Module UFMEQT-20-1

Stress and Dynamics

Dynamics – Revision Notes

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Abstract

This book covers some basics of physics, kinematics and mathematics. The material in this book will not be covered in the Dynamics lectures, and much of the content of the lectures will assume that the students have an understanding of this material. The students may refer to this material in order to aid understanding, and if necessary, can work through some of the problems and exercises at the end of each chapter to reinforce this knowledge.

Contents

1 Linear Motion

1.1 Introduction

The following section deals with linear motion with *constant acceleration*. These equations can only be applied when it is known that acceleration does not vary with respect to time.

- Linear motion is motion along a *straight line*, i.e. one-dimensional motion.
- In one-dimensional motion, the direction component of vector quantities is assumed to be in the positive x-direction in an x-y plane (with the exception of constant acceleration due to gravity – see Section 1.4.)
- Constant acceleration in a straight line is called *uniform acceleration.*
- The SI units for displacement are metres (m). Metres per second (m/s or ms^{-1}) are used for velocity, and metres per second per second ($\mathrm{m/s}^2$ or ms^{-2}) for acceleration.
- The variable t represents time, v represents velocity, and a represents acceleration. Displacement is generally represented by s, but sometimes x and y are used when we know that the displacement is along a certain axis.

1.2 Derivation of Kinematic Equations

Constant acceleration is important as it applies to many objects in nature. For example, an object in free-fall near the Earth's surface moves in the vertical direction with constant acceleration (neglecting air resistance). Constant acceleration is denoted as a.

Acceleration is the change in velocity over the change in time:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v_f - v_i}{t_f - t_i}$$

where the f and i subscripts denote 'final' and 'initial'. For convenience, let $t_i = 0$ and t_f be any arbitrary time t. Also, let $v_i = v_0$ and $v_f = v_1$. With this notation, the acceleration is expressed as this:

$$a = \frac{v_1 - v_0}{t}$$

or:

$$\boxed{v_1 = v_0 + at} \tag{1.1}$$

When dealing with constant acceleration, velocity increases linearly with time, as shown in Figure 1.1. (The equation for this line is equation 1.1). The average velocity is just the mean value of the initial and final velocities.

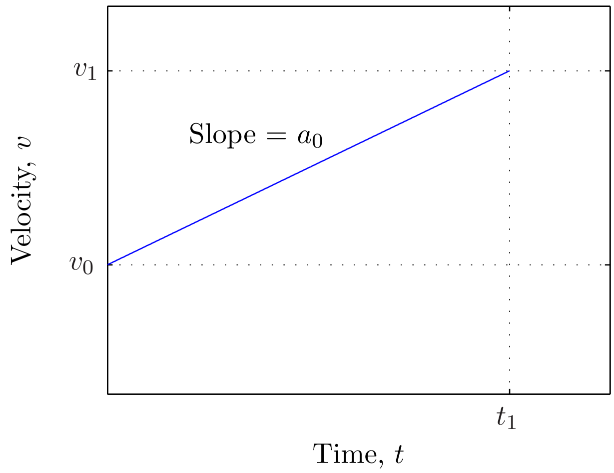


Figure 1.1: Velocity-time graph for constant acceleration

The average velocity is:

$$\boxed{v_{\text{av}} = \frac{v_0 + v_1}{2}} \tag{1.2}$$

The area under the graph in Figure 1.1 is the displacement, s:

$$\text{Area} = \text{average height} \times \text{base} \tag{1.3}$$

$$\therefore s = \text{average velocity} \times \text{time} = \left(\frac{v_0 + v_1}{2}\right) t$$

$$s = \frac{1}{2}(v_1 + v_0)t \tag{1.4}$$

By substituting the equation for v_1 (equation 1.1), an expression for the displacement as a function of time can be obtained:

$$s = \frac{1}{2}(v_0 + at + v_0)t$$

$$\boxed{s = v_0 t + \frac{1}{2}at^2} \tag{1.5}$$

Finally, an expression that does not contain time can be obtained by rearranging equation 1.1 to isolate t, then substituting this into the equation for displacement (equation 1.4):

$$v_1 = v_0 + at \quad \rightarrow \quad t = \frac{v_1 - v_0}{a}$$

$$s = \frac{1}{2}(v_1 + v_0)\left(\frac{v_1 - v_0}{a}\right) = \frac{v_1^2 - v_0^2}{2a}$$

$$\boxed{v_1^2 = v_0^2 + 2as} \tag{1.6}$$

1.3 Kinematic Equations for Constant Acceleration

The following equations apply when the initial displacement is zero:

$$s = 0 \qquad \text{at} \qquad t = 0$$

Notation:

Initial velocity	=	v_0	or	u
Final velocity	=	v_1	or	v
Average velocity	=	v_{av}		
Constantacceleration	=	a		
Time elapsed	=	t		
Displacement	=	s		

Kinematic equations	Alternative form	Equation number
$v_1 = v_0 + at$	$v = u + at$	1.1
$v_{\text{av}} = \frac{(v_0 + v_1)}{2}$	$v_{\text{av}} = \frac{(u + v)}{2}$	1.2
$s = \frac{1}{2}(v_0 + v_1)t$ $s = v_{\text{av}}t$	$s = \frac{1}{2}(u + v)t$	1.4
$s = v_0 t + \frac{1}{2}at^2$	$s = ut + \frac{1}{2}at^2$	1.5
$v_1^2 = v_0^2 + 2as$	$v^2 = u^2 + 2as$	1.6

Note:
- When speed is *increasing*, acceleration is *positive*
- When speed is *decreasing* (retardation or deceleration), the acceleration is *negative*

1.4 Free Fall Under Gravity

A particular case of constant acceleration in one dimension is when a body falls freely towards the earth. If air resistance is neglected, and the distances of fall is short, then the downward acceleration of a body when it is close to the surface of the earth is constant.

This acceleration due to gravity is represented by g where:

$$\boxed{g = 9.81\,\mathrm{m/s^2}}$$

Note: *It is important to consider the direction of the acceleration due to gravity.* For example, when considering the upward vertical motion of a body under the effects of gravity, then:

> $v_0 = 20\,\mathrm{m/s} \uparrow$ indicates an initial velocity of $20\,\mathrm{m/s}$ *upwards*. (So, vertical motion upwards is positive.)
>
> $a = 9.81\,\mathrm{m/s^2} \downarrow$ indicates a acceleration of $9.81\,\mathrm{m/s^2}$ *downwards*.

To be consistent with the direction of the velocity, the acceleration must be written as: $a = -g = -9.81\,\mathrm{m/s^2} \uparrow$. This indicates a *deceleration* or *retardation* of $9.81\,\mathrm{m/s^2}$ in the upwards direction. It generally makes sense for the positive vertical direction to be upwards. In this case, the acceleration due to gravity is:

$$\boxed{a = -g = -9.81\,\mathrm{m/s^2}}$$

It is important to remember that once an object has been released, it is moving under the influence of gravity only, regardless of its initial motion (this is defined as free fall). Once an object is in free fall, it will have an acceleration with a magnitude of g, directed downwards.

Since this is a particular case of motion in one dimension, all equations defined above apply to freely falling bodies, remembering that motion is in the vertical direction, and that the value of acceleration, $a = -g$.

1.5 Dot Notation

In this module, another form of notation used to signify motion is known as *dot notation*. A dot above a variable basically means *the derivative with respect to time*. Two dots above a variable means *the second deriviative with respect to time*.

So, assuming displacement is defined as x:

$$\begin{aligned}
\dot{x} &= \frac{\mathrm{d}x}{\mathrm{d}t} = \text{velocity} \\
\ddot{x} &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \text{acceleration}
\end{aligned}$$

So, another alternative form of the equations can be seen, when:

- s is replaced with x
- v is replaced with $\dot{x}$
- a is replaced with $\ddot{x}$

For example, equation 1.5 becomes:

$$x = \dot{x}t + \frac{1}{2}\ddot{x}t^2$$

1.6 Examples

1.6.1 Constant Acceleration

Question

A sports car starting from rest accelerates at a rate of $5.00\,\mathrm{m/s^2}$. What is the velocity of the car after it has travelled $30.0\,\mathrm{m}$?

Solution

The first step with these problems it to check that the units you are dealing with are consistent. Here, we have an acceleration figure in $\mathrm{m/s^2}$ and a distance in metres, so this is fine.

Next, the car is accelerating in a straight line, so we can assume that the car is travelling along the x-axis. The origin of the x-axis is at the initial location of the car, and the positive direction is to the right. Using this convention, velocity, accelerations and displacements to the right are positive.

Now, there are three bits of information given in the question. These are:

- The initial velocity, v_0, which is zero.
- The aceleration, a, which is $5.00\,\mathrm{m/s^2}$.
- The displacement, s, of $30.0\,\mathrm{m}$.

We are required to find out v_1.

Since we have v_0, a, and s, the most appropriate equation to use is equation 1.1:

$$v_1^2 = v_0^2 + 2as$$

Substituting in the numbers, results in:

$$v_1^2 = (0\,\mathrm{m/s})^2 + 2(5.00\,\mathrm{m/s^2})(30.0\,\mathrm{m}) = 300\,\mathrm{m^2/s^2}$$

$$v_1 = \sqrt{300\,\mathrm{m^2/s^2}} = 17.3\,\mathrm{m/s}$$

The velocity of the car after $30\,\mathrm{m}$ is $17.3\,\mathrm{m/s}$.

Question

A manufacturer of a high performance sports car claims that it can accelerate from 0 to 100.0 mph in 8.00 s. Assuming constant acceleration, determine:

(a) the acceleration of the car
(b) the displacement of the car in the first 8.00 s.

