Section 7.7

1. The eigenvalues and eigenvectors were found in Prob. 1, Section 7.5.

$$r_1 = -1, \ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \ r_2 = 2, \ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ 2\,e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} 2\,e^{2t} \\ e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-t} & 2\,e^{2t} \\ 2\,e^{-t} & e^{2t} \end{pmatrix}.$$

We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

So that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{3} \begin{pmatrix} -e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ -2e^{-t} + 2e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.$$

3. The eigenvalues and eigenvectors were found in Prob. 3, Section 7.5. The general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 + 3c_2 &= 0, \end{aligned}$$

to obtain $c_1 = 3/2, c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 3c_2 &= 1, \end{aligned}$$

to obtain $c_1 = -1/2, c_2 = 1/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} + \frac{3}{2}e^{-t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} 3e^{t} - e^{-t} & -e^{t} + e^{-t} \\ 3e^{t} - 3e^{-t} & -e^{t} + 3e^{-t} \end{pmatrix}.$$

5. The general solution, found in Prob. 3 , Section 7.6 , is given by

$$\mathbf{x} = c_1\begin{pmatrix} 5\,cos\,t \\ 2\,cos\,t + sin\,t \end{pmatrix} + c_2\begin{pmatrix} 5\,sin\,t \\ -\,cos\,t + 2\,sin\,t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 1 \\ 2c_1 - c_2 &= 0\,, \end{aligned}$$

resulting in $c_1 = 1/5\,, c_2 = 2/5$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} cos\,t + 2\,sin\,t \\ sin\,t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 0 \\ 2c_1 - c_2 &= 1\,, \end{aligned}$$

resulting in $c_1 = 0\,, c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -\,5\,sin\,t \\ cos\,t - 2\,sin\,t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} cos\,t + 2\,sin\,t & -\,5\,sin\,t \\ sin\,t & cos\,t - 2\,sin\,t \end{pmatrix}.$$

7. The general solution, found in Prob. 15 , Section 7.5 , is given by

$$\mathbf{x} = c_1\begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix} + c_2\begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 + c_2 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/2$, $c_2 = 3/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} \\ -3e^{2t} + 3e^{4t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$ require that

$$c_1 + c_2 = 0$$
$$3c_1 + c_2 = 1,$$

resulting in $c_1 = 1/2$, $c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} e^{2t} - e^{4t} \\ 3e^{2t} - e^{4t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} & e^{2t} - e^{4t} \\ -3e^{2t} + 3e^{4t} & 3e^{2t} - e^{4t} \end{pmatrix}.$$

8. The general solution, found in Prob. 5, Section 7.6, is given by

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} cos\, t \\ 2\, cos\, t + sin\, t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} sin\, t \\ -cos\, t + 2\, sin\, t \end{pmatrix}.$$

The specific solution corresponding to the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$ is

$$\mathbf{x} = e^{-t}\begin{pmatrix} cos\, t + 2\, sin\, t \\ 5 sin\, t \end{pmatrix}.$$

For the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, the solution is

$$\mathbf{x} = e^{-t}\begin{pmatrix} -sin\, t \\ cos\, t - 2 sin\, t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = e^{-t}\begin{pmatrix} cos\, t + 2\, sin\, t & -sin\, t \\ 5 sin\, t & cos\, t - 2 sin\, t \end{pmatrix}.$$

9. The general solution, found in Prob. 13, Section 7.5, is given by

$$\mathbf{x} = c_1\begin{pmatrix} 4e^{-2t} \\ -5e^{-2t} \\ -7e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{-t} \\ -4e^{-t} \\ -2e^{-t} \end{pmatrix} + c_3\begin{pmatrix} 0 \\ e^{2t} \\ -e^{2t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 1 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/2\,,\, c_2 = 1\,,\, c_3 = 3/2\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -2e^{-2t} + 3e^{-t} \\ \frac{5}{2}e^{-2t} - 4e^{-t} + \frac{3}{2}e^{2t} \\ \frac{7}{2}e^{-2t} - 2e^{-t} - \frac{3}{2}e^{2t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 1 \\ -7c_1 - 2c_2 - c_3 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/4\,,\, c_2 = 1/3\,,\, c_3 = 13/12\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{13}{12}e^{2t} \\ \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{13}{12}e^{2t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(3)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 1\,, \end{aligned}$$

