Section 5.7

2. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = -2 - x$ and $x^2 q(x) = 2 + x^2$. It follows that

$$p_0 = \lim_{x \to 0}(-2 - x) = -2$$
$$q_0 = \lim_{x \to 0}\left(2 + x^2\right) = 2$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - 2r + 2 = 0\,,$$

that is, $r^2 - 3r + 2 = 0$, with roots $r_1 = 2$ and $r_2 = 1$.

4. The coefficients $P(x)$, $Q(x)$, and $R(x)$ are analytic for all $x \in \mathbb{R}$. Hence there are *no* singular points.

5. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = 3\frac{sin\,x}{x}$ and $x^2 q(x) = -2$. It follows that

$$p_0 = \lim_{x \to 0} 3\,\frac{sin\,x}{x} = 3$$
$$q_0 = \lim_{x \to 0} -2 = -2$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + 3r - 2 = 0\,,$$

that is, $r^2 + 2r - 2 = 0$, with roots $r_1 = -1 + \sqrt{3}$ and $r_2 = -1 - \sqrt{3}$.

6. $P(x) = 0$ for $x = 0$ and $x = -2$. We note that $p(x) = x^{-1}(x+2)^{-1}/2$, and $q(x) = -(x+2)^{-1}/2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x \to 0} \frac{1}{2(x+2)} = \frac{1}{4}$$
$$q_0 = \lim_{x \to 0} \frac{-x^2}{2(x+2)} = 0$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{3}{4}r = 0$, with roots $r_1 = \frac{3}{4}$ and $r_2 = 0$. For the singularity at $x = -2$,

$$p_0 = \lim_{x \to -2} (x+2)p(x) = \lim_{x \to -2} \frac{1}{2x} = -\frac{1}{4}$$
$$q_0 = \lim_{x \to -2} (x+2)^2 q(x) = \lim_{x \to -2} \frac{-(x+2)}{2} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{5}{4}r = 0$, with roots $r_1 = \frac{5}{4}$ and $r_2 = 0$.

7. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = \frac{1}{2} + \frac{\sin x}{2x}$ and $x^2 q(x) = 1$. It follows that

$$p_0 = \lim_{x \to 0} xp(x) = 1$$
$$q_0 = \lim_{x \to 0} x^2 q(x) = 1$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + r + 1 = 0\,,$$

that is, $r^2 + 1 = 0$, with *complex conjugate* roots $r = \pm\, i$.

8. Note that $P(x) = 0$ only for $x = -1$. We find that $p(x) = 3(x-1)/(x+1)$, and $q(x) = 3/(x+1)^2$. It follows that

$$p_0 = \lim_{x \to -1} (x+1)p(x) = \lim_{x \to -1} 3(x-1) = -6$$
$$q_0 = \lim_{x \to -1} (x+1)^2 q(x) = \lim_{x \to -1} 3 = 3$$

and therefore $x = -1$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - 6r + 3 = 0\,,$$

that is, $r^2 - 7r + 3 = 0$, with roots $r_1 = \left(7 + \sqrt{37}\right)/2$ and $r_2 = \left(7 - \sqrt{37}\right)/2$.

10. $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x(x-2)^{-2}(x+2)^{-1}$, and $q(x) = 3(x-2)^{-1}(x+2)^{-1}$. For the singularity at $x = 2$,

$$\lim_{x \to 2} (x-2)p(x) = \lim_{x \to 2} \frac{2x}{x^2 - 4}\,,$$

which is *undefined*. Therefore $x = 0$ is an *irregular* singular point. For the singularity at $x = -2$,

$$p_0 = \lim_{x\to -2} (x+2)p(x) = \lim_{x\to -2} \frac{2x}{(x-2)^2} = -\frac{1}{4}$$
$$q_0 = \lim_{x\to -2} (x+2)^2 q(x) = \lim_{x\to -2} \frac{3(x+2)}{x-2} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{5}{4}r = 0$, with roots $r_1 = \frac{5}{4}$ and $r_2 = 0$.

11. $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x/(4 - x^2)$, and $q(x) = 3/(4 - x^2)$. For the singularity at $x = 2$,

$$p_0 = \lim_{x\to 2} (x-2)p(x) = \lim_{x\to 2} \frac{-2x}{x+2} = -1$$
$$q_0 = \lim_{x\to 2} (x-2)^2 q(x) = \lim_{x\to 2} \frac{3(2-x)}{x+2} = 0$$

and therefore $x = 2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - r = 0\,,$$

that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$. For the singularity at $x = -2$,

$$p_0 = \lim_{x\to -2} (x+2)p(x) = \lim_{x\to -2} \frac{2x}{2-x} = -1$$
$$q_0 = \lim_{x\to -2} (x+2)^2 q(x) = \lim_{x\to -2} \frac{3(x+2)}{2-x} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - r = 0\,,$$

that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$.

