{W(t)}dt \\ &= -\,2\,e^{-t} \end{aligned}$$

Therefore $Y(t) = te^{-t} + e^{-t}/2 - 2t\,e^{-t} = -\,te^{-t} + \,e^{-t}/2\,.$

17. Note that $g(x) = ln\,x\,$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 ln\,x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = x^3$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$\begin{aligned} u_1(x) &= -\int \frac{x^2 ln\,x(ln\,x)}{W(x)}dx \\ &= -\,(ln\,x)^3/3 \end{aligned}$$

$$\begin{aligned} u_2(x) &= \int \frac{x^2(\ln x)}{W(x)}dx \\ &= (\ln x)^2/2 \end{aligned}$$

Therefore $Y(x) = -x^2(\ln x)^3/3 + x^2(\ln x)^3/2 = x^2(\ln x)^3/6$.

19. First write the equation in *standard form*. Note that the forcing function becomes $g(x)/(1-x)$. The functions $y_1(x) = e^x$ and $y_2(x) = x$ are a fundamental set of solutions,
as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = (1-x)e^x$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$u_1(x) = -\int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)}d\tau$$

$$u_2(x) = \int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)}d\tau$$

Therefore

$$\begin{aligned} Y(x) &= -e^x\int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)}d\tau + x\int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)}d\tau \\ &= \int^x \frac{(xe^\tau - e^x\tau)g(\tau)}{(1-\tau)^2e^\tau}d\tau\,. \end{aligned}$$

20. First write the equation in *standard form*. The forcing function becomes $g(x)/x^2$. The functions $y_1(x) = x^{-1/2}\sin x$ and $y_2(x) = x^{-1/2}\cos x$ are a fundamental set of solutions. The Wronskian of the solutions is $W(y_1,y_2) = -1/x$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$u_1(x) = \int^x \frac{\cos\tau\,(g(\tau))}{\tau\sqrt{\tau}}d\tau$$

$$u_2(x) = -\int^x \frac{\sin\tau\,(g(\tau))}{\tau\sqrt{\tau}}d\tau$$

Therefore

$$\begin{aligned} Y(x) &= \frac{sin\,x}{\sqrt{x}}\int^x \frac{cos\,\tau\,(g(\tau))}{\tau\sqrt{\tau}}dt - \frac{cos\,x}{\sqrt{x}}\int^x \frac{sin\,\tau\,(g(\tau))}{\tau\sqrt{\tau}}d\tau \\ &= \frac{1}{\sqrt{x}}\int^x \frac{sin(x-\tau)\,g(\tau)}{\tau\sqrt{\tau}}d\tau\,. \end{aligned}$$

21. Let $y_1(t)$ and $y_2(t)$ be a fundamental set of solutions, and $W(t) = W(y_1, y_2)$ be the corresponding Wronskian. Any solution, $u(t)$, of the homogeneous equation is a linear combination $u(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$. Invoking the initial conditions, we require that

$$\begin{aligned} y_0 &= \alpha_1\,y_1(t_0) + \alpha_2\,y_2(t_0) \\ y_0' &= \alpha_1\,y_1'(t_0) + \alpha_2\,y_2'(t_0) \end{aligned}$$

Note that this system of equations has a unique solution, since $W(t_0) \neq 0$. Now consider the *nonhomogeneous* problem, $L[v] = g(t)$,with *homogeneous* initial conditions. Using the method of variation of parameters, the particular solution is given by

$$Y(t) = \,-\,y_1(t)\int_{t_0}^t \frac{y_2(s)\,g(s)}{W(s)}ds + y_2(t)\int_{t_0}^t \frac{y_1(s)\,g(s)}{W(s)}ds\,.$$

The general solution of the IVP (iii) is

$$\begin{aligned} v(t) &= \beta_1 y_1(t) + \beta_2 y_2(t) + Y(t) \\ &= \beta_1 y_1(t) + \beta_2 y_2(t) + y_1(t)u_1(t) + y_2(t)u_2(t) \end{aligned}$$

in which u_1 and u_2 are defined above. Invoking the initial conditions, we require that

$$\begin{aligned} 0 &= \beta_1 y_1(t_0) + \beta_2 y_2(t_0) + Y(t_0) \\ 0 &= \beta_1 y_1'(t_0) + \beta_2 y_2'(t_0) + Y'(t_0) \end{aligned}$$

Based on the definition of u_1 and u_2 , $Y(t_0) = 0$. Furthermore, since $y_1 u_1' + y_2 u_2' = 0$, it follows that $Y'(t_0) = 0$. Hence the only solution of the above system of equations is the *trivial solution*. Therefore $v(t) = Y(t)$. Now consider the function $y = u{+}v$. Then $L[y] = L[u+v] = L[u] + L[v] = g(t)$. That is, $y(t)$ is a solution of the nonhomogeneous
problem. Further, $y(t_0) = u(t_0) + v(t_0) = y_0$, and similarly, $y'(t_0) = y_0'$. By the uniqueness theorems, $y(t)$ is the unique solution of the initial value problem.