resulting in $c_1 = -1/4\,,\, c_2 = 1/3\,,\, c_3 = 1/12\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{1}{12}e^{2t} \\ \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{1}{12}e^{2t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} -2e^{-2t} + 3e^{-t} & -e^{-2t} + e^{-t} & -e^{-2t} + e^{-t} \\ \frac{5}{2}e^{-2t} - 4e^{-t} + \frac{3}{2}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{13}{12}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{1}{12}e^{2t} \\ \frac{7}{2}e^{-2t} - 2e^{-t} - \frac{3}{2}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{13}{12}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{1}{12}e^{2t} \end{pmatrix}.$$

12. The solution of the initial value problem is given by

$$\begin{aligned}\mathbf{x} &= \mathbf{\Phi}(t)\mathbf{x}(0)\\ &= \begin{pmatrix} e^{-t}cos\,2t & -2e^{-t}sin\,2t \\ \frac{1}{2}e^{-t}sin\,2t & e^{-t}cos\,2t \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}\\ &= e^{-t}\begin{pmatrix} 3cos\,2t - 2sin\,2t \\ \frac{3}{2}sin\,2t + cos\,2t \end{pmatrix}.\end{aligned}$$

13. Let

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}(t_0) = \begin{pmatrix} x_1^{(1)}(t_0) & \cdots & x_1^{(n)}(t_0) \\ \vdots & & \vdots \\ x_n^{(1)}(t_0) & \cdots & x_n^{(n)}(t_0) \end{pmatrix}$$

is a *scalar* matrix, which is invertible, since the solutions are linearly independent. Let $\mathbf{\Psi}^{-1}(t_0) = (c_{ij})$. Then

$$\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}.$$

The *j-th* column of the product matrix is

$$\left[\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\right]^{(j)} = \sum_{k=1}^{n} c_{kj}\,\mathbf{x}^{(k)},$$

which is a solution vector, since it is a linear combination of solutions. Furthermore, the columns are all linearly independent, since the vectors $\mathbf{x}^{(k)}$ are. Hence the product is a fundamental matrix. Finally, setting $t = t_0$, $\mathbf{\Psi}(t_0)\mathbf{\Psi}^{-1}(t_0) = \mathbf{I}$. This is precisely the definition of $\mathbf{\Phi}(t)$.

14. The fundamental matrix $\mathbf{\Phi}(t)$ for the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}$$

is given by

$$\mathbf{\Phi}(t) = \frac{1}{4}\begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}.$$

Direct multiplication results in

$$\begin{aligned}\mathbf{\Phi}(t)\mathbf{\Phi}(s) &= \frac{1}{16}\begin{pmatrix} 2e^{3t}+2e^{-t} & e^{3t}-e^{-t} \\ 4e^{3t}-4e^{-t} & 2e^{3t}+2e^{-t}\end{pmatrix}\begin{pmatrix} 2e^{3s}+2e^{-s} & e^{3s}-e^{-s} \\ 4e^{3s}-4e^{-s} & 2e^{3s}+2e^{-s}\end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} 8(e^{3t+3s}+e^{-t-s}) & 4(e^{3t+3s}-e^{-t-s}) \\ 16(e^{3t+3s}-e^{-t-s}) & 8(e^{3t+3s}+e^{-t-s})\end{pmatrix}.\end{aligned}$$

Hence

$$\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \frac{1}{4}\begin{pmatrix} 2e^{3(t+s)}+2e^{-(t+s)} & e^{3(t+s)}-e^{-(t+s)} \\ 4e^{3(t+s)}-4e^{-(t+s)} & 2e^{3(t+s)}+2e^{-(t+s)}\end{pmatrix}.$$