Solution

First, we have to make sure that the units are consistent. Since the final velocity is reported in miles per hour, this needs converting to m/s:

$$100\,\frac{\cancel{\text{miles}}}{\cancel{\text{hour}}} \times \frac{1609.33\,\text{metres}}{1\,\cancel{\text{mile}}} \times \frac{1\,\cancel{\text{hour}}}{3600\,\text{seconds}} = 44.70\,\text{m/s}$$

From, the question we have the following information:

- The initial velocity, v_0 is zero
- The final velocity, v_1 is 44.70 m/s
- The time to accelerate, t is 8.00 s

(a) The most appropriate equation to calculate the acceleration is therefore equation 1.1:

$$v_1 = v_0 + at \quad \rightarrow \quad a = \frac{v_1 - v_0}{t}$$

Substituting in the values:

$$a = \frac{44.70\,\text{m/s} - 0\,\text{m/s}}{8.0\,\text{s}} = 5.59\,\text{m/s}^2$$

(b) The displacement travelled by the car in the first 8.00 seconds can be found using equation 1.5:

$$s = \frac{1}{2}(5.59\,\text{m/s}^2)(8.00\,\text{s})^2 = 179\,\text{m}$$

1.6.2 Free Fall Under Gravity

Question

A golf ball is released from rest at the top of a very tall building. Neglecting air resistance, calculate the position and velocity of the ball after 1.00, 2.00 and 3.00 s.

Answer

We choose our coordinates so that the starting point of the ball is at the origin ($y = 0$ at $t = 0$) and remember that y is normally defined as positive in the upwards direction. We know that $v_0 = 0$, and $a = -g = -9.81\,\text{m/s}^2$.

The most appropriate equations to use are equations 1.1 and 1.5:

$$v_1 = v_0 + at = -9.81t$$
$$s = y = v_0 t + \frac{1}{2}at^2 = \frac{1}{2}(-9.81\,\mathrm{m/s^2})t^2$$

Therefore, at $t = 1.00s$:

$$\begin{aligned} v_1 &= (-9.81\,\mathrm{m/s^2})(1.00\,\mathrm{s}) = -9.81\,\mathrm{m/s} \\ y &= \frac{1}{2}(-9.81\,\mathrm{m/s^2})(1.00\,\mathrm{s})^2 = -4.91\,\mathrm{m} \end{aligned}$$

Likewise, at $t = 2.00\,\mathrm{s}$:

$$\begin{aligned} v_1 &= (-9.81\,\mathrm{m/s^2})(2.00\,\mathrm{s}) = -19.6\,\mathrm{m/s} \\ y &= \frac{1}{2}(-9.81\,\mathrm{m/s^2})(2.00\,\mathrm{s})^2 = -19.6\,\mathrm{m} \end{aligned}$$

At $t = 3.00\,\mathrm{s}$, $v_0 = -29.4\,\mathrm{m/s}$ and $y = -44.1\,\mathrm{m}$.
Note: the minus sign for v_0 indicates that the velocity vector is directed downwards, and the minus sign for y indicates displacement in the negative y-direction.

Question

A stone is thrown from the top of a building with an initial velocity of 20.0 m/s straight upward. The building is 50.0 m high, and the stone just misses the edge of the roof on its way down. Determine:

(a) the time needed for the stone to reach its maximum height
(b) the maximum height
(c) the time needed for the stone to return to the level of the thrower
(d) the velocity of the stone at this instant
(e) the velocity and position of the stone at $t = 5.00\,\mathrm{s}$
(f) the velocity of the stone just before it hits the ground.

Answer

(a) To find the time necessary for the stone to reach the maximum height, use equation 1.1:

$$v_1 = v_0 + at \quad \rightarrow \quad t = \frac{v_1 - v_0}{a}$$

knowing that $v_1 = 0$ at the maximum height. $v_0 = 20\,\mathrm{m/s}$, $a = -9.81\,\mathrm{m/s^2}$:

$$t = \frac{0\,\mathrm{m/s} - 20.0\,\mathrm{m/s}}{-9.81\,\mathrm{m/s^2}} = 2.04\,\mathrm{s}$$

(b) The maximum height can be determined using equation 1.5:

$$s = v_0 t + \frac{1}{2}at^2$$

where s is replaced by y_{max}, (which will be the height as measured from the thrower's position). Substituting the value for time found in part (a) results in:

$$y_{\text{max}} = (20.0\,\text{m/s})(2.04\,\text{s}) + \frac{1}{2}(-9.81\,\text{m/s}^2)(2.04\,\text{s})^2 = 20.4\,\text{m}$$

(c) When the stone is back at the height of the thrower, the value for displacement, y, is zero. Again, using equation 1.5 (with y replacing s):

$$y = v_0 t + \frac{1}{2}at^2$$

and setting $y = 0$, a value for t can be determined:

$$0 = (20.0)t + \frac{1}{2}(-9.81)t^2 = t(20.0 - 4.90t)$$

This is a quadratic equation and has two solutions for t. One solution is $t = 0$, which corresponds to the time at which the stone starts its motion. The other solution is $t = 4.08\,\text{s}$, which is the answer to this part.

(d) The value for t found in (c) can be inserted equation 1.1 to find v_1.

$$v_1 = (20.0\,\text{m/s}) + (-9.81\,\text{m/s}^2)(4.08\,\text{s}) = -20.0\,\text{m/s}$$

which is directed downwards. Note that the velocity of the stone when it arrives back at its original height is equal in magnitude to the stone's initial velocity but opposite in direction.

(e) To find the velocity after 5.00 s, again use equation 1.1:

$$v_1 = (20.0\,\text{m/s}) + (-9.81\,\text{m/s}^2)(5.00\,\text{s}) = -29.1\,\text{m/s}$$

To find the position (equation 1.5):

$$y = (20.0\,\text{m/s})(5.00\,\text{s}) + \frac{1}{2}(-9.81\,\text{m/s}^2)(5.00\,\text{s}) = -22.6\,\text{m}$$

which is 22.6 m below the height of the thrower.

(f) To find the velocity of the stone just before it hits the ground, we use equation 1.6 since we know: $v_0 = 20,0\,\text{m/s}$, $a = -9.81\,\text{m/s}^2$ and $s = y = -50.0\,\text{m}$ (50 metres below the thrower). Substituting these values into the equation:

$$v_1^2 = (20.0\,\text{m/s})^2 + 2(-9.81\,\text{m/s}^2)(-50.0\,\text{m}) = 1381\,\text{m}^2/\text{s}^2$$

$$v_1 = \sqrt{1381\,\text{m}^2/\text{s}^2} = 37.1\,\text{m/s}$$

1.7 Exercises

1.7.1 Constant Acceleration

1. An athlete swims the length of a 50.0 m pool in 20.0 s and makes the return trip to the starting position in 22.0 s. Determine the swimmer's average velocities in:
   (a) the first half of the swim
   (b) the second half of the swim and
   (c) the round trip.
   (Ans: (a) 2.50 m/s (b) -2.27 m/s (c) 0)
2. A speedy tortoise can run with a speed of 10.0 cm/s, and a hare can run 20 times as fast. In a race, they both start at the same time, but the hare stops to rest for 2.0 minutes. The tortoise wins by a shell (20 cm).
   (a) How long does the race take?
   (b) What is the length of the race?
   (Ans: (a) 126 s (b) 12.6 m)
3. Two cars travel in the same direction along a straight road, one at a constant speed of 55 mph and other at 70 mph.
   (a) Assuming that they start at the same point, how much sooner does the faster car arrive at a destination 10 miles away?
   (b) How far must the faster car travel before it has a 15 minute lead on the slower car?
   (Ans: (a) 2.34 minutes (b) 64.17 miles)
4. Jules Verne in 1865 proposed sending men to the Moon by firing a space capsule from a 220 metre long cannon with a final velocity of 10.97 km/s. What would have been the unrealistically large acceleration experienced by the space travellers during launch? *(Ans: 273502 m/s^2)*
5. A car travelling initially at 7.0 m/s accelerates at the rate of 0.8 m/s^2 for an interval of 2.0 s. What is its velocity at the end of the acceleration? *(Ans: 8.60 m/s^2)*
6. A driver in a car travelling at a speed of 60 mph sees a deer 100 m away on the road. Calculate the minimum constant acceleration that is necessary for the car to stop without hitting the deer (assuming that the deer does not move in the meantime). *(Ans: -3.60 m/s^2)*
7. A Cessna aircraft has a lift-off speed of 120 km/h.
   (a) What minimum constant acceleration does this require if the aircraft is to be airborne after a take off run of 240 m?
   (b) How long does it take the aircraft to become airborne?
   (Ans. (a) 2.32 m/s^2 (b) 14.4 s)
8. A jet plane lands with a velocity of 100 m/s and can accelerate at a maximum rate of -5.0 m/s^2 as it comes to rest.
   (a) From the instant it touches the runway, what is the minimum time needed before it can come to rest?

(b) Can this plan land on a small island airport where the runway is 0.80 km long?

(Ans. (a) 20 s (b) No: minimum distance required is 1000 m)

1.7.2 Free Fall Under Gravity

10. A ball is thrown vertically upward with a speed of 25.0 m/s.
    (a) How high does it rise?
    (b) How long does it take to reach its highest point?
    (c) How long does it take to hit the ground after it reaches its highest point?
    (d) What is its speed when it returns to the level from which it started?

    (Ans. (a) 31.9 m (b) 2.55 s (c) 2.55 s (d) 25 m/s)
11. A peregrin falcon dives at a pigeon at rest. The falcon starts downward from rest and falls with freefall acceleration. If the pigeon is 76.0 m below the initial position of the falcon, how long does it take the falcon to reach the pigeon? *(Ans. 3.94 s)*
12. A rocket moves upwards, starting from rest with an acceleration of 29.4 m/s^2 for 4.00 s. It runs out of fuel at the end of this 4.00 s and continues to move upward. How high does it rise? *Ans. 941 m*
13. A pebble is dropped into a deep well, and 3.0 s later the sound of a splash is heard as the pebble reaches the bottom of the well. The speed of sound in air is about 340 m/s.
    (a) How long does it take the pebble to hit the water?
    (b) How long does it take for the sound to reach the observer?
    (c) What is the depth of the well?