12. $P(x) = 0$ for $x = 0$ and $x = -3$. We note that $p(x) = -2x^{-1}(x+3)^{-1}$, and $q(x) = -1/(x+3)^2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x\to 0} x\,p(x) = \lim_{x\to 0} \frac{-2}{x+3} = -\frac{2}{3}$$
$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{-x^2}{(x+3)^2} = 0$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - \frac{2}{3}r = 0\,,$$

that is, $r^2 - \frac{5}{3}r = 0$, with roots $r_1 = \frac{5}{3}$ and $r_2 = 0$. For the singularity at $x = -3$,

$$p_0 = \lim_{x \to -3} (x+3)p(x) = \lim_{x \to -3} \frac{-2}{x} = \frac{2}{3}$$
$$q_0 = \lim_{x \to -3} (x+3)^2 q(x) = \lim_{x \to -3} (-1) = -1$$

and therefore $x = -3$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + \frac{2}{3}r - 1 = 0\,,$$

that is, $r^2 - \frac{1}{3}r - 1 = 0$, with roots $r_1 = \left(1 + \sqrt{37}\right)/6$ and $r_2 = \left(1 - \sqrt{37}\right)/6$.

$13(a)$. Note the $p(x) = 1/x$ and $q(x) = -1/x$. Furthermore, $x\,p(x) = 1$ and $x^2 q(x) = -x$. It follows that

$$p_0 = \lim_{x \to 0}(1) = 1$$
$$q_0 = \lim_{x \to 0}(-x) = 0$$

and therefore $x = 0$ is a *regular* singular point.

(b). The indicial equation is given by

$$r(r-1) + r = 0\,,$$

that is, $r^2 = 0$, with roots $r_1 = r_2 = 0$.

(c). Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^{n+1} + \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

After adjusting the indices in the *first* series, we obtain

$$a_1 - a_0 + \sum_{n=1}^{\infty} [n(n+1)a_{n+1} + (n+1)a_{n+1} - a_n]x^n = 0\,.$$

Setting the coefficients equal to *zero*, it follows that for $n \geq 0$,

$$a_{n+1} = \frac{a_n}{(n+1)^2}\,.$$

So for $n \geq 1$,

$$a_n = \frac{a_{n-1}}{n^2} = \frac{a_{n-2}}{n^2(n-1)^2} = \cdots = \frac{1}{(n!)^2}\,a_0\,.$$

With $a_0 = 1$, one solution is

$$y_1(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \cdots + \frac{1}{(n!)^2}\,x^n + \cdots.$$

For a second solution, set $y_2(x) = y_1(x)\,ln\,x \,+\, b_1x + b_2x^2 + \cdots + b_nx^n + \cdots$. Substituting into the ODE, we obtain

$$L[y_1(x)]\cdot ln\,x + 2\,y_1'(x) + L\left[\sum_{n=1}^{\infty} b_n\,x^n\right] = 0\,.$$

Since $L[y_1(x)] = 0$, it follows that

$$L\left[\sum_{n=1}^{\infty} b_n\,x^n\right] = \,-\,2\,y_1'(x)\,.$$

More specifically,

$$b_1 + \sum_{n=1}^{\infty}[n(n+1)b_{n+1} + (n+1)b_{n+1} - b_n]x^n =$$
$$= \,-\,2 - x - \frac{1}{6}x^2 - \frac{1}{72}x^3 - \frac{1}{1440}x^4 - \cdots.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} b_1 &= \,-\,2 \\ 4b_2 - b_1 &= \,-\,1 \\ 9b_3 - b_2 &= \,-\,1/6 \\ 16b_4 - b_3 &= \,-\,1/72 \\ &\vdots \end{aligned}$$

Solving these equations for the coefficients, $b_1 = \,-\,2,\ b_2 = \,-\,3/4,\ b_3 = \,-\,11/108,$ $b_4 = \,-\,25/3456\,, \cdots$. Therefore a *second* solution is

$$y_2(x) = y_1(x)\,ln\,x + \left[\,-\,2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \cdots\right].$$

14(a). Here $x\,p(x) = 2x$ and $x^2q(x) = 6\,xe^x$. Both of these functions are *analytic* at $x = 0$, therefore $x = 0$ is a *regular* singular point. Note that $p_0 = q_0 = 0$.