$15(a)$. Let s be arbitrary, but *fixed*, and t variable. Similar to the argument in Prob. 13, the *columns* of the matrix $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ are linear combinations of fundamental solutions. Hence the columns of $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ are also solution of the system of equations. Further, setting $t=0$, $\mathbf{\Phi}(0)\mathbf{\Phi}(s)=\mathbf{I}\,\mathbf{\Phi}(s)=\mathbf{\Phi}(s)$. That is, $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ is a solution of the initial value problem $\mathbf{Z}'=\mathbf{AZ}$, with $\mathbf{Z}(0)=\mathbf{\Phi}(s)$. Now consider the change of variable $\tau=t+s$. Let $\mathbf{W}(\tau)=\mathbf{Z}(\tau-s)$. The given initial value problem can be reformulated as

$$\frac{d}{d\tau}\mathbf{W}=\mathbf{AW}\,,\text{ with }\mathbf{W}(s)=\mathbf{\Phi}(s)\,.$$

Since $\mathbf{\Phi}(t)$ is a fundamental matrix satisfying $\mathbf{\Phi}'=\mathbf{A\Phi}$, with $\mathbf{\Phi}(0)=\mathbf{I}$, it follows that

$$\begin{aligned}\mathbf{W}(\tau) &= \left[\mathbf{\Phi}(\tau)\mathbf{\Phi}^{-1}(s)\right]\mathbf{\Phi}(s) \\ &= \mathbf{\Phi}(\tau).\end{aligned}$$

That is, $\mathbf{\Phi}(t+s)=\mathbf{\Phi}(\tau)=\mathbf{W}(\tau)=\mathbf{Z}(t)=\mathbf{\Phi}(t)\mathbf{\Phi}(s)$.

(b). Based on Part (a), $\mathbf{\Phi}(t)\mathbf{\Phi}(-t)=\mathbf{\Phi}(t+(-t))=\mathbf{\Phi}(0)=\mathbf{I}$. Hence

$$\mathbf{\Phi}(-t)=\mathbf{\Phi}^{-1}(t).$$

(c). It also follows that $\mathbf{\Phi}(t-s)=\mathbf{\Phi}(t+(-s))=\mathbf{\Phi}(t)\mathbf{\Phi}(-s)=\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$.

16. Let $\mathbf{A}$ be a *diagonal matrix*, with $\mathbf{A}=[a_1\mathbf{e}^{(1)},a_2\mathbf{e}^{(2)},\cdots,a_n\mathbf{e}^{(n)}]$. Note that for any positive integer, k,

$$\mathbf{A}^k=\left[a_1^k\,\mathbf{e}^{(1)},a_2^k\,\mathbf{e}^{(2)},\cdots,a_n^k\,\mathbf{e}^{(n)}\right].$$

It follows, from basic matrix algebra, that

$$\mathbf{I}+\sum_{k=1}^{m}\mathbf{A}^k\frac{t^k}{k!}=\begin{pmatrix}\sum\limits_{k=0}^{m}a_1^k\frac{t^k}{k!} & 0 & \cdots & 0\\ 0 & \sum\limits_{k=0}^{m}a_2^k\frac{t^k}{k!} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & \sum\limits_{k=0}^{m}a_n^k\frac{t^k}{k!}\end{pmatrix}.$$

It can be shown that the partial sums on the left hand side converge for all t. Taking the limit (as $m\to\infty$) on both sides of the equation, we obtain

$$exp(\mathbf{A}t)=\begin{pmatrix}e^{a_1t} & 0 & \cdots & 0\\ 0 & e^{a_2t} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & e^{a_nt}\end{pmatrix}.$$

Alternatively, consider the system $\mathbf{x}'=\mathbf{A}\mathbf{x}$. Since ODEs are *uncoupled*, the vectors $\mathbf{x}^{(j)}=exp(a_jt)\,\mathbf{e}^{(j)},\ j=1,2,\cdots n$, are a set of linearly independent solutions. Hence the matrix

$$\mathbf{X}=\left[exp(a_1t)\,\mathbf{e}^{(1)},exp(a_2t)\,\mathbf{e}^{(2)},\cdots,exp(a_nt)\,\mathbf{e}^{(n)}\right]$$

is a *fundamental matrix*. Finally, since $\mathbf{X}(0)=\mathbf{I}$, it follows that

$$\left[exp(a_1t)\,\mathbf{e}^{(1)},exp(a_2t)\,\mathbf{e}^{(2)},\cdots,exp(a_nt)\,\mathbf{e}^{(n)}\right]=\mathbf{\Phi}(t)=exp(\mathbf{A}t)\,.$$

