Section 6.5

2. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Applying the initial conditions,

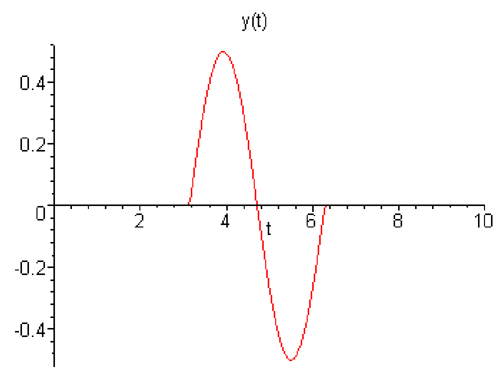
$$s^2 Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4} = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}\,.$$

Applying Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{2}sin(2t-2\pi)u_{\pi}(t) - \frac{1}{2}sin(2t-4\pi)u_{2\pi}(t) \\ &= \frac{1}{2}sin(2t)[\,u_{\pi}(t) - u_{2\pi}(t)\,]\,. \end{aligned}$$



4. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s\,y(0) - y'(0) - Y(s) = -20\,e^{-3s}.$$

Applying the initial conditions,

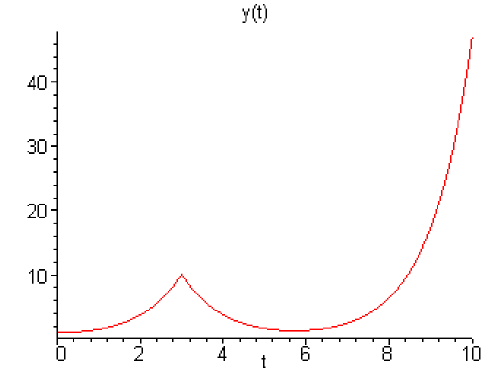
$$s^2 Y(s) - Y(s) - s = -20\,e^{-3s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{s^2-1} - \frac{20\,e^{-3s}}{s^2-1}\,.$$

Using a *table of transforms*, and Theorem 6.3.1 , the solution of the IVP is

$$y(t) = cosh\,t - 20\,sinh(t-3)u_3(t)\,.$$



6. Taking the initial conditions into consideration, the transform of the ODE is

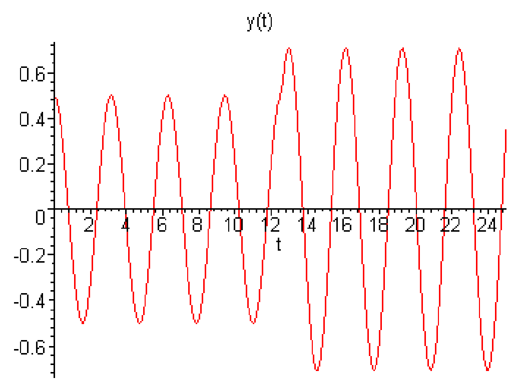
$$s^2\,Y(s) + 4Y(s) - s/2 = e^{-4\pi s}.$$

Solving for the transform,

$$Y(s) = \frac{s/2}{s^2+4} + \frac{e^{-4\pi s}}{s^2+4}\,.$$

Using a *table of transforms*, and Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{2}cos\,2t + \frac{1}{2}sin(2t-8\pi)u_{4\pi}(t) \\ &= \frac{1}{2}cos\,2t + \frac{1}{2}sin(2t)\,u_{4\pi}(t)\,. \end{aligned}$$



8. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 4Y(s) = 2\,e^{-(\pi/4)s}.$$

Applying the initial conditions,

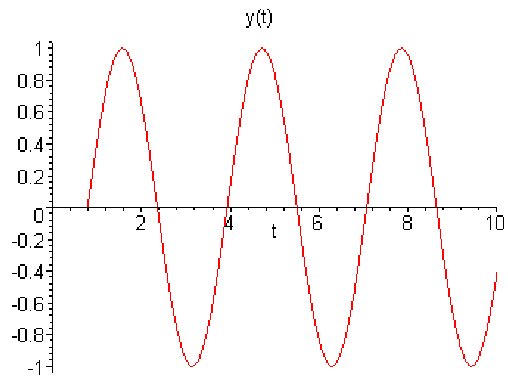
$$s^2\,Y(s) + 4Y(s) = 2\,e^{-(\pi/4)s}.$$