(b). The indicial equation is given by

$$r(r-1)=0\,,$$

that is, $r^2-r=0$, with roots $r_1=1$ and $r_2=0$.

(c). In order to find the solution corresponding to $r_1=1$, set $y=x\sum_{n=0}^{\infty}a_nx^n$. Upon substitution into the ODE, we have

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+1}\,x^{n+1}+2\sum_{n=0}^{\infty}(n+1)a_nx^{n+1}+6\,e^x\sum_{n=0}^{\infty}a_nx^{n+1}=0\,.$$

After adjusting the indices in the *first* two series, and expanding the *exponential* function,

$$\sum_{n=1}^{\infty}n(n+1)a_n\,x^n+2\sum_{n=1}^{\infty}n\,a_{n-1}x^n+6\,a_0x+(6a_0+6a_1)x^2+$$
$$+(6a_2+6a_1+3a_0)x^3+(6a_3+6a_2+3a_1+a_0)x^4+\cdots=0\,.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned}2a_1+2a_0+6a_0&=0\\6a_2+4a_1+6a_0+6a_1&=0\\12a_3+6a_2+6a_2+6a_1+3a_0&=0\\20a_4+8a_3+6a_3+6a_2+3a_1+a_0&=0\\&\vdots\end{aligned}$$

Setting $a_0=1$, solution of the system results in $a_1=-4$, $a_2=17/3$, $a_3=-47/12$, $a_4=191/120\,,\cdots$. Therefore one solution is

$$y_1(x)=x-4x^2+\frac{17}{3}x^3-\frac{47}{12}x^4+\cdots.$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x)=a\,y_1(x)\,ln\,x\;+1+c_1x+c_2x^2+\cdots+c_nx^n+\cdots.$$

Substituting into the ODE, we obtain

$$a\,L[y_1(x)]\cdot ln\,x+2a\,y_1'(x)+2a\,y_1(x)-a\frac{y_1(x)}{x}+L\left[1+\sum_{n=1}^{\infty}c_n\,x^n\right]=0\,.$$

Since $L[y_1(x)]=0$, it follows that

$$L\left[1+\sum_{n=1}^{\infty}c_n\,x^n\right]=-2a\,y_1'(x)-2a\,y_1(x)+a\frac{y_1(x)}{x}\,.$$

More specifically,

$$\sum_{n=1}^{\infty} n(n+1)c_{n+1}x^n + 2\sum_{n=1}^{\infty} n\, c_n x^n + 6\ + (6+6c_1)x +$$
$$+\,(6c_2+6c_1+3)x^2 + \cdots \,=\, -\,a + 10ax - \frac{61}{3}ax^2 + \frac{193}{12}ax^3 + \cdots .$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned}
6 &= \,-\,a \\
2c_2 + 8c_1 + 6 &= 10a \\
6c_3 + 10c_2 + 6c_1 + 3 &= \,-\,\frac{61}{3}a \\
12c_4 + 12c_3 + 6c_2 + 3c_1 + 1 &= \frac{193}{12}a \\
&\vdots
\end{aligned}$$

Solving these equations for the coefficients, $a = \,-\,6$. In order to solve the remaining equations, set $c_1 = 0$. Then $c_2 = \,-\,33,\ c_3 = 449/6$, $c_4 = \,-\,1595/24\,, \cdots$. Therefore a *second* solution is

$$y_2(x) = \,-\,6\,y_1(x)\,ln\,x + \left[1 - 33x^2 + \frac{449}{6}x^3 - \frac{1595}{24}x^4 + \cdots\right].$$

$15(a)$. Note the $p(x) = 6x/(x-1)$ and $q(x) = 3x^{-1}(x-1)^{-1}$. Furthermore, $x\,p(x) = 6x^2/(x-1)$ and $x^2q(x) = 3x/(x-1)$. It follows that

$$\begin{aligned}
p_0 &= \lim_{x\to 0}\frac{6x^2}{x-1} = 0 \\
q_0 &= \lim_{x\to 0}\frac{3x}{x-1} = 0
\end{aligned}$$

and therefore $x = 0$ is a *regular* singular point.

