Section 2.7

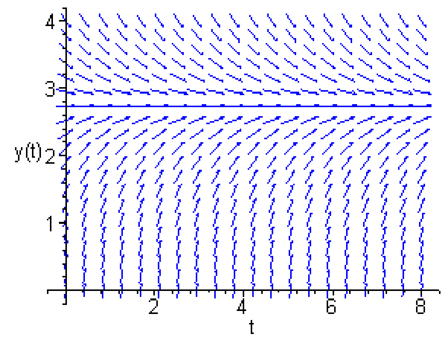
$2(a)$. The Euler formula is $y_{n+1} = y_n + h(2y_n - 1) = (1 + 2h)y_n - h$.

(d). The differential equation is *linear*, with solution $y(t) = (1 + e^{2t})/2$.

$4(a)$. The Euler formula is $y_{n+1} = (1 - 2h)y_n + 3h\, cos\, t_n$.

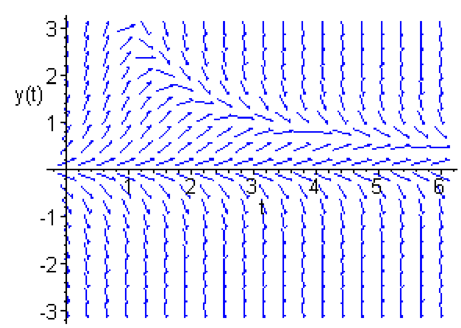
(d). The exact solution is $y(t) = (6cos\, t + 3sin\, t - 6\, e^{-2t})/5$.

5.



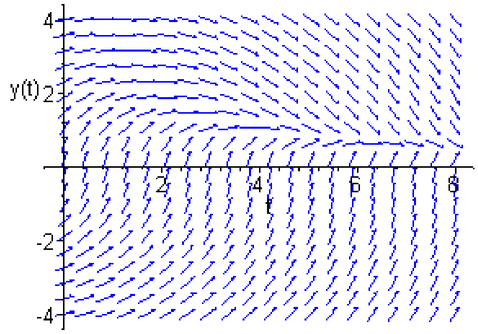
All solutions seem to converge to $\phi(t) = 25/9$.

6.



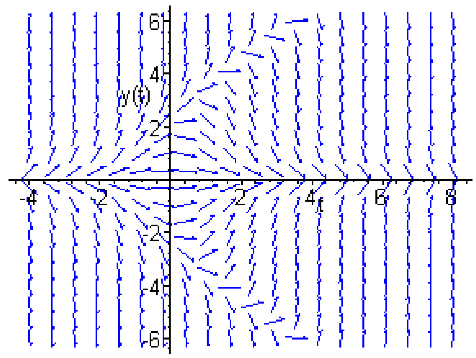
Solutions with *positive* initial conditions seem to converge to a specific function. On the other hand, solutions with *negative* coefficients decrease without bound. $\phi(t) = 0$ is an equilibrium solution.

7.



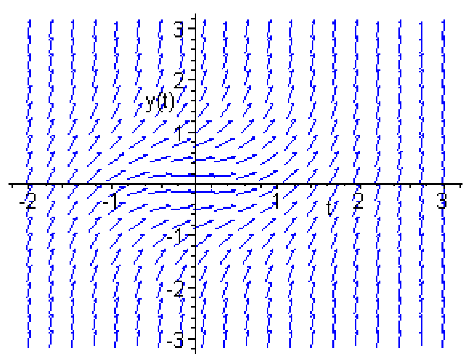
All solutions seem to converge to a specific function.

8.



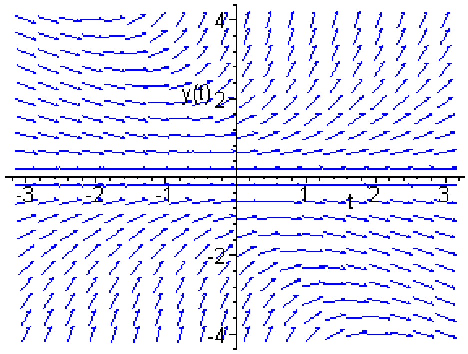
Solutions with initial conditions to the *'left'* of the curve $t = 0.1y^2$ seem to diverge. On the other hand, solutions to the *'right'* of the curve seem to converge to *zero*. Also, $\phi(t)$ is an equilibrium solution.