Solving for the transform,

$$Y(s) = \frac{2\,e^{-(\pi/4)s}}{s^2+4}\,.$$

Applying Theorem 6.3.1 , the solution of the IVP is

$$y(t) = sin\Big(2t - \frac{\pi}{2}\Big)u_{\pi/4}(t) = \;-\,cos(2t)\,u_{\pi/4}(t)\,.$$



9. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + Y(s) = \frac{e^{-(\pi/2)s}}{s} + 3\,e^{-(3\pi/2)s} - \frac{e^{-2\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-(\pi/2)s}}{s(s^2+1)} + \frac{3\,e^{-(3\pi/2)s}}{s^2+1} - \frac{e^{-2\pi s}}{s(s^2+1)}\,.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Hence

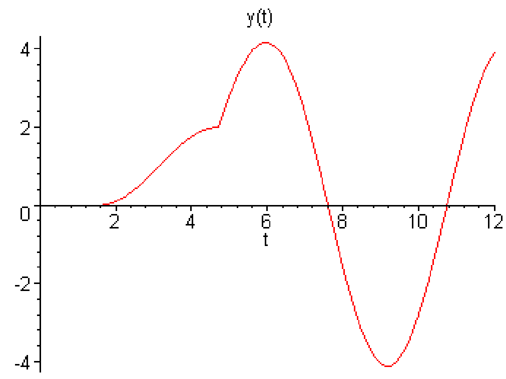
$$Y(s) = \frac{e^{-(\pi/2)s}}{s} - \frac{s\,e^{-(\pi/2)s}}{s^2+1} + \frac{3\,e^{-(3\pi/2)s}}{s^2+1} - \frac{e^{-2\pi s}}{s} + \frac{s\,e^{-2\pi s}}{s^2+1}\,.$$

Based on Theorem 6.3.1 , the solution of the IVP is

$$\begin{aligned} y(t) &= u_{\pi/2}(t) - cos\Big(t - \frac{\pi}{2}\Big)u_{\pi/2}(t) + 3\,sin\Big(t - \frac{3\pi}{2}\Big)u_{3\pi/2}(t) - \\ &\quad - u_{2\pi}(t) + cos(t-2\pi)u_{2\pi}(t)\,. \end{aligned}$$

That is,

$$y(t) = [1 - sin(t)]\,u_{\pi/2}(t) + 3\,cos(t)\,u_{3\pi/2}(t) - [1 - cos(t)]\,u_{2\pi}(t)\,.$$



10. Taking the transform of both sides of the ODE,

$$2s^2Y(s) + sY(s) + 4Y(s) = \int_0^\infty e^{-st}\,\delta\Big(t - \frac{\pi}{6}\Big) sin\, t\, dt$$
$$= \frac{1}{2}\, e^{-(\pi/6)s}.$$

Solving for the transform,

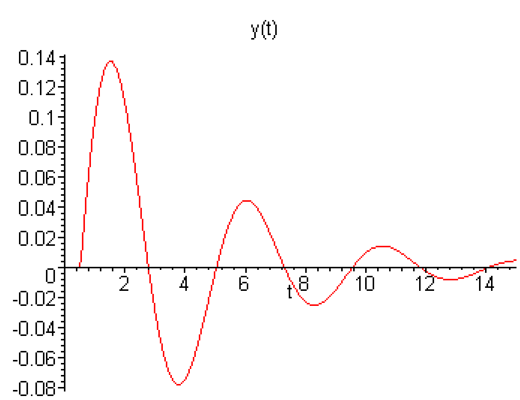
$$Y(s) = \frac{e^{-(\pi/6)s}}{2(2s^2 + s + 4)}\,.$$