(b). The indicial equation is given by

$$r(r-1) = 0\,,$$

that is, $r^2 - r = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c). In order to find the solution corresponding to $r_1 = 1$, set $y = x\sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n\, x^{n+1} - \sum_{n=1}^{\infty} n(n+1)a_n\, x^n + \\ + 6\sum_{n=0}^{\infty}(n+1)a_n x^{n+2} + 3\sum_{n=0}^{\infty} a_n x^{n+1} = 0\,.$$

After adjusting the indices, it follows that

$$\sum_{n=2}^{\infty} n(n-1)a_{n-1}\, x^n - \sum_{n=1}^{\infty} n(n+1)a_n\, x^n + \\ + 6\sum_{n=2}^{\infty}(n-1)a_{n-2} x^n + 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

That is,

$$-2a_1 + 3a_0 + \sum_{n=2}^{\infty}[-n(n+1)a_n + (n^2-n+3)a_{n-1} + 6(n-1)a_{n-2}]x^n = 0.$$

Setting the coefficients equal to *zero*, we have $a_1 = 3a_0/2$, and for $n \geq 2$,

$$n(n+1)a_n = \left(n^2-n+3\right)a_{n-1} + 6(n-1)a_{n-2}\,.$$

If we assign $a_0 = 1$, then we obtain $a_1 = 3/2$, $a_2 = 9/4$, $a_3 = 51/16$, $\cdots$.
Hence one solution is

$$y_1(x) = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \frac{51}{16}x^4 + \frac{111}{40}x^5 + \cdots.$$

The exponents differ by an *integer*. So for a second solution, set

$$y_2(x) = a\, y_1(x)\, ln\, x + 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$$

Substituting into the ODE, we obtain

$$2ax\, y_1'(x) - 2a\, y_1'(x) + 6ax\, y_1(x) - a\, y_1(x) + a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] = 0,$$

since $L[y_1(x)] = 0$. It follows that

$$L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] = 2a\, y_1'(x) - 2ax\, y_1'(x) + a\, y_1(x) - 6ax\, y_1(x) - a\frac{y_1(x)}{x}\,.$$

Now

$$L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] = 3 + (-2c_2 + 3c_1)x + (-6c_3 + 5c_2 + 6c_1)x^2 + \\ + (-12c_4 + 9c_3 + 12c_2)x^3 + (-20c_5 + 15c_4 + 18c_3)x^4 + \cdots.$$

Substituting for $y_1(x)$, the *right hand side* of the ODE is

$$a+\frac{7}{2}ax+\frac{3}{4}ax^2+\frac{33}{16}ax^3-\frac{867}{80}ax^4-\frac{441}{10}ax^5+\cdots.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 3 &= a \\ -2c_2+3c_1 &= \frac{7}{2}a \\ -6c_3+5c_2+6c_1 &= \frac{3}{4}a \\ -12c_4+9c_3+12c_2 &= \frac{33}{16}a \\ &\vdots \end{aligned}$$

We find that $a=3$. In order to solve the second equation, set $c_1=0$. Solution of the remaining equations results in $c_2=-21/4$, $c_3=-19/4$, $c_4=-597/64$, $\cdots$. Hence a second solution is

$$y_2(x)=3\,y_1(x)\,ln\,x+\left[1-\frac{21}{4}x^2-\frac{19}{4}x^3-\frac{597}{64}x^4+\cdots\right].$$

16(a). After multiplying both sides of the ODE by x, we find that $x\,p(x)=0$ and $x^2q(x)=x$. Both of these functions are *analytic* at $x=0$, hence $x=0$ is a *regular* singular point.

(b). Furthermore, $p_0=q_0=0$. So the indicial equation is $r(r-1)=0$, with roots $r_1=1$ and $r_2=0$.

(c). In order to find the solution corresponding to $r_1=1$, set $y=x\sum_{n=0}^{\infty}a_nx^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty}n(n+1)a_n\,x^n+\sum_{n=0}^{\infty}a_nx^{n+1}=0\,.$$

That is,

$$\sum_{n=1}^{\infty}\left[n(n+1)a_n+a_{n-1}\right]x^n=0\,.$$

Setting the coefficients equal to *zero*, we find that for $n\geq 1$,

$$a_n=\frac{-\,a_{n-1}}{n(n+1)}\,.$$

It follows that

$$a_n = \frac{-\,a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \cdots = \frac{(-1)^n a_0}{(n!)^2(n+1)}\,.$$

Hence one solution is

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \frac{1}{2880}x^5 + \cdots.$$

The exponents differ by an *integer*. So for a second solution, set

$$y_2(x) = a\,y_1(x)\,ln\,x\, + 1 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots.$$