    (This is a tough one!) *(Ans. (a) 2.88 s (b) 0.120 s (c) 40.7 m)*

2 Angular Motion

2.1 Introduction

In section 1, linear motion in a single dimension was considered. Angular motion is used to describe motion of an object rotating around an axis.

With linear motion, the components of displacement, velocity and acceleration were considered. Angular motion has similar components: angular displacement, angular velocity and angular acceleration.

2.2 Components

2.2.1 Angular Displacement

Angular displacement, often signified by the Greek letter, θ (theta), is the angle turned through by a rotating body, and is measured in *radians*. See the Additional Notes section for an explanation of radians. Convention dictates that angular displacement is positive in the anti-clockwise direction, with $\theta = 0$ being aligned with the horizontal (on the right), as shown in Figure 2.1.

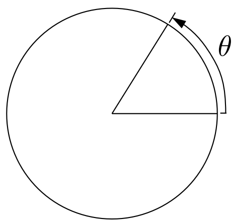


Figure 2.1: Angular displacement

2.2.2 Angular Velocity

As with linear motion, where velocity is the change in displacement divided by the change in time, angular velocity is the change in angular displacement divided by the change in time, and is often denoted by the Greek letter ω (omega). Its units are *radians per second*.

$$\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\mathrm{d}\theta}{\mathrm{d}t} \quad [\mathrm{rad/s}]$$

2.2.3 Angular Acceleration

As with linear motion, acceleration is the change in velocity over the change in time, angular acceleration is the change in angular velocity of the change in time, and is denoted by the Greek letter α (alpha). Its units are *radians per second per second.*

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \quad [\mathrm{rad/s^2}]$$

2.3 Equations for Angular Motion

In Section 1, a process was undertaken to derive a set of equations to be used when dealing with constant linear acceleration problems. An identical process can be undertaken with angular motion, assuming constant *angular* acceleration, where a_0 is replaced by α, v_0 and v_1 are replaced by ω_0 and ω_1, and s is replaced by θ:

Initial angular velocity $= \omega_0$	Angular acceleration $= \alpha$
Final angular velocity $= \omega_1$	Angular displacement $= \theta$
Average angular velocity $= \omega_{\mathrm{av}}$	Time elapsed $= t$

Thus, a similar table of equations can be created:

Kinematic equations	Equation number	Linear Analogy
$\omega_1 = \omega_0 + \alpha t$	2.1	$v_1 = v_0 + at$
$\omega_{\mathrm{av}} = \frac{(\omega_0 + \omega_1)}{2}$	2.2	$v_{\mathrm{av}} = \frac{v_0 + v_1}{2}$
$\theta = \frac{1}{2}(\omega_0 + \omega_1)t$ $\theta = \omega_{\mathrm{av}} t$	2.3	$s = \frac{1}{2}(v_0 + v_1)t$ $s = v_{\mathrm{av}} t$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	2.4	$s = v_0 t + \frac{1}{2}at^2$
$\omega_1^2 = \omega_0^2 + 2\alpha\theta$	2.5	$v_1^2 = v_0^2 + 2as$

2.4 Linear and Angular Motion

It is possible to relate angular and linear motion. Figure 2.2 shows a spinning disk with a point, P, on the edge. The disk is spinning with an angular velocity of ω about the centre of the disk, O. The radius of the disk is r.

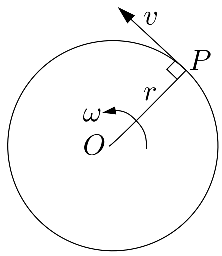


Figure 2.2: Relating Angular and Linear Velocity

In one revolution, the line OP will travel through an angle of 2π radians, and point P will travel a distance of one circumference of the disk, i.e. $2\pi r$.

The time taken for one revolution can be determined:

$$\omega = \frac{\text{angular displacement}}{\text{time taken}} = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2\pi}{t} \quad \rightarrow \quad t = \frac{\text{angular displacement}}{\text{angular velocity}} = \frac{2\pi}{\omega}$$

Then the linear velocity of point P, v as shown in Figure 2.2, is given by:

$$v = \frac{\text{distance moved}}{\text{time taken}} = \frac{2\pi r}{\left(\dfrac{2\pi}{\omega}\right)}$$

$$\boxed{v_t = \omega r} \tag{2.6}$$

which is the *instantaneous tangential velocity* of point P, as denoted by the subscript t.

Linear and angular acceleration can be related in a similar way. Figure 2.3 shows line OP has an angular acceleration of α.

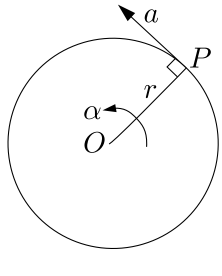


Figure 2.3: Relating Angular and Linear Acceleration

P will have an *instantaneous tangential acceleration* of:

$$\boxed{a_t = \alpha r} \tag{2.7}$$

Note that these equations are valid when ω is measured in radians per second, and α in radians per second per second. Other measures of angular motion, such as degrees per second or revolutions per minute can not to be used with these equations.

2.5 Centripetal Acceleration

When the velocity of an object travelling in a straight line changes, for example from 10 m/s to 20 m/s, an acceleration has occurred due to the change in the *magnitude* of the velocity.

Because velocity is a vector, having both direction and magnitude, a change in direction of the velocity can also result in an acceleration. Consider a disk with a line OP rotating around point O with a constant angular velocity. P is therefore following a circular path, as shown in Figure 2.4.

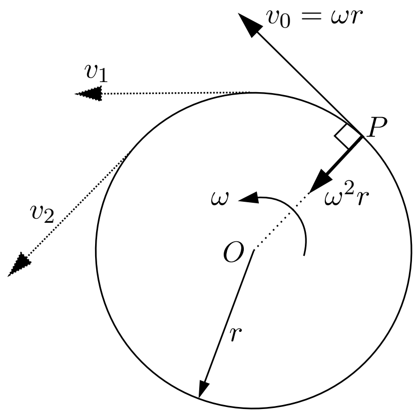


Figure 2.4: Centripetal Acceleration

As determined above, the linear velocity of point P directly proportional to the angular velocity. Although the *magnitude* of this velocity does not change (ω is constant), the direction of the velocity is changing continuously, with the rate of this directional change equal to the angular velocity, ω. This results in an acceleration known as *centripetal* acceleration.

As is known, acceleration is the rate of change of velocity. In this case, acceleration is given by:

$$\text{Acceleration} = \text{Magnitude of velocity } \times \text{ Rate of change of direction} = v\omega$$

Therefore, centripetal acceleration, given by a_c:

$$\boxed{a_c = v\omega = \omega^2 r = \frac{v^2}{r}} \tag{2.8}$$

The direction of centripetal acceleration is *always towards the centre of the circular path*, as shown in Figure 2.4.

Note the distinction between tangential acceleration and centripetal acceleration. The tangential component of acceleration is due to changing speed, while the centripetal component of acceleration is due to changing direction. If both of these components exist simultaneously, the tangential acceleration is tangent to the circular path, and the centripetal acceleration is pointing towards the centre of the circular path. These components are therefore perpendicular to each other (forming the sides of a right angle triangle), thus the *total acceleration* can be found using the Pythagorean theorem:

$$\boxed{a = \sqrt{a_t^2 + a_c^2}} \tag{2.9}$$

2.6 Examples

2.6.1 Constant Angualar Acceleration

Question

A bicycle wheel rotates with a constant angular acceleration of $3.5\,\mathrm{rad/s^2}$. The initial angular speed of the wheel is $2.0\,\mathrm{rad/s}$ at $t_0 = 0$.

(a) through what angle does the wheel rotate in $2.0\,\mathrm{s}$?
(b) what is the angular speed at $t = 2.0\,\mathrm{s}$?
(c) Using the results from part (a) find again the angular speed at $t = 2.0\,\mathrm{s}$.

Answer

(a) Since we are given $\omega_0 = 2.0\,\mathrm{rad/s}$ and $\alpha = 3.5\,\mathrm{rad/s^2}$, use equation 2.4:

$$\begin{aligned}\theta &= \omega_0 + \frac{1}{2}\alpha t^2 \\ \theta &= (2.0\,\mathrm{rad/s})(2.0\,\mathrm{s}) + \frac{1}{2}(3.5\,\mathrm{rad/s^2})(2.0\,\mathrm{s})^2 \\ &= 11\,\mathrm{rad} = 630^\circ\end{aligned}$$

(b) Using the equation 2.1 for ω_1:

$$\omega_1 = \omega_0 + \alpha t = 2.0\,\mathrm{rad/s} + (3.5\,\mathrm{rad/s^2})(2.0\,\mathrm{s}) = 9.0\,\mathrm{rad/s}$$

(c) With a value for θ, equation 2.5 can be used:

$$\omega_1^2 = \omega_0^2 + 2\alpha\theta = (2.0\,\mathrm{rad/s})^2 + 2(3.5\,\mathrm{rad/s^2})(11\,\mathrm{rad}) = 81.0\,\mathrm{rad^2/s^2}$$

$$\omega_1 = \sqrt{81.0\,\mathrm{rad^2/s^2}} = 9.0\,\mathrm{rad/s}$$

2.6.2 Angular and Linear Motion

Question

A compact disc is inserted to a player and rotates up to angular speed of 48.0 rad/s in 0.25 s.

(a) What is the angular acceleration of the disc, assuming the angular acceleration is uniform?
(b) How many rotations does the disc make while coming up to speed.
(c) What is the tangential acceleration of a point on the edge of a CD (diameter is 120 mm)
(d) The linear reading speed (scanning velocity) of the laser is 1.2 m/s. If the radius of the inside track of the CD is 25 mm, verify that 48 rad/s is the correct angular velocity.
(e) The linear scanning velocity of a CD is constant. Find the angular velocity (in revolutions per minute) of the CD when the laser is reading a track at the outer edge of the CD (radius of 58 mm).
(f) What is the angular acceleration of the disc, if it takes 74 minutes to play the CD from start to finish. Note: the laser starts reading from the centre (where radius is 25 mm) outwards (until the radius is 58 mm).