First write

$$\frac{1}{2(2s^2 + s + 4)} = \frac{\frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{31}{16}}\,.$$

It follows that

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{\sqrt{31}} e^{-(t-\pi/6)/4} \cdot\, sin\frac{\sqrt{31}}{4}\Big(t - \frac{\pi}{6}\Big) u_{\pi/6}(t)\,.$$



11. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + 2s\,Y(s) + 2\,Y(s) = \frac{s}{s^2+1} + e^{-(\pi/2)s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-(\pi/2)s}}{s^2+2s+2}\,.$$

Using partial fractions,

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{1}{5}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right].$$

We can also write

$$\frac{s+4}{s^2+2s+2} = \frac{(s+1)+3}{(s+1)^2+1}\,.$$

Let

$$Y_1(s) = \frac{s}{(s^2+1)(s^2+2s+2)}\,.$$

Then

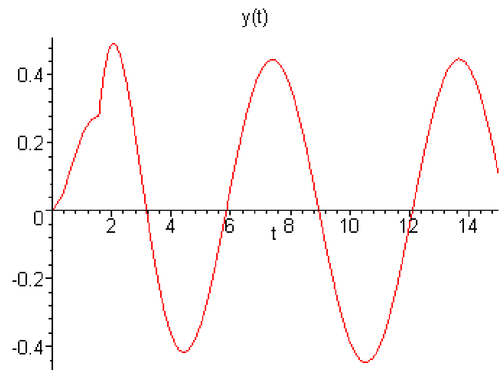
$$\mathcal{L}^{-1}[Y_1(s)] = \frac{1}{5}\,cos\,t + \frac{2}{5}\,sin\,t - \frac{1}{5}\,e^{-t}[cos\,t + 3\,sin\,t].$$

Applying Theorem 6.3.1 ,

$$\mathcal{L}^{-1}\left[\frac{e^{-(\pi/2)s}}{s^2+2s+2}\right] = e^{-\left(t-\frac{\pi}{2}\right)}\,sin\Big(t-\frac{\pi}{2}\Big)u_{\pi/2}(t)\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) \; &= \frac{1}{5}\,cos\,t + \frac{2}{5}\,sin\,t - \frac{1}{5}\,e^{-t}[cos\,t + 3\,sin\,t] - \\ &- e^{-\left(t-\frac{\pi}{2}\right)}\,cos(t)\,u_{\pi/2}(t)\,. \end{aligned}$$



12. Taking the initial conditions into consideration, the transform of the ODE is

$$s^4 Y(s) - Y(s) = e^{-s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-s}}{s^4 - 1}.$$

Using partial fractions,

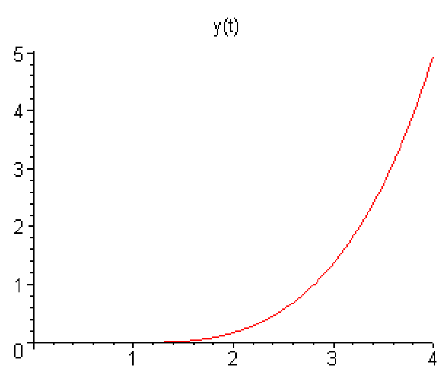
$$\frac{1}{s^4 - 1} = \frac{1}{2}\left[\frac{1}{s^2 - 1} - \frac{1}{s^2 + 1}\right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^4 - 1}\right] = \frac{1}{2} sinh\, t - \frac{1}{2} sin\, t\,.$$