Substituting into the ODE, we obtain

$$a\,L[y_1(x)]\cdot ln\,x + 2a\,y_1'(x) - a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = 0\,.$$

Since $L[y_1(x)] = 0$, it follows that

$$L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = -\,2a\,y_1'(x) + a\frac{y_1(x)}{x}\,.$$

Now

$$\begin{aligned} L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] &= 1 + (2c_2 + c_1)x + (6c_3 + c_2)x^2 + (12c_4 + c_3)x^3 + \\ &\quad + (20c_5 + c_4)x^4 + (30c_6 + c_5)x^5 + \cdots. \end{aligned}$$

Substituting for $y_1(x)$, the *right hand side* of the ODE is

$$-\,a + \frac{3}{2}ax - \frac{5}{12}ax^2 + \frac{7}{144}ax^3 - \frac{1}{320}ax^4 + \cdots.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 1 &= -\,a \\ 2c_2 + c_1 &= \frac{3}{2}a \\ 6c_3 + c_2 &= -\,\frac{5}{12}a \\ 12c_4 + c_3 &= \frac{7}{144}a \\ &\vdots \end{aligned}$$

Evidently, $a = -\,1$. In order to solve the *second* equation, set $c_1 = 0$. We then find that $c_2 = -\,3/4$, $c_3 = 7/36$, $c_4 = -\,35/1728$, $\cdots$. Therefore a second solution is

$$y_2(x) = -\,y_1(x)\,ln\,x + \left[1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \cdots\right].$$

$19(a)$. After dividing by the leading coefficient, we find that

$$p_0 = \lim_{x\to 0} x\,p(x) = \lim_{x\to 0} \frac{\gamma - (1+\alpha+\beta)x}{1-x} = \gamma\,.$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{-\,\alpha\beta\,x}{1-x} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. The indicial equation is $r(r-1) + \gamma\, r = 0$, with roots $r_1 = 1-\gamma$ and $r_2 = 0$.

(b). For $x = 1$,

$$p_0 = \lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1} \frac{-\,\gamma + (1+\alpha+\beta)x}{x} = 1 - \gamma + \alpha + \beta\,.$$

$$q_0 = \lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1} \frac{\alpha\beta(x-1)}{x} = 0\,.$$

Hence $x = 1$ is a *regular* singular point. The indicial equation is

$$r^2 - (\gamma - \alpha - \beta)\,r = 0\,,$$

with roots $r_1 = \gamma - \alpha - \beta$ and $r_2 = 0$.

(c). Given that $r_1 - r_2$ is not a positive integer, we can set $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution into the ODE results in

$$x(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + [\gamma - (1+\alpha+\beta)x]\sum_{n=1}^{\infty} n\,a_n x^{n-1} - \alpha\beta\sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \gamma\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n -$$
$$-\,(1+\alpha+\beta)\sum_{n=1}^{\infty} n\,a_n x^n - \alpha\beta\sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining the series, we obtain

$$\gamma\, a_1 - \alpha\beta\, a_0 + [(2+2\gamma)a_2 - (1+\alpha+\beta+\alpha\beta)a_1]x + \sum_{n=2}^{\infty} A_n\, x^n = 0\,,$$

in which

$$A_n = (n+1)(n+\gamma)a_{n+1} - [n(n-1) + (1+\alpha+\beta)n + \alpha\beta]a_n \, .$$

Note that $n(n-1) + (1+\alpha+\beta)n + \alpha\beta = (n+\alpha)(n+\beta)$. Setting the coefficients equal to *zero*, we have $\gamma\, a_1 - \alpha\beta\, a_0 = 0$, and

$$a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)}\, a_n$$

for $n \geq 1$. Hence one solution is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 2!}x^2 + \\ + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 3!}x^3 + \cdots .$$

Since the nearest other singularity is at $x = 1$, the radius of convergence of $y_1(x)$ will be *at least* $\rho = 1$.

