Section 2.4

2. Considering the roots of the coefficient of the leading term, the ODE has unique solutions on intervals *not* containing 0 or 4. Since $2 \in (0\,,4)$, the initial value problem has a unique solution on the interval $(0\,,4)$.

3. The function $tan\,t$ is discontinuous at *odd multiples* of $\frac{\pi}{2}$. Since $\frac{\pi}{2} < \pi < \frac{3\pi}{2}$, the initial value problem has a unique solution on the interval $\left(\frac{\pi}{2}\,,\frac{3\pi}{2}\right)$.

5. $p(t) = 2t/(4 - t^2)$ and $g(t) = 3t^2/(4 - t^2)$. These functions are discontinuous at $x = \pm 2$. The initial value problem has a unique solution on the interval $(\,-2\,,2)$.

6. The function $ln\,t$ is defined and continuous on the interval $(0\,,\infty)$. Therefore the initial value problem has a unique solution on the interval $(0\,,\infty)$.

7. The function $f(t\,,y)$ is continuous everywhere on the plane, *except* along the straight line $y = -2t/5$. The partial derivative $\partial f/\partial y = -7t/(2t+5y)^2$ has the *same* region of continuity.

9. The function $f(t\,,y)$ is discontinuous along the coordinate axes, and on the hyperbola $t^2 - y^2 = 1$. Furthermore,

$$\frac{\partial f}{\partial y} = \frac{\pm 1}{y(1-t^2+y^2)} - 2\frac{y\,ln|ty|}{(1-t^2+y^2)^2}$$

has the *same* points of discontinuity.

10. $f(t\,,y)$ is continuous everywhere on the plane. The partial derivative $\partial f/\partial y$ is also continuous everywhere.

12. The function $f(t\,,y)$ is discontinuous along the lines $t = \pm k\,\pi$ and $y = -1$. The partial derivative $\partial f/\partial y = cot(t)/(1+y)^2$ has the *same* region of continuity.

14. The equation is separable, with $dy/y^2 = 2t\,dt$. Integrating both sides, the solution is given by $y(t) = y_0/(1 - y_0t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \leq 0$, solutions are defined for *all* t.

15. The equation is separable, with $dy/y^3 = -\,dt$. Integrating both sides and invoking the initial condition, $y(t) = y_0/\sqrt{2y_0t+1}$. Solutions exist as long as $2y_0t + 1 > 0$, that is, $2y_0t > -1$. If $y_0 > 0$, solutions exist for $t > -1/2y_0$. If $y_0 = 0$, then the solution $y(t) = 0$ exists for all t. If $y_0 < 0$, solutions exist for $t < -1/2y_0$.

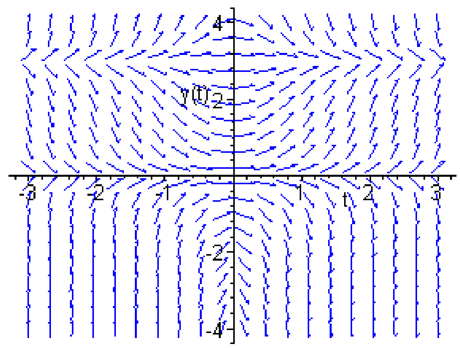
16. The function $f(t\,,y)$ is discontinuous along the straight lines $t = -1$ and $y = 0$. The partial derivative $\partial f/\partial y$ is discontinuous along the same lines. The equation is

separable, with $y\,dy = t^2\,dt/(1+t^3)$. Integrating and invoking the initial condition, the solution is $y(t) = \left[\frac{2}{3}ln|1+t^3| + y_0^2\right]^{1/2}$. Solutions exist as long as

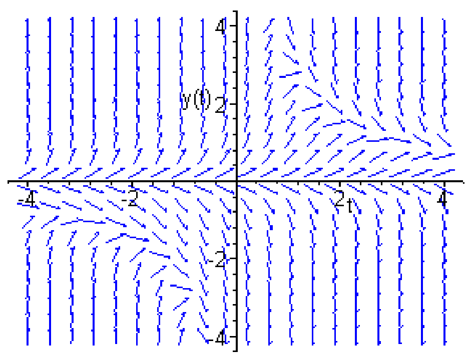
$$\frac{2}{3}ln|1+t^3| + y_0^2 \geq 0\,,$$

that is, $y_0^2 \geq -\frac{2}{3}ln|1+t^3|$. For all y_0 (it can be verified that $y_0 = 0$ yields a valid solution, even though Theorem 2.4.2 does not guarantee one) , solutions exists as long as $|1+t^3| \geq exp(-3y_0^2/2)$. From above, we must have $t > -1$. Hence the inequality may be written as $t^3 \geq exp(-3y_0^2/2) - 1$. It follows that the solutions are valid for $[exp(-3y_0^2/2) - 1]^{1/3} < t < \infty$.

17.

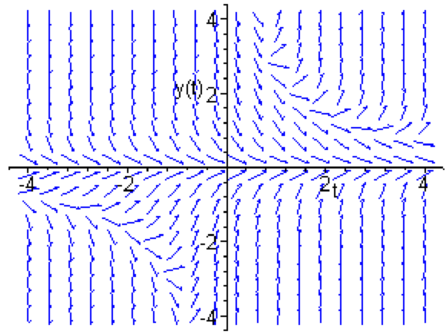


18.