9.



All solutions seem to diverge.

10.



Solutions with *positive* initial conditions increase without bound. Solutions with *negative*
initial conditions decrease without bound. Note that $\phi(t) = 0$ is an equilibrium solution.

11. The Euler formula is $y_{n+1} = y_n - 3h\sqrt{y_n} + 5h$. The initial value is $y_0 = 2$.

12. The iteration formula is $y_{n+1} = (1+3h)y_n - h\,t_n\,y_n^2$. $(t_0, y_0) = (0\,, 0.5)$.

14. The iteration formula is $y_{n+1} = (1 - h\,t_n\,)y_n + h\,y_n^3\,/10$. $(t_0, y_0) = (0\,, 1)$.

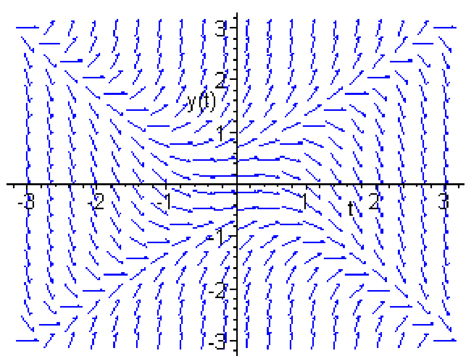
17. The Euler formula is

$$y_{n+1} = y_n + \frac{h(y_n^2 + 2t_n\,y_n)}{3 + t_n^2}.$$

The initial point is $(t_0, y_0) = (1\,, 2)$.

$18(a)$. See Problem 8.

$19(a)$.



(b). The iteration formula is $y_{n+1} = y_n + h\,y_n^2 - h\,t_n^2$. The critical value of α appears to be near $\alpha_0 \approx 0.6815$. For $y_0 > \alpha_0$, the iterations diverge.

$20(a)$. The ODE is *linear*, with general solution $y(t) = t + c\,e^t$. Invoking the specified initial condition, $y(t_0) = y_0$, we have $y_0 = t_0 + c\,e^{t_0}$. Hence $c = (y_0 - t_0)e^{-t_0}$. Thus the solution is given by $\phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

(b). The Euler formula is $y_{n+1} = (1+h)y_n + h - h\,t_n$. Now set $k = n+1$.

(c). We have $y_1 = (1+h)y_0 + h - h\,t_0 = (1+h)y_0 + (t_1 - t_0) - h\,t_0$. Rearranging the terms, $y_1 = (1+h)(y_0 - t_0) + t_1$. Now suppose that $y_k = (1+h)^k(y_0 - t_0) + t_k$, for some $k \geq 1$. Then $y_{k+1} = (1+h)y_k + h - h\,t_k$. Substituting for y_k, we find that $y_{k+1} = (1+h)^{k+1}(y_0 - t_0) + (1+h)\,t_k + h - h\,t_k = (1+h)^{k+1}(y_0 - t_0) + t_k + h$. Noting that $t_{k+1} = t_k + k$, the result is verified.

(d). Substituting $h = (t - t_0)/n$, with $t_n = t$,

$$y_n = \left(1 + \frac{t - t_0}{n}\right)^n (y_0 - t_0) + t\,.$$

Taking the limit of both sides, as $n \to \infty$, and using the fact that $\lim_{n\to\infty}(1 + a/n)^n = e^a$, pointwise convergence is proved.

21. The exact solution is $\phi(t) = e^t$. The Euler formula is $y_{n+1} = (1+h)y_n$. It is easy to see that $y_n = (1+h)^n y_0 = (1+h)^n$. Given $t > 0$, set $h = t/n$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(1 + t/n)^n = e^t$.

23. The exact solution is $\phi(t) = t/2 + e^{2t}$. The Euler formula is $y_{n+1} = (1+2h)y_n + h/2 - h\,t_n$. Since $y_0 = 1$, $y_1 = (1+2h) + h/2 = (1+2h) + t_1/2$. It is easy to show by mathematical induction, that $y_n = (1+2h)^n + t_n/2$. For $t > 0$, set $h = t/n$ and thus $t_n = t$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}[(1 + 2t/n)^n + t/2] = e^{2t} + t/2$. Hence pointwise convergence is proved.