Answer

(a) For this, use $\omega_1 = \omega_0 + \alpha t$, and the fact that $\omega_0 = 0$ at $t = 0$ results in:

$$\alpha = \frac{\omega_1}{t} = \frac{48.0\,\mathrm{rad/s}}{0.25\,\mathrm{s}} = 192\,\mathrm{rad/s^2}$$

(b) Equation 2.4 for θ needs to be used here:

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}(192\,\mathrm{rad/s^2})(0.25)^2 = 6\,\mathrm{rad}$$

In revolutions, divide by 2π: 0.95 revolutions.

(c) The relation $v_t = \omega r$ (equation 2.6) and the given values lead to:

$$v_t = \omega r = (48.0)\left(\frac{0.120}{2}\right) = 2.88\,\mathrm{m/s}$$

(d) In this part, take $v_t = 1.2\,\mathrm{m/s}$, $r = 0.025\,\mathrm{m}$ and use the relatinoship:

$$v_t = \omega r \quad \rightarrow \quad \omega = \frac{v_t}{r} = \frac{1.2\,\mathrm{m/s}}{0.025\,\mathrm{m}} = 48.0\,\mathrm{rad/s}$$

(e) Again, $v_t = 1.2\,\mathrm{m/s}$, now $r = 0.058\,\mathrm{m}$:

$$v_t = \omega r \quad \rightarrow \quad \omega = \frac{v_t}{r} = \frac{1.2\,\mathrm{m/s}}{0.058\,\mathrm{m}} = 20.7\,\mathrm{rad/s}$$

In revolutions per minute (rpm), following the conversion:

$$20.7\,\frac{\cancel{\text{rad}}}{\cancel{\text{sec}}} \times \frac{1\,\text{rev}}{2\pi\,\cancel{\text{rad}}} \times \frac{60\,\cancel{\text{sec}}}{1\,\text{min}} = 198\,\text{rpm}$$

(f) Firstly, calculate the number of seconds in 74 minutes:

$$74\,\cancel{\text{min}} \times \frac{60\,\text{sec}}{1\,\cancel{\text{min}}} = 4440\,\text{sec}$$

From part (d), $\omega_0 = 48\,\text{rad/s}$, and from part (e), $\omega_1 = 20.7\,\text{rad/s}$. So using equation 2.1:

$$\omega_1 = \omega_0 + \alpha t \quad \rightarrow \quad \alpha = \frac{\omega_1 - \omega_0}{t} = \frac{20.7\,\text{rad/s} - 48\,\text{rad/s}}{4440\,\text{s}} = -6.15 \times 10^{-3}\,\text{rad/s}^2$$

2.6.3 Centripetal Acceleration

Question

A test car moves at a constant speed of 10 m/s around a circular road of radius 50 m. For the car, determine:

(a) the centripetal acceleration of the car
(b) the angular speed of the car

Answer

(a) From equation 2.8, the magnitude of the centripetal acceleration of the car is found to be:

$$a_c = \frac{v^2}{r} = \frac{(10\,\text{m/s})^2}{50\,\text{m}} = 2.0\,\text{m/s}^2$$

By definition, the direction of $\mathbf{a}_c$ is towards the centre of the curvature of the road.

(b) The angular speed of the car can be found using equation 2.6 which gives:

$$v_t = r\omega \quad \rightarrow \quad \omega = \frac{v_t}{r} = \frac{10\,\text{m/s}}{50\,\text{m}} = -0.20\,\text{rad/s}$$

2.7 Exercises

2.7.1 Constant Angular Acceleration

1. Find the angular speed of the earth about the Sun in radians per second and degrees per day. *(Ans. 1.99×10^{-7} rad/s; 0.986 deg/day)*
2. A potter's wheel moves from rest to an angular speed of 0.20 rev/s in 30 s. Find its angular acceleration in radians per second per second. *(Ans. 4.2×10^{-2} rad/s^2)*
3. A dentist's drill starts from rest. After 3.20 s of constant angular acceleration it turns at a rate of 2.51×10^4 rev/min.

(a) Find the drill's angular acceleration.
(b) Determine the angle (in radians) through which the drill rotates during this period.

(Ans. (a) 8.21×10^2 rad/s^2 (b) 4.21×10^3 rad)

4. The driver of a car travelling at 30.0 m/s applies the brakes and unergoes a constant negative acceleration of 2.00 m/s^2. How many revolutions does each tyre make before the car comes to a stop, assuming that the car does not skid and that the tyres have radii of 0.300 m? *(Ans. 119 revolutions)*
5. A car has a wheel of effective diameter of 500 mm, and a brake drum with a diameter of 200 mm. If the car is travelling 72 km/hr, calculate the velocity of a point, P, on the edge of the brake drum when
   (a) point P is vertically above the centre of the wheel
   (b) when point P is vertically below the centre of the wheel.
   (Ans. (a) 28 m/s (b) 12 m/s)

2.7.2 Centripetal Acceleration

7. A flywheel of 460 mm diameter rotates at a constant speed of 2500 rev/min. Calculate
   (a) the linear speed speed of a point on the rim of the wheel
   (b) the centripetal acceleration of a point on the rim of the wheel
   (Ans. (a) 60.21 m/s (b) 15764 m/s^2)
8. A car having wheels with a rolling diameter of 500 mm travels at 108 km/h around a horizontal curved track having a radius of 50 m. Calculate
   (a) the angular velocity of the wheels
   (b) the centripetal acceleration of a point on the rim of a wheel
   (c) the centripetal acceleration of the car as a whole.
   (Ans. (a) 120 rad/s (b) 3600 m/s^2 (c) 18 m/s^2)
9. What is the least speed in rev/min at which a bucket of water must be swing around in a vertical circle of radius 800 mm for none of the water to be spilled. *(Ans. 33.44 rev/min)*
10. Calculate the maximum speed in km/h at which a train may travel around a bend of 100 m radius if the maximum centripetal acceleration must not exceed 0.1 g (i.e. 0.1×9.81 m/s^2) in order to maintain passenger comfort. *(Ans. 35.66 km/h)*

3 Scalars and Vectors

3.1 Introduction

Every physical quantity encountered in engineering can be categorised as either a *scalar* or a *vector* quantity.

- **Scalar**: A scalar is a quantity that can be completely specified by its magnitude and appropriate units; a scalar has only a magnitude and no direction.
  Examples: temperature, mass, work, energy, distance travelled, speed, time interval.
- **Vector**: A vector is a quantity that requires both a magnitude and a direction to be fully specified.
  Examples: force, displacement[1], velocity, acceleration.

An example of a vector quantity is force. If you are told that someone is going to exert a force of 10 N on an object, that is not enough information to let you know what will happen to the object. The effect of the force of 10 N exerted horizontally is different from the effect of a force of 10 N exerted vertically upwards. In other words, you need to know the *direction* of the force as well as the magnitude.

Vector quantities are designated in **boldface** in text books and underlined type or with an arrow over the letter in hand-written work. For example $\mathbf{a}$, $\underline{a}$ or $\overrightarrow{a}$ denotes an acceleration vector, with $\mathbf{v}$, $\underline{v}$ or $\overrightarrow{v}$ signifying a velocity vector.

The following symbols are normally used:

$\mathbf{r}$	position
$\boldsymbol{\Delta}\mathbf{r}$	change in position (displacement)
$\mathbf{v}$	velocity
$\mathbf{a}$	acceleration
$\mathbf{F}$	Force

The magnitude of a vector is represented in *italic* type (such as A). Likewise, italic type is used to represent scalars.

3.2 Vector Representation in 2-D

A vector quantity is represented graphically in two-dimensions by an arrow-headed line, as shown in Figure 3.1. The length of the line defines the magnitude, and the arrow

[1] Note the difference between displacement and distance travelled—a body moving from $x = 0\,\text{m}$ to $x = 10\,\text{m}$ and back again in a specific time period travels a *distance* of 20 m. It's *displacement* over the same time period is zero. Likewise, it's speed is distance/time, while velocity is displacement/time.

gives the direction.

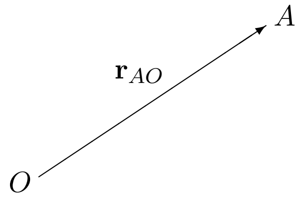


Figure 3.1: A displacement vector

The position of point O is given by the position vector $\mathbf{r}_O$ and the position of point A is given by the position vector $\mathbf{r}_A$. The displacement of A relative to O is given by:

$$\boxed{\mathbf{r}_{AO} = \mathbf{r}_A - \mathbf{r}_O}$$

Equal vectors have the same magnitude and the same direction. They do not need to be applied to the same point. Figure 3.2 demonstrates this: $\mathbf{r}_{AO}$ and $\mathbf{r}_{BP}$ are equal.

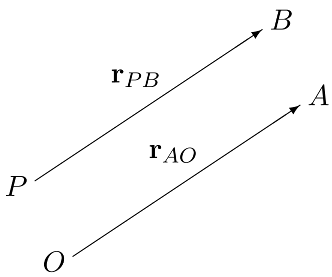


Figure 3.2: A displacement vector

3.3 Vector Addition and Resultant Vector

Vectors of the same units can be added together to obtain a *resultant* vector. For example, consider vectors $\mathbf{A}$ and $\mathbf{B}$, as shown in Figure 3.3. With the origin at O, these vectors can be added together by drawing the second vector at the end point of the first (a — the triangle method of addition) or by placing both vectors starting at the origin (b — the parallelogram method of addition). In Figure 3.3(a), the resultant vector, $\mathbf{R}$, is a vector joining the start point of vector $\mathbf{A}$ to the end point of $\mathbf{B}$. In Figure 3.3(b), the resultant is the diagonal of the parallelogram formed with $\mathbf{A}$ and $\mathbf{B}$ as its sides.