Applying Theorem 6.3.1 , the solution of the IVP is

$$y(t) = \frac{1}{2}[sinh(t-1) - sin(t-1)]u_1(t)\,.$$



$14(a)$. The Laplace transform of the ODE is

$$s^2\,Y(s)+\frac{1}{2}s\,Y(s)+\,Y(s)=e^{-s}.$$

Solving for the transform of the solution,

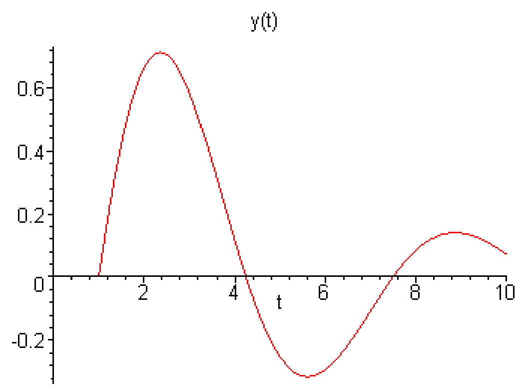
$$Y(s)=\frac{e^{-s}}{s^2+s/2+1}\,.$$

First write

$$\frac{1}{s^2+s/2+1}=\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\,.$$

Taking the inverse transform and applying both *shifting theorems*,

$$y(t)=\frac{4}{\sqrt{15}}e^{-(t-1)/4}\,sin\frac{\sqrt{15}}{4}(t-1)\,u_1(t)\,.$$



(b). As shown on the graph, the maximum is attained at some $t_1>2$. Note that for $t>2$,

$$y(t)=\frac{4}{\sqrt{15}}e^{-(t-1)/4}\,sin\frac{\sqrt{15}}{4}(t-1)\,.$$

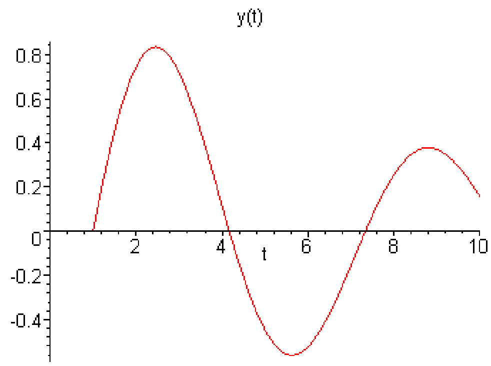
Setting $y'(t)=0$, we find that $t_1\approx 2.3613$. The maximum value is calculated as $y(2.3613)\approx 0.71153$.

(c). Setting $\gamma=1/4$, the transform of the solution is

$$Y(s)=\frac{e^{-s}}{s^2+s/4+1}\,.$$

Following the same steps, it follows that

$$y(t) = \frac{8}{3\sqrt{7}} e^{-(t-1)/8}\, sin \frac{3\sqrt{7}}{8}(t-1)\, u_1(t)\,.$$



Once again, the maximum is attained at some $t_1 > 2$. Setting $y'(t) = 0$, we find that $t_1 \approx 2.4569$, with $y(t_1) \approx 0.8335$.

(d). Now suppose that $0 < \gamma < 1$. Then the transform of the solution is

$$Y(s) = \frac{e^{-s}}{s^2 + \gamma s + 1}\,.$$

First write

$$\frac{1}{s^2 + \gamma s + 1} = \frac{1}{(s + \gamma/2)^2 + (1 - \gamma^2/4)}\,.$$

It follows that

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + \gamma s + 1}\right] = \frac{2}{\sqrt{4 - \gamma^2}}\, e^{-\gamma t/2} sin\left(\sqrt{1 - \gamma^2/4} \cdot t\right).$$

Hence the solution is

$$y(t) = h(t-1)\, u_1(t)\,.$$

The solution is nonzero only if $t > 1$, in which case $y(t) = h(t-1)$. Setting $y'(t) = 0$
,
we obtain

$$tan\left[\sqrt{1 - \gamma^2/4} \cdot (t-1)\right] = \frac{1}{\gamma}\sqrt{4 - \gamma^2}\,,$$

that is,

$$\frac{tan\left[\sqrt{1 - \gamma^2/4} \cdot (t-1)\right]}{\sqrt{1 - \gamma^2/4}} = \frac{2}{\gamma}\,.$$