23. A fundamental set of solutions is $y_1(t) = cos\,t$ and $y_2(t) = sin\,t$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = 1$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^t \frac{cos(s)\,sin(t) - cos(t)\,sin(s)}{W(s)}g(s)ds \\ &= \int_{t_0}^t [cos(s)\,sin(t) - cos(t)\,sin(s)]g(s)ds\,. \end{aligned}$$

Finally, we have $cos(s)\,sin(t) - cos(t)\,sin(s) = sin(t-s)$.

24. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = e^{bt}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = (b-a)exp[(a+b)t]$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{W(s)} g(s)ds \\ &= \frac{1}{b-a}\int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{exp[(a+b)s]} g(s)ds\,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a}\int_{t_0}^{t} \left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)ds\,.$$

26. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{W(s)} g(s)ds \\ &= \frac{1}{b-a}\int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{exp[(a+b)s]} g(s)ds\,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a}\int_{t_0}^{t} \left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)ds\,.$$

26. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{te^{as+at} - s\,e^{at+as}}{W(s)} g(s)ds \\ &= \int_{t_0}^{t} \frac{(t-s)e^{as+at}}{e^{2as}} g(s)ds\,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \int_{t_0}^{t} (t-s)e^{a(t-s)} g(s)ds\,.$$

27. Depending on the values of a, b and c, the operator $aD^2 + bD + c$ can have *three* types of fundamental solutions.

(i) The characteristic roots $r_{1,2} = \alpha\,, \beta\,;\ \alpha \neq \beta\,.$ $y_1(t) = e^{\alpha t}$ and $y_2(t) = e^{\beta t}$.

$$K(t) = \frac{1}{\beta - \alpha}\left[e^{\beta t} - e^{\alpha t}\right].$$

(ii) The characteristic roots $r_{1,2} = \alpha\,,\beta\,;\ \alpha = \beta\,.\ \ y_1(t) = e^{\alpha t}$ and $y_2(t) = te^{\alpha t}$.

$$K(t) = t\,e^{\alpha t}.$$

(iii) The characteristic roots $r_{1,2} = \lambda \pm i\,\mu\,.\ y_1(t) = e^{\lambda t}cos\,\mu t$ and $y_2(t) = e^{\lambda t}sin\,\mu t.$

$$K(t) = \frac{1}{\mu}e^{\lambda t}sin\,\mu t\,.$$

28. Let $y(t) = v(t)y_1(t)$, in which $y_1(t)$ is a solution of the *homogeneous equation*. Substitution into the given ODE results in

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)[v'y_1 + vy_1'] + q(t)vy_1 = g(t)\,.$$

By assumption, $y_1'' + p(t)y_1 + q(t)y_1 = 0$, hence $v(t)$ must be a solution of the ODE

$$v''y_1 + [2y_1' + p(t)y_1]v' = g(t)\,.$$

Setting $w = v'$, we also have $w'y_1 + [2y_1' + p(t)y_1]w = g(t)$.

30. First write the equation as $y'' + 7t^{-1}y + 5t^{-2}\,y = t^{-1}$. As shown in Prob. 28, the function $y(t) = t^{-1}v(t)$ is a solution of the given ODE as long as v is a solution of

$$t^{-1}\,v'' + [\,-2t^{-2} + 7t^{-2}]v' = t^{-1}\,,$$

that is, $v'' + 5t^{-1}\,v' = 1$. This ODE is *linear and first order* in v'. The integrating factor is $\mu = t^5$. The solution is $v' = t/6 + c\,t^{-5}$. Direct integration now results in $v(t) = t^2/12 + c_1t^{-4} + c_2$. Hence $y(t) = t/12 + c_1t^{-5} + c_2t^{-1}$.

31. Write the equation as $y'' - t^{-1}(1+t)y + t^{-1}\,y = t\,e^{2t}$. As shown in Prob. 28, the function $y(t) = (1+t)v(t)$ is a solution of the given ODE as long as v is a solution of

$$(1+t)\,v'' + \left[2 - t^{-1}(1+t)^2\right]v' = t\,e^{2t},$$

that is, $v'' - \frac{1+t^2}{t(t+1)}\,v' = \frac{t}{t+1}e^{2t}$. This equation is first order linear in v', with integrating factor $\mu = t^{-1}(1+t)^2e^{-t}$. The solution is $v' = (t^2e^{2t} + c_1te^t)/(1+t)^2$. Integrating, we obtain $v(t) = e^{2t}/2 - e^{2t}/(t+1) + c_1e^t/(t+1) + c_2$. Hence the solution of the original ODE is $y(t) = (t-1)e^{2t}/2 + c_1e^t + c_2(t+1)$.

32. Write the equation as $y'' + t(1-t)^{-1}y - (1-t)^{-1}\,y = 2(1-t)\,e^{-t}$. The function $y(t) = e^tv(t)$ is a solution to the given ODE as long as v is a solution of