The resultant displacement is written as:

$$\boxed{\mathbf{R} = \mathbf{A} + \mathbf{B}}$$

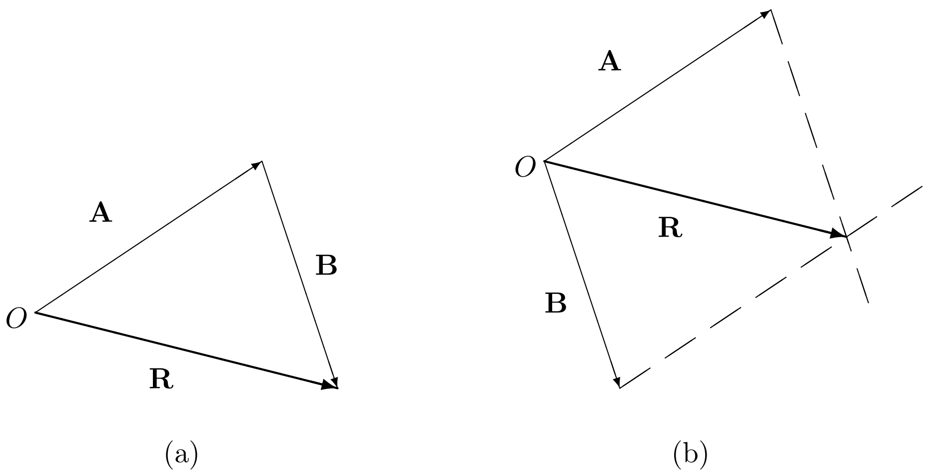


Figure 3.3: Vector addition — graphical method

Note that when two vectors are added, the sum is independent of the order of the addition. So $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Notice that in Figure 3.3(a), if $\mathbf{A}$ and $\mathbf{B}$ are displacement vectors, a particle following these vectors would travel a total *distance* of the magnitude of $\mathbf{A}$ plus the magnitude of $\mathbf{B}$. But its *displacement* is the magnitude of $\mathbf{R}$.

The *negative* of vector $\mathbf{A}$ is defined as the vector when added to $\mathbf{A}$ results in zero for the vector sum. This means that $\mathbf{A}$ and $-\mathbf{A}$ have the same magnitude but opposite directions.

3.4 Restultant of Two Perpendicular Vectors

Consider a vector, $\mathbf{A}$, in a rectangular coordinate system, as shown in Figure 3.4. Note that $\mathbf{A}$ can be expressed as the sum of two vectors, $\mathbf{A}_x$ parallel to the x axis, and $\mathbf{A}_y$ parallel to the y axis. That is:

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$$

$\mathbf{A}_x$ and $\mathbf{A}_y$ are component vectors of $\mathbf{A}$. The projection of $\mathbf{A}$ along the x axis, A_x, is called the x component of $\mathbf{A}$ and the projection of $\mathbf{A}$ along the y axis is called the y component of $\mathbf{A}$. From the definitions of sine and cosine of an angle:

$$\cos\theta = \frac{A_x}{A} \quad \rightarrow \quad A_x = A\cos\theta \tag{3.1}$$

$$\sin\theta = \frac{A_y}{A} \quad \rightarrow \quad A_y = A\sin\theta \tag{3.2}$$

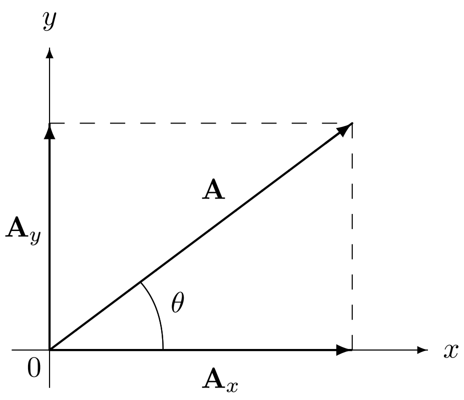


Figure 3.4: A displacement vector

As can be seen in Figure 3.4, these components form two sides of a right triangle, the hypotenuse of which has the magnitude of A.

Thus, from the Pythagorean theorem and the definition of the tangent:

The magnitude of **A**:

$$\boxed{A = \sqrt{A_x^2 + A_y^2}} \tag{3.3}$$

The direction of **A**:

$$\boxed{\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)} \tag{3.4}$$

Note that the angle, θ will be the angle the vector makes with the positive x axis, in the anti-clockwise direction.

To add vectors algebraically, the x and y components of each vector need to be determined (using the same coordinate system). Next, add all the x components together to determine the resultant x component. Similarly, add all the y components together. Since the resultant x and y components are perpendicular, the equations above can be used to determine the magnitude and direction of the resultant vector.

3.5 Examples

Question

Find the horizontal and vertical components of the 100 m diplacement of a superhero who flies down from the top of a tall building at an angle of -36.9° from the horizontal.

Answer

The triangle formed by the displacement and its componetns is shown in Figure 3.5. $A = 100\,\text{m}$ and $\theta = -36.9^\circ$ (negative as it is measured clockwise from the x-axis).

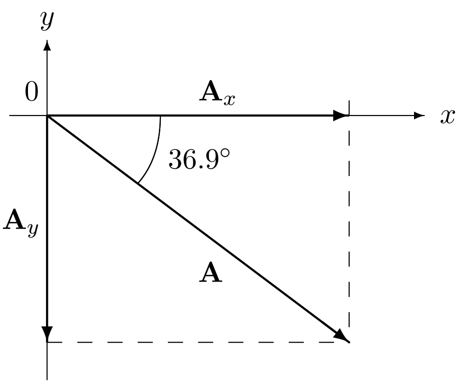


Figure 3.5: The displacement of a superhero

The vertical y component of A is therefore (equation 3.2):

$$A_y = A\sin\theta = (100\,\text{m})\sin(-36.9^\circ) = -60\,\text{m}$$

Note that $\sin(-\theta) = -\sin\theta$. The negative sign for A_y reflects the fact that displacement in the y direction is downward from the origin.

The x component of displacement is (equation 3.1):

$$A_x = A\cos\theta = (100\,\text{m})\cos(-36.9^\circ) = 80\,\text{m}$$

Note that $\cos(-\theta) = \cos\theta$. Also, from inspection of the figure, you should be able to see that A_x is positive in this case.

Question

A hiker begins a trip by walking 25.0 km due southeast from the base camp on the first day. On the second day, the hiker walks 40.0 km in a direction 60.0° north of east.

(a) Determine the components of the hiker's displacements in the first and second days.
(b) Determine the components of the hiker's total displacement for the trip
(c) Determine the magnitude and direction of the total displacement

Answer

It is useful to set the displacement vectors on the first and second days as **A** and **B**, respectively.

(a) Displacement **A** has a magnitude of 25.0 km and is 45.0° south of east. Its components are therefore:

$$A_x = A\cos(-45.0^\circ) = (25.0\,\text{km})(0.707) = 17.7\,\text{km}$$
$$A_y = A\sin(-45.0^\circ) = -(25.0\,\text{km})(0.707) = -17.7\,\text{km}$$

The negative value of A_y indicates that the y coordinate decreased in this displacement.
The second day's displacement, **B**, has a magnitude of 40.0 km and is 60.0° north of east. Its components are:

$$B_x = B\cos 60.0^\circ = (40.0\,\text{km})(0.500) = 20.0\,\text{km}$$
$$B_y = B\sin 60.0^\circ = (40.0\,\text{km})(0.866) = 34.6\,\text{km}$$

(b) The resultant displacement for the trip, $\mathbf{R} = \mathbf{A} + \mathbf{B}$, has components given by:

$$R_x = A_x + B_x = 17.7\,\text{km} + 20.0\,\text{km} = 37.7\,\text{km}$$
$$R_y = A_y + B_y = -17.7\,\text{km} + 34.6\,\text{km} = 16.9\,\text{km}$$

(c) The magnitude and displacement of **R** can be found using the results of part (b):

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(37.7\,\text{km})^2 + (16.9\,\text{km})^2} = 41.3\,\text{km}$$
$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{16.9\,\text{km}}{37.7\,\text{km}}\right) = 24.1^\circ$$

The magnitude is 41.3 km, directed 24.1° north of east from the base camp.

3.6 Exercises

Hint: It helps if you draw a vectory diagram for each of these problems (similar to Figure 3.5).

1. A student walks 4.5 km due north and then 5.8 km due south. Calculate the resultant displacement of the student, that is, how far and in what direction the student is from the starting point. *(Ans. 1.3 km due south)*
2. A man walks 3 km north and then 4 km east. Calculate the resultant displacement of the man, i.e. how far and in what direction he is from his starting point. *(Ans. 5 km at 36.9°)*
3. An aircraft flies 500 km due south and then 600 km due west because the pilot is lost. Calculate the resultant displacement of the aircraft. *(Ans. 781 km,* 219.8° *anti-clockwise from due east (39.8° south of west))*
4. I walk 800 m in a direction 140° and then 600 m in a direction 50°. Calculate my resultant displacement. *(Ans. 946.9 m, 76.9° north of west (or 103.1° anti-clockwise from due east))*

5. A man in a daze in a maze walks 800 m west, then 700 m south, followed by 500 m east, and then 600 m north. Calculate the resultant displacement of the man. *(Ans. 316.2 m, 18.4° south of west (or 198.4° anti-clockwise from east))*
6. An assembly robot moves 1 m in a direction 30°, followed by 800 mm in a direction 120°, and then 1.6 m in a direction 210°. Calculate the resultant displacement of the robot. *(Ans. 1 m, 23.1° north of west (or 156.9° anti-clockwise from east))*

Note: if you are having difficulty trying to suss out the angles for the vector directions, Figure 5.4 in Section 5 might help you.