As $\gamma \to 0$, we obtain the *formal* equation $tan(t-1) = \infty$. Hence $t_1 \to 1 + \frac{\pi}{2}$. Setting $t = \pi/2$ in $h(t)$, and letting $\gamma \to 0$, we find that $y_1 \to 1$. These conclusions agree with the case $\gamma = 0$, for which it is easy to show that the solution is

$$y(t) = sin(t-1)\, u_1(t)\,.$$

$15(a)$. See Prob. 14. It follows that the solution of the IVP is

$$y(t) = \frac{4k}{\sqrt{15}} e^{-(t-1)/4}\, sin \frac{\sqrt{15}}{4}(t-1)\, u_1(t)\,.$$

This function is a *multiple* of the answer in Prob. $14(a)$. Hence the peak value occurs at $t_1 \approx 2.3613$. The maximum value is calculated as $y(2.3613) \approx 0.71153\, k$. We find that the appropriate value of k is $k_1 = 2/0.71153 \approx 2.8108$.

(b). Based on Prob. $14(c)$, the solution is

$$y(t) = \frac{8\,k}{3\sqrt{7}} e^{-(t-1)/8}\, sin \frac{3\sqrt{7}}{8}(t-1)\, u_1(t)\,.$$

Since this function is a *multiple* of the solution in Prob. $14(c)$, we have $t_1 \approx 2.4569$, with $y(t_1) \approx 0.8335\, k$. The solution attains a value of $y = 2$, for $k_1 = 2/0.8335$, that is, $k_1 \approx 2.3995$.

(c). Similar to Prob. $14(d)$, for $0 < \gamma < 1$, the solution is

$$y(t) = h(t-1)\, u_1(t)\,,$$

in which

$$h(t) = \frac{2\,k}{\sqrt{4-\gamma^2}}\, e^{-\gamma t/2} sin\Big(\sqrt{1-\gamma^2/4} \cdot t\Big).$$

It follows that $t_1 - 1 \to \pi/2$. Setting $t = \pi/2$ in $h(t)$, and letting $\gamma \to 0$, we find that $y_1 \to k$. Requiring that the *peak value* remains at $y = 2$, the limiting value of k is $k_1 = 2$. These conclusions agree with the case $\gamma = 0$, for which it is easy to show that the solution is

$$y(t) = k\, sin(t-1)\, u_1(t)\,.$$

$16(a)$. Taking the initial conditions into consideration, the transformation of the ODE is

$$s^2\, Y(s) + Y(s) = \frac{1}{2k}\left[\frac{e^{-(4-k)s}}{s} - \frac{e^{-(4+k)s}}{s}\right].$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{2k}\left[\frac{e^{-(4-k)s}}{s(s^2+1)} - \frac{e^{-(4+k)s}}{s(s^2+1)}\right].$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

Now let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - cos\, t\,.$$

Applying Theorem 6.3.1 , the solution is

$$\phi(t\,,k) = \frac{1}{2k}[h(t-4+k)\,u_{4-k}(t) - h(t-4-k)\,u_{4+k}(t)].$$

That is,

$$\begin{aligned}\phi(t\,,k) \;&= \frac{1}{2k}[\,u_{4-k}(t) - \,u_{4+k}(t)] - \\ &- \frac{1}{2k}[cos(t-4+k)\,u_{4-k}(t) - cos(t-4-k)\,u_{4+k}(t)].\end{aligned}$$

(b). Consider various values of t. For any fixed $t < 4$, $\phi(t\,,k) = 0$, as long as $4 - k > t$. If $t \geq 4$, then for $4 + k < t$,

$$\phi(t\,,k) = \,-\frac{1}{2k}[cos(t-4+k) - cos(t-4-k)\,].$$

It follows that

$$\begin{aligned}\lim_{k\to 0}\phi(t\,,k) \;&=\; \lim_{k\to 0}\; -\frac{cos(t-4+k) - cos(t-4-k)}{2k} \\ &= sin(t-4)\,.\end{aligned}$$