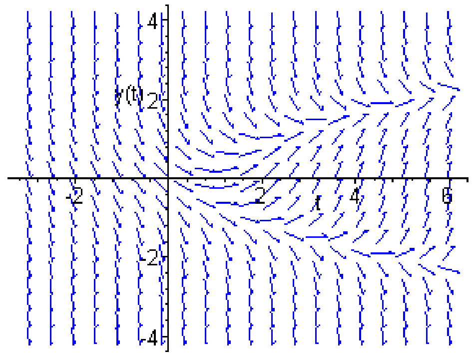
Based on the direction field, and the differential equation, for $y_0 < 0$, the slopes *eventually* become negative, and hence solutions tend to $-\infty$. For $y_0 < 0$, solutions increase without bound if $t_0 < 0$. Otherwise, the slopes *eventually* become negative, and solutions tend to *zero*. Furthermore, $y_0 = 0$ is an *equilibrium solution*. Note that slopes are *zero* along the curves $y = 0$ and $ty = 3$.

19.



For initial conditions (t_0, y_0) satisfying $ty < 3$, the respective solutions all tend to *zero*. Solutions with initial conditions *above or below* the hyperbola $ty = 3$ eventually tend to $\pm\infty$. Also, $y_0 = 0$ is an *equilibrium solution*.

20.



Solutions with $t_0 < 0$ all tend to $-\infty$. Solutions with initial conditions (t_0, y_0) to the *right* of the parabola $t = 1 + y^2$ asymptotically approach the parabola as $t \to \infty$. Integral curves with initial conditions *above* the parabola (and $y_0 > 0$) also approach the curve. The slopes for solutions with initial conditions *below* the parabola (and $y_0 < 0$) are all negative. These solutions tend to $-\infty$.

21. Define $y_c(t) = \frac{2}{3}(t-c)^{3/2}u(t-c)$, in which $u(t)$ is the Heaviside step function. Note that $y_c(c) = y_c(0) = 0$ and $y_c\left(c + (3/2)^{2/3}\right) = 1$.

(a). Let $c = 1 - (3/2)^{2/3}$.

(b). Let $c = 2 - (3/2)^{2/3}$.

(c). Observe that $y_0(2) = \frac{2}{3}(2)^{3/2}$, $y_c(t) < \frac{2}{3}(2)^{3/2}$ for $0 < c < 2$, and that $y_c(2) = 0$ for $c \geq 2$. So for any $c \geq 0$, $\pm y_c(2) \in [-2, 2]$.

26(a). Recalling Eq. (35) in Section 2.1,

$$y = \frac{1}{\mu(t)}\int \mu(s)g(s)\,ds + \frac{c}{\mu(t)}.$$

It is evident that $y_1(t) = \frac{1}{\mu(t)}$ and $y_2(t) = \frac{1}{\mu(t)}\int \mu(s)g(s)\,ds$.

(b). By definition, $\frac{1}{\mu(t)} = exp\big(-\int p(t)dt\big)$. Hence $y_1' = -p(t)\,\frac{1}{\mu(t)} = -p(t)y_1$. That is, $y_1' + p(t)y_1 = 0$.

(c). $y_2' = \left(-p(t)\,\frac{1}{\mu(t)}\right)\int_0^t \mu(s)g(s)\,ds + \left(\frac{1}{\mu(t)}\right)\mu(t)g(t) = -p(t)y_2 + g(t)$. That is, $y_2' + p(t)y_2 = g(t)$.

30. Since $n = 3$, set $v = y^{-2}$. It follows that $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$ and $\frac{dy}{dt} = -\frac{y^3}{2}\frac{dv}{dt}$. Substitution into the differential equation yields $-\frac{y^3}{2}\frac{dv}{dt} - \varepsilon y = -\sigma y^3$, which further results in $v' + 2\varepsilon v = 2\sigma$. The latter differential equation is linear, and can be written as $(e^{2\varepsilon t})' = 2\sigma$. The solution is given by $v(t) = 2\sigma t\, e^{-2\varepsilon t} + ce^{-2\varepsilon t}$. Converting back to the original dependent variable, $y = \pm v^{-1/2}$.

31. Since $n = 3$, set $v = y^{-2}$. It follows that $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$ and $\frac{dy}{dt} = -\frac{y^3}{2}\frac{dv}{dt}$. The differential equation is written as $-\frac{y^3}{2}\frac{dv}{dt} - (\Gamma cos\,t + T)y = \sigma y^3$, which upon further substitution is $v' + 2(\Gamma cos\,t + T)v = 2$. This ODE is linear, with integrating factor $\mu(t) = exp\big(2\int(\Gamma cos\,t + T)dt\big) = exp(-2\Gamma sin\,t + 2Tt)$. The solution is

$$v(t) = 2exp(2\Gamma sin\,t - 2Tt)\int_0^t exp(-2\Gamma sin\,\tau + 2T\tau)d\tau + c\,exp(-2\Gamma sin\,t + 2Tt).$$

Converting back to the original dependent variable, $y = \pm v^{-1/2}$.

33. The solution of the initial value problem $y_1' + 2y_1 = 0$, $y_1(0) = 1$ is $y_1(t) = e^{-2t}$. Therefore y$(1^-) = y_1(1) = e^{-2}$. On the interval $(1, \infty)$, the differential equation is $y_2' + y_2 = 0$, with $y_2(t) = ce^{-t}$. Therefore $y(1^+) = y_2(1) = ce^{-1}$. Equating the limits $y(1^-) = y(1^+)$, we require that $c = e^{-1}$. Hence the global solution of the initial value problem is

$$y(t) = \begin{cases} e^{-2t}, & 0 \le t \le 1 \\ e^{-1-t}, & t > 1 \end{cases}.$$

Note the discontinuity of the derivative

$$y(t) = \begin{cases} -2e^{-2t}, & 0 < t < 1 \\ -e^{-1-t}, & t > 1 \end{cases}.$$