4 Projectiles in 2-D Motion

4.1 Introduction

In Section 1 of these revision notes, objects moved in one-dimension along straight line paths, such as along the x axis. Projectile motion involves objects that move in a plane, as in the object can move in both the x and y directions resulting in two dimensional motion. Techniques covered in Section 3 involving vector manipulation can be used here to solve problems associated with projectiles. Anyone who has observed a ball in motion, an arrow fired from a bow (or, for that matter, any object thrown into the air) has observed projectile motion.

Three assumptions are made to help analyse projectiles:

- The free-fall acceleration vector, $\mathbf{g}$, has a magnitude of $9.81\,\mathrm{m/s^2}$, and has a direction in the negative y direction (downwards) ($\theta = 270°$).
- The effect of air resistance is negligible
- The rotation of the Earth does not affect the motion.

It is helpful to use a standard cartesian coordinate system where the x axis is horizontal, and positive moving to the right, and the y axis is vertical, with positive motion moving upwards as shown in Figure 4.1.

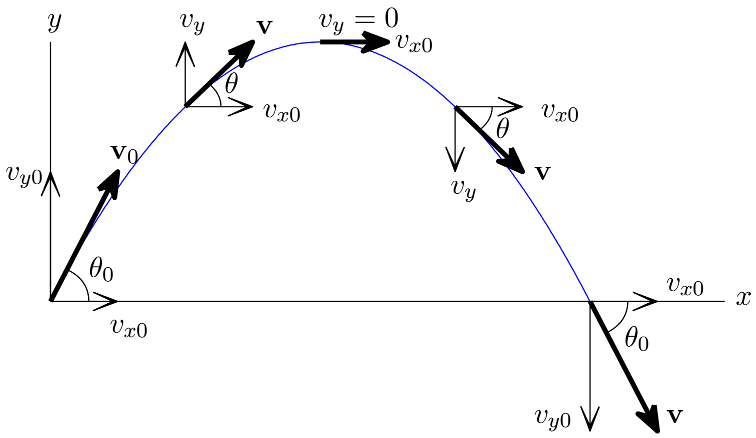


Figure 4.1: Parabolic trajectory of a particle that leaves the origin with a velocity of v_0

Using the assumptions and this coordinate system, we can say that:

- The acceleration in the y direction is $-g$ (note the *italic* type, signifying the magnitude of $\mathbf{g}$), just as in free-fall.
- The acceleration in the x direction is zero (because air resistance is neglected).

The path a projectile takes is known as its *trajectory*, and this curved path is a parabola, as shown in Figure 4.1. Note the velocity, $\mathbf{v}$, changes with time. However, the x component of velocity, v_x, remains constant with respect to time (neglecting air resistance). Also, $v_y = 0$ at the peak.

4.2 Velocity Components

Referring to Figure 4.1, a particle is launched with an initial velocity, $\mathbf{v}_0$ at angle θ_0 to the horizontal. The initial horizontal component of the velocity is:

$$\boxed{v_{x0} = v_0 \cos\theta_0} \tag{4.1}$$

Similarly, the initial vertical component of the velocity is:

$$\boxed{v_{y0} = v_0 \sin\theta_0} \tag{4.2}$$

To analyse projectile motion, it is important to seperate the motion into the two components, x (horizontal) and y (vertical) motion, and solve each part separately. It makes sense to deal with the vertical motion first.

4.3 Vertical Motion (Height)

Acceleration is experienced in the vertical direction and has a value of $-g$ (assuming upwards is positive). The initial velocity is v_{y0}. So, using the equations developed in Section 1, an equation relating velocity as a function of time can be determined:

$$\boxed{v_y = v_{y0} - gt} \tag{4.3}$$

Similarly, for the vertical height as a function of time:

$$\boxed{y = v_{y0}t - \frac{1}{2}gt^2} \tag{4.4}$$

From Figure 4.1, it can be seen that the vertical height, y, is zero at the initial point of the particle (when time, $t = 0$), but also when the particle hits the ground. Factorising the above equation:

$$y = t(v_{y0} - \frac{1}{2}gt) \quad \rightarrow \quad y = 0 \text{ when } \begin{cases} t = 0 \\ v_{y0} - \frac{1}{2}gt = 0 \end{cases}$$

So, by rearranging the non-zero solution, the total time for the motion:

$$\boxed{t_{\text{total}} = \frac{2v_{y0}}{g}} \tag{4.5}$$

Note: this equation is only correct when the trajectory is across a horizontal plane (launch and landing are at the same height).

From Figure 4.1, it can be seen that the vertical velocity, $v_y = 0$ at the peak of the trajectory, i.e. the point of maximum height. By substituting $v_y = 0$ into the equation relating velocity and time (equation 4.4), the time taken for the motion in the upwards direction can be determined:

$$v_y = v_{y0} - gt \quad \rightarrow \quad 0 = v_{y0} - gt$$

$$\boxed{t_{\text{up}} = \frac{v_{y0}}{g}} \tag{4.6}$$

Notice that this is half of the total time. This is always true when the trajectory is across a horizontal plane (launch and landing are at the same height).

Now that t_{up} is known, substituting this in the equation for height the maximum height, y_{max} can be determined:

$$y = v_{y0}t - \frac{1}{2}gt^2 \quad \rightarrow \quad y_{\text{max}} = v_{y0}\frac{v_{y0}}{g} - \frac{1}{2}g\left(\frac{v_{y0}}{g}\right)^2$$

$$y_{\text{max}} = \frac{v_{y0}^2}{g} - \frac{v_{y0}^2}{2g}$$

$$\boxed{y_{\text{max}} = \frac{v_{y0}^2}{2g}} \tag{4.7}$$

4.4 Horizontal Motion (Range)

Because there is no acceleration in the x direction (i.e. $a_x = 0$), the velocity in the x direction *remains constant*. Thus, the velocity in the x direction is always equal to the initial horizontal direction:

$$v_x = v_{x0} = v_0 \cos\theta_0$$

Since we know that distance equals velocity times time, we can determine the horizontal position as a function of time:

$$\boxed{x = v_{x0}t = (v_0 \cos\theta_0)t} \tag{4.8}$$

By substituting the value found for t_{total} found when examining the vertical motion, into this equation, the total distance (the range) can be determined (for a trajectory across a horizontal plane):

$$\boxed{x_{\text{max}} = v_{x0}\frac{2v_{y0}}{g}} \tag{4.9}$$

4.5 Maximum Range

Using the equations derived above, the angle required to maximise the range across a horizontal plane can be determined.

Combining these equations:

$$\left.\begin{aligned} v_{y0} &= v_0 \sin\theta_0 \\ v_{x0} &= v_0 \cos\theta_0 \\ \\ x_{\text{max}} &= v_{x0}\frac{2v_{y0}}{g} \end{aligned}\right\} \quad \rightarrow \quad x_{\text{max}} = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g}$$

Using the relationship: $2\sin\theta\cos\theta = \sin 2\theta$ gives:

$$\boxed{x_{\text{max}} = \frac{v_0^2 \sin 2\theta_0}{g}} \tag{4.10}$$

The sine function has a maximum of one, which occurs when the angle is $\frac{\pi}{2}$ or 90°. Therefore, x_{max} occurs is a maximum when $\sin 2\theta_0 = 1$, which occurs when $2\theta_0 = 90°$. Hence, the maximum range occurs when $\theta_0 = 45°$.

4.6 Trajectory Across a Non-Horizontal Plane

The equations derived above for t_{total} and x_{max} are only valid when the launch point and the landing point have the same vertical position. If these values are different, for example when firing an arrow over a cliff (see example) the analysis will need the following techniques to be completed separately from each other:

1. Consider the vertical motion in the upwards direction to calculate the t_{up}.
2. Consider the vertical motion in the downwards direction to calculate the t_{down}. These are the same calculations as used in freefall problems (See section 1.6.2 of these revision notes).
3. Use the value of $t_{\text{up}} + t_{\text{down}}$ to calculate t_{total} and therefore the range with equation 4.8 the range.

Alternatively, you can use equation 4.4 and by entering the final height and solving for t (using the quadratic equation), a value for t_{total} can be directly determined (see the third example).

Any information required to analyse the upwards and downwards vertical motion separately will be provided.

4.7 Example

Question

A rescue plane drops a package of emergency rations to a stranded party of explorers. The plane is travelling horizontally at 40.0 m/s at a height of 100 m above the ground.

(a) Where does the package strike the ground relative to the point at which it was released?

(b) What are the horizontal and vertical components of the velocity of the package just before it hits the ground?

Answer

(a) It makes sense to set the coordinate system for this problem with the positive x direction to the right and the positive y direction upwards. The plan is travelling to the right at 40.0 m/s at a height of 100 m above the ground (at $y = 0\,\mathrm{m}$). Consider first the horizontal motion of the package. The only equation available is equation 4.8:

$$x = v_{x0}t$$

The initial x component of the package velocity is the same as the velocity of the plan when the package was released, 40.0 m/s. Thus we have:

$$x = (40.0\,\mathrm{m/s})t$$

It is clear that if t is known, x can be determined. To find t, the equations associated with objects in freefall can be used. The instant the package hits the ground, y has a value of $-100\,\mathrm{m}$, and because the initial velocity of the package in the vertical direction is zero, t can be determined from equation 4.4:

$$\begin{aligned} y &= -\frac{1}{2}gt^2 \\ -100\,\mathrm{m} &= -\frac{1}{2}(9.81\,\mathrm{m/s^2})t^2 \\ t^2 &= 20.4\,\mathrm{s^2} \\ t &= 4.51\,\mathrm{s} \end{aligned}$$

To determine x, the value found is substituted into equation 4.8:

$$x = (40.0\,\mathrm{m/s})(4.51\,\mathrm{s}) = 180\,\mathrm{m}$$

The package hits the ground 180 m (in the positive x direction) from the point from where it was released.