Hence

$$\lim_{k\to 0}\phi(t\,,k) = sin(t-4)\,u_4(t)\,.$$

(c). The Laplace transform of the differential equation

$$y'' + y = \delta(t-4),$$

with $y(0) = y'(0) = 0$, is

$$s^2\,Y(s) + \,Y(s) = e^{-4s}.$$

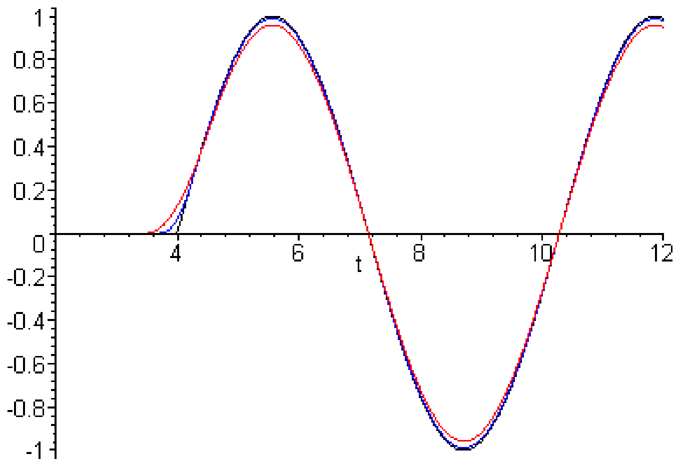
Solving for the transform of the solution,

$$Y(s) = \frac{e^{-4s}}{s^2+1}\,.$$

It follows that the solution is

$$\phi_0(t) = sin(t-4)\,u_4(t)\,.$$

(d).



$18(b)$. The transform of the ODE (given the specified initial conditions) is

$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{20}(-1)^{k+1}e^{-k\pi s}.$$

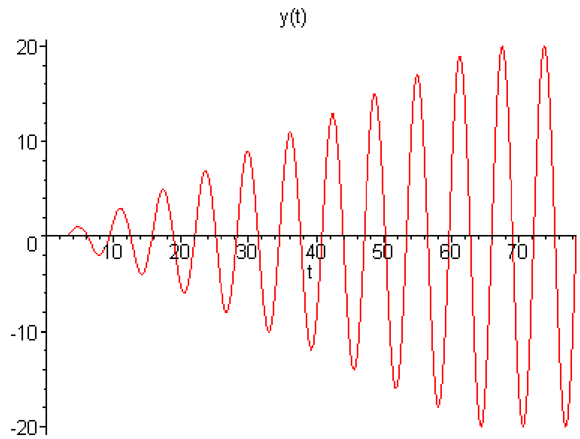
Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+1}\;\sum_{k=1}^{20}(-1)^{k+1}e^{-k\pi s}.$$

Applying Theorem 6.3.1 , term-by-term,

$$\begin{aligned} y(t) &= \sum_{k=1}^{20}(-1)^{k+1} sin(t-k\pi)\,u_{k\pi}(t) \\ &= -\,sin(t)\cdot\sum_{k=1}^{20}u_{k\pi}(t)\,. \end{aligned}$$

(c).



$19(b)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{20} e^{-(k\pi/2)s}.$$

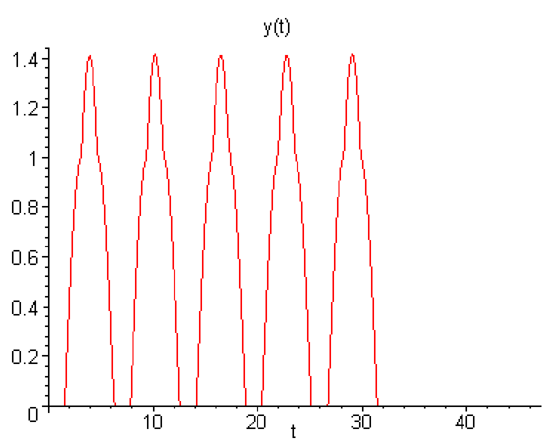
Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+1}\;\sum_{k=1}^{20} e^{-(k\pi/2)s}.$$

Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{20} sin\left(t - \frac{k\pi}{2}\right) u_{k\pi/2}(t)\,.$$

(c).