$17(a)$. Assuming that $\mathbf{x}=\phi(t)$ is a solution, then $\phi'=\mathbf{A}\phi$, with $\phi(0)=\mathbf{x}^0$. Integrate both sides of the equation to obtain

$$\phi(t)-\phi(0)=\int_0^t\mathbf{A}\phi(s)ds\,.$$

Hence

$$\phi(t)=\mathbf{x}^0+\int_0^t\mathbf{A}\phi(s)ds\,.$$

(b). Proceed with the iteration

$$\phi^{(i+1)}(t)=\mathbf{x}^0+\int_0^t\mathbf{A}\phi^{(i)}(s)ds\,.$$

With $\phi^{(0)}(t)=\mathbf{x}^0$, and noting that $\mathbf{A}$ is a *constant* matrix,

$$\begin{aligned}\phi^{(1)}(t) &= \mathbf{x}^0 + \int_0^t \mathbf{A}\mathbf{x}^0 ds \\ &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t\,.\end{aligned}$$

That is, $\phi^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0$.

(c). We then have

$$\begin{aligned}\phi^{(2)}(t) &= \mathbf{x}^0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}t)\mathbf{x}^0 ds \\ &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t + \mathbf{A}^2\mathbf{x}^0\frac{t^2}{2} \\ &= \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2}\right)\mathbf{x}^0.\end{aligned}$$

Now suppose that

$$\phi^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0.$$

It follows that

$$\int_0^t \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0 ds =$$

$$\begin{aligned}&= \mathbf{A}\left(\mathbf{I}t + \mathbf{A}\frac{t^2}{2} + \mathbf{A}^2\frac{t^3}{3!} + \cdots + \mathbf{A}^n\frac{t^{n+1}}{(n+1)!}\right)\mathbf{x}^0 \\ &= \left(\mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \cdots + \mathbf{A}^{n+1}\frac{t^n}{n!}\right)\mathbf{x}^0.\end{aligned}$$

Therefore

$$\phi^{(n+1)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^{n+1}\frac{t^{n+1}}{(n+1)!}\right)\mathbf{x}^0.$$

By induction, the asserted form of $\phi^{(n)}(t)$ is valid for all $n \geq 0$.

(d). Define $\phi^{(\infty)}(t) = \lim_{n\to\infty}\phi^{(n)}(t)$. It can be shown that the limit does exist. In fact,

$$\phi^{(\infty)}(t) = exp(\mathbf{A}t)\mathbf{x}^0.$$

Term-by-term differentiation results in

$$\begin{aligned}\frac{d}{dt}\phi^{(\infty)}(t) &= \frac{d}{dt}\left(\mathbf{I}+\mathbf{A}t+\mathbf{A}^2\frac{t^2}{2}+\cdots+\mathbf{A}^n\frac{t^n}{n!}+\right)\mathbf{x}^0 \\ &= \left(\mathbf{A}+\mathbf{A}^2 t+\cdots+\mathbf{A}^n\frac{t^{n-1}}{(n-1)!}+\right)\mathbf{x}^0 \\ &= \mathbf{A}\left(\mathbf{I}+\mathbf{A}t+\mathbf{A}^2\frac{t^2}{2}+\cdots+\mathbf{A}^{n-1}\frac{t^{n-1}}{(n-1)!}+\right)\mathbf{x}^0.\end{aligned}$$

That is,

$$\frac{d}{dt}\phi^{(\infty)}(t)=\mathbf{A}\phi^{(\infty)}(t).$$

Furthermore, $\phi^{(\infty)}(0)=\mathbf{x}^0$. Based on *uniqueness* of solutions, $\phi(t)=\phi^{(\infty)}(t)$.