(b) The horizontal component of the velocity is already known, since this remains constant at 40.0 m/s throughout its fall.

The vertical component of the velocity just before the package hits the gorund may be found, again, using the equation associated with free fall. In this case, knowing that the initial vertical velocity is zero, equation 4.3 can be used:

$$v_y = v_{y0} - gt = 0 - (-9.81\,\text{m/s}^2)(4.51\,\text{s}) = -44.1\,\text{m/s}$$

The components of velocity are 40.0 m/s in the horizontal direction, and −44.1 m/s in the vertical direction.

Question

A long jumper leaves the ground at an angle of 20.0° to the horizontal at a speed of 11.0 m/s.

(a) How far does he jump? (Assume that the motion of the long jumper is equivalent to that of a particle.)

(b) What is the maximum height reached?

Answer

(a) The long jumper's horizontal motion is described by using equation 4.8:

$$x = (v_0 \cos\theta_0)t = (11.0\,\text{m/s})(\cos 20.0^\circ)t$$

The value of x can be found if t_{total}, the total duration of the jump, is known. Time, t_{total} can be found with equations 4.5.

$$\begin{aligned} t_{\text{total}} &= \frac{2v_{y0}}{g} = \frac{2v_0 \sin\theta_0}{g} = \frac{2(11.0\,\text{m/s})(\sin 20.0^\circ)}{9.81\,\text{m/s}^2} \\ t_{\text{total}} &= 0.767\,\text{s} \end{aligned}$$

Substituting this into equation 4.8:

$$x = (11.0\,\text{m/s})(\cos 20.0^\circ)(0.767\,\text{s}) = 7.93\,\text{m}$$

Note: equations 4.9 or 4.10 could have also been used directly to find this value.

(b) The maximum height can be found using equation 4.7

$$\begin{aligned} y_{\text{max}} &= \frac{v_{y0}^2}{2g} = \frac{(v_0 \sin\theta_0)^2}{2g} = \frac{[(11.0\,\text{m/s})\sin 20.0^\circ]^2}{2(9.81\,\text{m/s}^2)} \\ y_{\text{max}} &= 0.721\,\text{m} \end{aligned}$$

Question

A stone is thrown upward from the top of a building at an angle of 30.0° to the horizontal and with an initial speed of 20.0 m/s. The height of the building is 45.0 m.

(a) How long is the stone in flight?

(b) What is the speed of the stone just before it strikes the ground?

(c) Where does the stone strike the ground?

Answer

(a) Setting the origin of the coordinates at the top of the building, the ground is at $y = -45\,\text{m}$. Knowing this, the total time can be found using the equation 4.4. Note that the equation derived in the text for t_{total} only works when the trajectory is across a horizontal plane (the launch and the landing are at the same level). So, from equation 4.4:

$$y = v_{y0}t - \frac{1}{2}gt^2 \quad \rightarrow \quad 0 = \frac{1}{2}gt^2 - v_0 \sin\theta_0 t + y$$
$$0 = \frac{1}{2}(9.81\,\text{m/s})t^2 - [(20.0\,\text{m/s})\sin 30°]t - 45.0\,\text{m}$$
$$0 = 4.905t^2 - 10t - 45$$

This is a quadratic equation, where $a = 4.905$, $b = -10$ and $c = -45$ so the values for t are:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{(-10)^2 - 4(4.905)(-45)}}{2(4.905)} = \begin{cases} 4.22\,\text{s} \\ -2.18\,\text{s} \end{cases}$$

Logic dictates that the total time cannot be before the stone has been launched, so the total time for the flight is 4.22 s.

(b) The y component of the velocity just before the stone strikes the ground can be obtained using equation 4.3 with $t = 4.22\,\text{s}$:

$$v_y = v_{y0} - gt = v_0 \sin\theta_0 - gt = (20.0\,\text{m/s})(\sin 30°) - (9.81\,\text{m/s}^2)(4.22\,\text{s})$$
$$v_y = -31.4\,\text{m/s}$$

The x component of the initial velocity remains constant throughout the flight, therefore a value can be found for v_x as it hits the ground:

$$v_x = v_{x0} = v_0 \cos\theta_0 = (20.0\,\text{m/s})(\cos 30.0°) = 17.3\,\text{m/s}.$$

Using the Pythagorean theorem, the total velocity can be found:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{17.3^2 + (-31.4)^2}\,\text{m/s} = 35.9\,\text{m/s}$$

(c) The range of the the throw can be determined by using equation 4.8 when $t = 4.22\,\text{s}$:

$$x = v_{x0}t = (v_0 \cos\theta_0)t = (20.0\,\text{m/s})(\cos 30°)(4.22\,\text{s}) = 73.1\,\text{m}$$

4.8 Exercises

1. A student stands on the top floor of the Eiffel tower, and throws a coin horizontally over the the edge of the balcony at a speed of 18 m/s. The height of the top floor of the Eiffel tower is 163 m.

(a) How long after being released does the coin hit the ground?
(b) What is the coin's speed and angle of impact with the ground?

Ans. (a) 5.76 s (b) 59.3 m/s and 72.3° with respect to the ground (from the ground between the base of the Eiffel tower to the impact site)

2. A student practicing some tennis serves accidentally hits the ball so that it leaves the racket horizontally at a speed of 36 m/s at a height of 2.5 m above the ground. If the student is standing at the baseline, how far beyond the service line (on the other side of the court, 18.29 m away) does the ball land? *(Ans. 7.41 m beyond the service line)*
3. A ball is thrown straight upward and returns to the thrower's hand after 3.00 s in the air. A second ball is thrown at an angle of 30.0° with the horizontal. At what speed must the second ball be thrown so that it reaches the same height as the one thrown vertically? *(Ans. 29.4 m/s)*
4. A golfball with an initial sped of 50 m/s lands exactly 240 m downrange on a level course.

   (a) Neglecting air friction, what *two* projection angles would achieve this result?
   (b) What is the maximum height achieved by the ball, using the two angles determined in (a)?

   Ans. (a) 35.1° or 54.9° (b) 42 m or 85 m respectively

5 Additional Notes

5.1 Introduction

This section is designed as a reference chapter. There are no exercises to complete in this section. The content is here to aid your understanding of the preceding sections, as well as the content of the module. The topics included here cover some basics of trigonometry, differentiation and integration, and some tips to do with unit conversion.

5.2 Trigonometry

5.2.1 Radians

A radian is defined as the angle, θ between two radial lines that create an arc length, s equal to the radius of the circle, r, as shown in Figure 5.1.

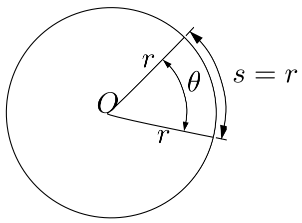


Figure 5.1: Definition of a radian

The circumference of a circle is defined as $2\pi r$. If this distance is divided by 1 radian, r, the total number of radians in one circle is:

$$\frac{2\pi r}{r} = 2\pi$$

One complete circle is 360°, so it can be said that:

$$360^\circ = 2\pi$$

To convert between degrees and radians, use the following conversion factors:

$$\text{degrees to radians} \quad \rightarrow \quad \text{multiply by: } \frac{\pi\,\text{rad}}{180\,\text{deg}}$$

$$\text{radians to degrees} \quad \rightarrow \quad \text{multiply by: } \frac{180\,\text{deg}}{\pi\,\text{rad}}$$

For example:

$$45^\circ = (45\,\cancel{\text{deg}})\left(\frac{\pi\,\text{rad}}{180\,\cancel{\text{deg}}}\right) = \frac{\pi}{4}\,\text{radians}$$

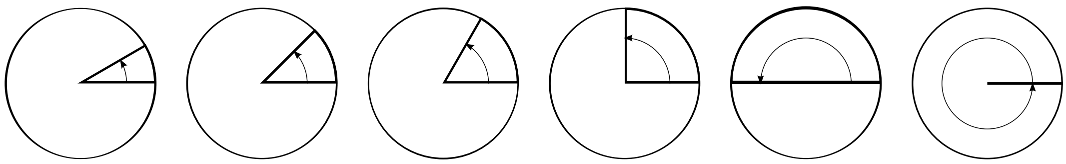
and

$$-\frac{\pi}{2}\,\text{radians} = \left(-\frac{\pi}{2}\,\cancel{\text{rad}}\right)\left(\frac{180\,\text{deg}}{\pi\,\cancel{\text{rad}}}\right) = -90^\circ$$

Remember:

- Radians have no units, so radians are assumed unless an angle is signified by the degree symbol °.
- To convert between radians and number of revolutions, divide the quantity by 2π.

It is useful to remember the conversion of common angles as shown in Figure 5.2. Use the conversion ratio for other angles.



$30^\circ = \frac{\pi}{6}$ $\quad 45^\circ = \frac{\pi}{4}$ $\quad 60^\circ = \frac{\pi}{3}$ $\quad 90^\circ = \frac{\pi}{2}$ $\quad 180^\circ = \pi$ $\quad 360^\circ = 2\pi$

Figure 5.2: Radian and degree measure for several common angles

5.2.2 The Trigonometric Functions

A common approach to the study of trigonometry is to define the trigonometric functions as ratios of two sides of a right angle triangle, as shown in Figure 5.3.