$20(b)$. The transform of the ODE (given the specified initial conditions) is

$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{20}(-1)^{k+1}e^{-(k\pi/2)s}.$$

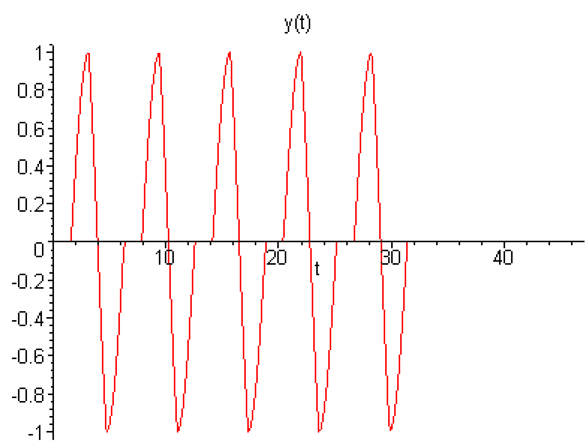
Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{20}(-1)^{k+1}\frac{e^{-(k\pi/2)s}}{s^2+1}.$$

Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{20}(-1)^{k+1}sin\left(t-\frac{k\pi}{2}\right)u_{k\pi/2}(t)\,.$$

(c).



$22(b)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + Y(s) = \sum_{k=1}^{40}(-1)^{k+1}e^{-(11k/4)s}.$$

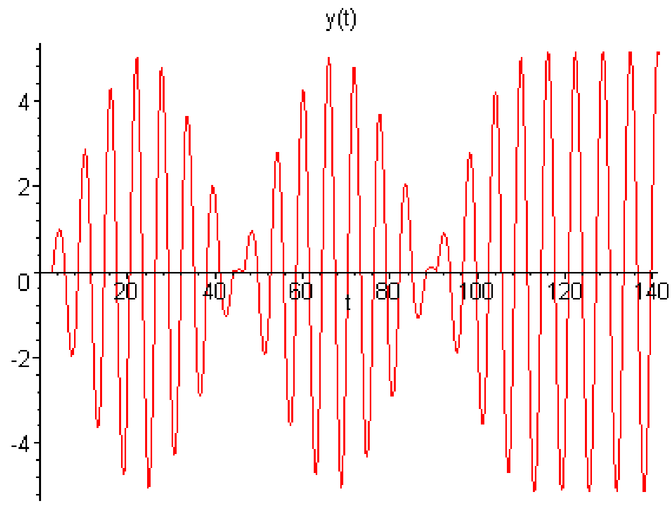
Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{40}(-1)^{k+1}\frac{e^{-(11k/4)s}}{s^2+1}.$$

Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{40} (-1)^{k+1} sin\left(t - \frac{11k}{4}\right) u_{11k/4}(t)\,.$$

(c).



$23(b)$. The transform of the ODE (given the specified initial conditions) is

$$s^2\,Y(s) + 0.1s\,Y(s) + \,Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{20} \frac{e^{-k\pi s}}{s^2 + 0.1s + 1}.$$

First write

$$\frac{1}{s^2 + 0.1s + 1} = \frac{1}{\left(s + \frac{1}{20}\right)^2 + \frac{399}{400}}\,.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 0.1s + 1}\right] = \frac{20}{\sqrt{399}}\,e^{-t/20} sin\left(\frac{\sqrt{399}}{20}\,t\right).$$