(d). Given that $r_1 - r_2$ is not a positive integer, we can set $y = x^{1-\gamma}\sum_{n=0}^{\infty} b_n x^n$. Then Substitution into the ODE results in

$$x(1-x)\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma-1} + \\ + [\gamma - (1+\alpha+\beta)x]\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - \alpha\beta\sum_{n=0}^{\infty}a_n x^{n+1-\gamma} = 0.$$

That is,

$$\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n+1-\gamma} + \\ + \gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n+1-\gamma} - \alpha\beta\sum_{n=0}^{\infty}a_n x^{n+1-\gamma} = 0.$$

After adjusting the indices,

$$\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=1}^{\infty}(n-\gamma)(n-1-\gamma)a_{n-1} x^{n-\gamma} + \\ + \gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=1}^{\infty}(n-\gamma)a_{n-1} x^{n-\gamma} - \alpha\beta\sum_{n=1}^{\infty}a_{n-1} x^{n-\gamma} = 0.$$

Combining the series, we obtain

$$\sum_{n=1}^{\infty} B_n\, x^{n-\gamma} = 0\, ,$$

in which

$$B_n = n(n+1-\gamma)b_n - [(n-\gamma)(n-\gamma+\alpha+\beta)+\alpha\beta]b_{n-1}.$$

Note that $(n-\gamma)(n-\gamma+\alpha+\beta)+\alpha\beta = (n+\alpha-\gamma)(n+\beta-\gamma)$. Setting $B_n = 0$, it follows that for $n \geq 1$,

$$b_n = \frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{n(n+1-\gamma)}\, b_{n-1}.$$

Therefore a second solution is

$$y_2(x) = x^{1-\gamma}\left[1 + \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{(2-\gamma)1!}x + \right. \\ \left. + \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{(2-\gamma)(3-\gamma)2\,!}x^2 + \cdots\right].$$

(e). Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4\frac{1}{\xi}\left(1-\frac{1}{\xi}\right)\frac{d^2y}{d\xi^2} + \left\{2\xi^3\frac{1}{\xi}\left(1-\frac{1}{\xi}\right) - \xi^2\left[\gamma-(1+\alpha+\beta)\frac{1}{\xi}\right]\right\}\frac{dy}{d\xi} - \alpha\beta\, y = 0\,.$$

That is,

$$\left(\xi^3-\xi^2\right)\frac{d^2y}{d\xi^2} + \left[2\xi^2 - \gamma\,\xi^2 + (\,-1+\alpha+\beta)\xi\right]\frac{dy}{d\xi} - \alpha\beta\ y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{(2-\gamma)\,\xi + (\,-1+\alpha+\beta)}{\xi^2-\xi} \text{ and } q(\xi) = \frac{-\,\alpha\beta}{\xi^3-\xi^2}\,.$$

It follows that

$$p_0 = \lim_{\xi\to 0}\xi\, p(\xi) = \lim_{\xi\to 0}\frac{(2-\gamma)\,\xi + (\,-1+\alpha+\beta)}{\xi-1} = 1-\alpha-\beta\,,$$

$$q_0 = \lim_{\xi\to 0}\xi^2 q(\xi) = \lim_{\xi\to 0}\frac{-\,\alpha\beta}{\xi-1} = \alpha\beta\,.$$

Hence $\xi = 0$ $(x=\infty)$ is a *regular* singular point. The indicial equation is

$$r(r-1) + (1-\alpha-\beta)r + \alpha\beta = 0\,,$$

or $r^2 - (\alpha+\beta)r + \alpha\beta = 0$. Evidently, the roots are $r = \alpha$ and $r = \beta$.

21(a). Note that

$$p(x) = \frac{\alpha}{x^s} \text{ and } q(\xi) = \frac{\beta}{x^t}\,.$$

It follows that

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} \alpha\, x^{1-s},$$

$$\lim_{\xi\to 0} \xi^2 q(\xi) = \lim_{\xi\to 0} \beta\, x^{2-s}.$$

Hence if $s > 1$ or $t > 2$, one or both of the limits does not exist. Therefore $x = 0$ is an *irregular* singular point.

(c). Let $y = a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots$. Write the ODE as

$$x^3 y'' + \alpha\, x^2 y' + \beta\, y = 0\,.$$

Substitution of the assumed solution results in

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+1} + \alpha\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} + \beta\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Adjusting the indices, we obtain

$$\sum_{n=1}^{\infty}(n-1+r)(n+r-2)a_{n-1}x^{n+r} + \alpha\sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n+r} + \beta\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Combining the series,

$$\beta\, a_0 + \sum_{n=1}^{\infty} A_n\, x^{n+r} = 0\,,$$

in which $A_n = \beta\, a_n + (n-1+r)(n+r+\alpha-2)a_{n-1}$. Setting the coefficients equal to *zero*, we have $a_0 = 0$. But for $n \geq 1$,

$$a_n = \frac{(n-1+r)(n+r+\alpha-2)}{\beta}\, a_{n-1}.$$

Therefore, regardless of the value of r, it follows that $a_n = 0$, for $n = 1, 2, \cdots$.