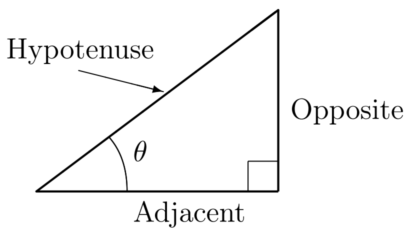


Figure 5.3: Sides of a right triangle

Six functions are defined (Sine, Cosine, Tangent, Cosecant, Secant and Cotangent, abbreviated a sin, cos, etc.) in Table 5.1. These are correct for right angle trianglges where:

$$0 < \theta < \frac{\pi}{2}$$

$\sin\theta = \dfrac{\text{opp.}}{\text{hyp.}}$	$\cos\theta = \dfrac{\text{adj.}}{\text{hyp.}}$	$\tan\theta = \dfrac{\text{opp.}}{\text{adj.}}$
$\csc\theta = \dfrac{\text{hyp.}}{\text{opp.}}$	$\sec\theta = \dfrac{\text{hyp.}}{\text{adj.}}$	$\cot\theta = \dfrac{\text{adj.}}{\text{opp.}}$

Table 5.1: Definition of the Six Trigonometric Functions

It is sometimes helpful to remember sine, cosine and tangent as SOH-CAH-TOA (Sine-Opposite-Hypotenuse, Cosine-Adjacent-Hypotenuse, Tanget-Opposite-Adjacent).

The degree and radian measures of common angles (less than $\frac{\pi}{2}$ or 90°) are given in Table 5.2, along with corresponding values of the sine, cosine and tangent.

Degrees	0°	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{3}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

Table 5.2: Definition of the Six Trigonometric Functions

For angles greater than $\frac{\pi}{2}$, a right angle triangle can be formed, and the same rules apply, however the signs of the resultant values may be different. The quadrant signs of the sine, cosine and tangent functions are shown in Figure 5.4.

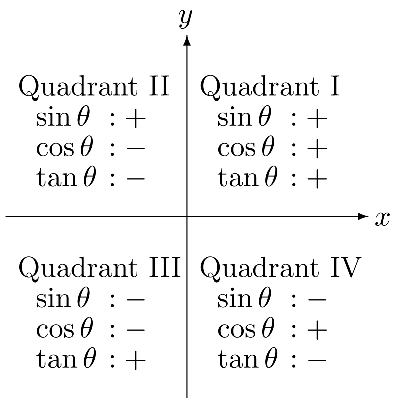


Figure 5.4: Sides of a right triangle

5.2.3 The Trigonometric Identities

The trigonometric identities, as shown in Table 5.3 are direct consequences of the definitions. (Note: ϕ also represents an angle, and $\sin^2\theta = (\sin\theta)^2$)

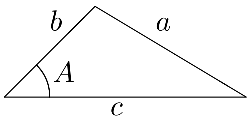
Pythagorean identities	**Reduction formulae**	
$\sin^2\theta + \cos^2\theta = 1$	$\sin(-\theta) = -\sin\theta$	$\sin\theta = -\sin(\theta - \pi)$
$\tan^2\theta + 1 = \sec^2\theta$	$\cos(-\theta) = \cos\theta$	$\cos\theta = -\cos(\theta - \pi)$
$\cot^2\theta + 1 = \csc^2\theta$	$\tan(-\theta) = -\tan\theta$	$\tan\theta = \tan(\theta - \pi)$
Sum/difference of two angles	**Half-angles**	**Double angles**
$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$	$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$	$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\cos\theta$	$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$	$\cos 2\theta = 2\cos^2\theta - 1$
$\tan(\theta \pm \phi) = \dfrac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$		
Law of cosines	**Reciprocal identities**	**Quotient Identities**
$a^2 = b^2 + c^2 - 2bc\cos A$	$\csc\theta = \dfrac{1}{\sin\theta}$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
	$\cot\theta = \dfrac{1}{\tan\theta}$	

Table 5.3: Trigonometric Identities

5.3 Derivatives and Integrals

The following two sections identify common differentiation rules and integration formulae. These tables should are not exhaustive (although should suffice for this module), but a useful resource for solving (symbolically) tough differentiation and integration formulas is Wolfram|Alpha (`www.wolframalpha.com`), which is a bit like Google for maths. For example, to differente $f(x) = x^2 + 3$, go to Wolfram|Alpha and enter '`diff(x^2 + 3)`'. The result comes out as $2x$. For integration of the same function, enter '`integral(x^2 + 3)`' which comes out to be $\frac{x^3}{3} + 3x + C$. This website can also show the steps required if you need further explanation.

5.3.1 Basic Differentiation Rules

Table 5.4 contains basic differentiation rules. Two notations are used to denote the derivative of a function with respect to x:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] = f'(x)$$

Constant rule:	$\frac{\mathrm{d}}{\mathrm{d}x}[c]$	$= 0$
Power rule ($n \neq 0, 1$):	$\frac{\mathrm{d}}{\mathrm{d}x}[x^n]$	$= nx^{n-1}$
Power rule ($n = 1$):	$\frac{\mathrm{d}}{\mathrm{d}x}[x]$	$= 1$
Constant multiple rule:	$\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)]$	$= c\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]$
Sum and Difference rule:	$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)]$	$= f'(x) \pm g'(x)$
Sine function:	$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x]$	$= \cos x$
Cosine function:	$\frac{\mathrm{d}}{\mathrm{d}x}[\cos x]$	$= -\sin x$
Logarithmic rule:	$\frac{\mathrm{d}}{\mathrm{d}x}[\ln x]$	$= \frac{1}{x}$
Exponential rule:	$\frac{\mathrm{d}}{\mathrm{d}x}[e^x]$	$= e^x$
Product rule:	$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)]$	$= f(x)g'(x) + g(x)f'(x)$
Quotient rule:	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right]$	$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
Chain rule:	$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))]$	$= f'(g(x))g'(x)$

Table 5.4: Basic differentiation rules

5.3.2 Integration Table

Note that for the following, C is the constant of integration. The formulae in Table 5.5 apply for integration.

$$
\begin{aligned}
\int 0 \,\mathrm{d}x &= C \\
\int k \,\mathrm{d}x &= kx + C \\
\int kf(x) \,\mathrm{d}x &= k \int f(x) \,\mathrm{d}x \\
\int [f(x) \pm g(x)] \,\mathrm{d}x &= \int f(x) \,\mathrm{d}x \pm \int g(x) \,\mathrm{d}x \\
\int x^n \,\mathrm{d}x &= \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \\
\int \sin x \,\mathrm{d}x &= -\cos x + C \\
\int \cos x \,\mathrm{d}x &= \sin x + C \\
\int \frac{1}{x} \,\mathrm{d}x &= \ln x + C \\
\int e^x \,\mathrm{d}x &= e^x + C
\end{aligned}
$$

Table 5.5: Basic integration rules

This list is in now way exhaustive, as there are many other integrals of functions that have specific forms as well as methods to solve (for example, integration by parts). The basics shown here will be generally sufficient for Level 1 Dynamics.

5.4 Unit Conversion

In many engineering problems, the units that in a given problem are not necessarily SI units (metres, seconds, kilograms). It is often necessary to convert the values to the correct units. For example, if you have a speed in miles per hour, a time in minutes, and wish to calculate the acceleration in $\mathrm{m/s^2}$, it will be necessary to convert the speed in to m/s and time to seconds for the equations (derived in section 1) to work correctly.

The simplest way to perform these conversions is to multiply the values by a conversion factor so that the units cancel. For example, knowing that there are 1000 in 1 kilometre, to convert 45 kilometres to metres, you should do the following:

$$45\,\cancel{\text{kilometres}} \times \frac{1000\,\mathrm{m}}{1\,\cancel{\text{kilometres}}} = 45000\,\text{metres}$$

The units we want are the numerator of the conversion factor, with the units we want to cancel are on the denominator.

When the units of a value are a quotient of non-SI units (for example, miles per hour), then two conversions need to take place: one to convert the units in the numerator and one to convert the units in the denominator. For example, 60 miles per hour becomes:

$$60\,\text{miles/hour} = 60\,\frac{\cancel{\text{miles}}}{\cancel{\text{hour}}} \times \frac{1609.33\,\text{metres}}{1\,\cancel{\text{miles}}} \times \frac{1\,\cancel{\text{hour}}}{3600\,\text{seconds}} = 26.82\,\text{m/s}$$

As can be seen, the conversion factor for the units in the denominator (hours, in this case) is such a that the numerator is the units we wish to cancel, and the units in the denominator are those we require (seconds in this case).

Table 5.6 provide some conversion factors for the Si units most required in dynamics (length, time, force).

Length (metre)	Time (second)	Force (Newton)
1 inch = 0.0254 m	1 minute = 60 s	1 kgf = 9.81 N
1 foot = 0.3048 m	1 hour = 3600 s	1 lbf = 4.448 N
1 yard = 0.9144 m	1 day = 86400 s	
1 km = 1000 m	1 year = 3.1577×10^7 s	
1 mile = 1609.33 m		

Table 5.6: Common conversion factors to the SI units for length, time and force

For example, to convert the imperial value of gravity of $32.2\,\text{ft/s}^2$ to the SI unit version, we do the following:

$$32.2\,\text{ft/s}^2 = 32.2\,\frac{\cancel{\text{feet}}}{\text{seconds}^2} \times \frac{0.3048\,\text{metres}}{1\,\cancel{\text{feet}}} = 9.81\,\text{m/s}^2$$

Another example: a conveyor belt moving at 24 inches per minute:

$$24\,\text{in/min} = 24\,\frac{\cancel{\text{inches}}}{\cancel{\text{minute}}} \times \frac{0.0254\,\text{metres}}{1\,\cancel{\text{inches}}} \times \frac{1\,\cancel{\text{minute}}}{60\,\text{seconds}} = 0.0102\,\text{m/s}$$

If one of the units is raised to a power, then the conversion needs to be done twice (or raised to a power of two). For example, converting 29 pounds per square inch (psi) to the SI units of pressure, N/m^2 (or Pascals):

$$29\,\text{psi} = 29\,\frac{\cancel{\text{lbf}}}{\cancel{\text{inches}^2}} \times \frac{4.448\,\text{Newtons}}{1\,\cancel{\text{lbf}}} \times \left(\frac{1\,\cancel{\text{inches}}}{0.0254\,\text{metres}}\right)^2 = 2 \times 10^5\,\text{N/m}^2$$