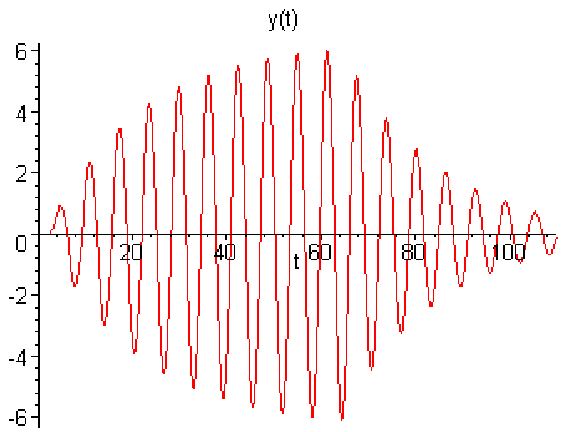
Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} h(t - k\pi)\, u_{k\pi}(t)\,,$$

in which

$$h(t) = \frac{20}{\sqrt{399}} e^{-t/20} sin\left(\frac{\sqrt{399}}{20}\,t\right).$$

(c).



$24(b)$. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2\,Y(s) + 0.1s\,Y(s) + \,Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{15} \frac{e^{-(2k-1)\pi s}}{s^2 + 0.1s + 1}.$$

As shown in Prob. 23 ,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 0.1s + 1}\right] = \frac{20}{\sqrt{399}} e^{-t/20} sin\left(\frac{\sqrt{399}}{20}\,t\right).$$

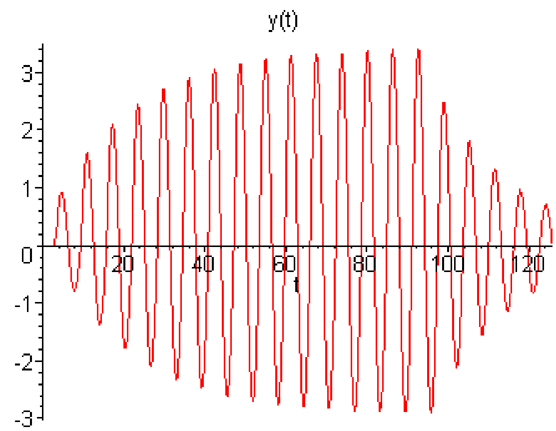
Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{15} h[t - (2k-1)\pi]\, u_{(2k-1)\pi}(t)\,,$$

in which

$$h(t) = \frac{20}{\sqrt{399}} e^{-t/20} sin\left(\frac{\sqrt{399}}{20}\, t \right).$$

(c).



$25(a)$. A fundamental set of solutions is $y_1(t) = e^{-t} cos\, t$ and $y_2(t) = e^{-t} sin\, t$. Based on Prob. 22 , in Section 3.7 , a particular solution is given by

$$y_p(t) = \int_0^t \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W(y_1\,, y_2)(s)} f(s)ds\,.$$

In the given problem,

$$\begin{aligned} y_p(t) &= \int_0^t \frac{e^{-s-t}[cos(s)sin(t) - sin(s)cos(t)]}{exp(\,-2s)} f(s)ds\,. \\ &= \int_0^t e^{-(t-s)} sin(t-s) f(s)ds\,. \end{aligned}$$

Given the specified initial conditions,

$$y(t) = \int_0^t e^{-(t-s)} sin(t-s) f(s)ds\,.$$

(b). Let $f(t) = \delta(t-\pi)$. It is easy to see that if $t < \pi$, $y(t) = 0$. If $t > \pi$,

$$\int_0^t e^{-(t-s)} sin(t-s)\delta(s-\pi)ds = e^{-(t-\pi)} sin(t-\pi)\,.$$

Setting $t = \pi + \varepsilon$, and letting $\varepsilon \to 0$, we find that $y(\pi) = 0$. Hence

$$y(t) = e^{-(t-\pi)} sin(t-\pi)\, u_\pi(t)\,.$$

(c). The Laplace transform of the solution is

$$\begin{aligned} Y(s) &= \frac{e^{-\pi s}}{s^2+2s+2} \\ &= \frac{e^{-\pi s}}{(s+1)^2+1}\,. \end{aligned}$$

Hence the solutions agree.