Dynamics Summary

1 One-Dimensional Motion

1.1 Definitions

t = Time (s)
s = Distance traveled (m)
v = Velocity (m/s)
a = Acceleration (m/s^2)
F_{res} = Resultant force (N)
x_0 = The value of any variable x at $t = 0$

1.2 Basic Equations

These are the basic equations which should be known by heart:

$$\mathbf{v} = \frac{d\mathbf{s}}{dt} \tag{1.1}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{s}}{dt^2} \tag{1.2}$$

$$\mathbf{a}\, d\mathbf{s} = \mathbf{v}\, d\mathbf{v} \tag{1.3}$$

$$\mathbf{F_{res}} = m\mathbf{a} \tag{1.4}$$

1.3 t as independent variable

If t is an independent variable, the following integrals can be derived from the basic equations:

$$v(t) = \int_{t_0}^{t} a(\bar{t})\, d\bar{t} + v_0 \tag{1.5}$$

$$s(t) = \int_{t_0}^{t} v(\bar{t})\, d\bar{t} + s_0 \tag{1.6}$$

1.4 s as independent variable

If s is an independent variable, the following integrals can be derived from the basic equations:

$$v(s) = \sqrt{2 \cdot \int_{s_0}^{s} a(\bar{s})\, d\bar{s} + v_0^2} \tag{1.7}$$

$$t(s) = \int_{s_0}^{s} \frac{1}{v(\bar{s})} d\bar{s} + t_0 \tag{1.8}$$

1.5 v as independent variable

If v is an independent variable, the following integrals can be derived from the basic equations:

$$s(v) = \int_{v_0}^{v} \frac{v}{a(\bar{v})} d\bar{v} + s_0 \tag{1.9}$$

$$t(v) = \int_{v_0}^{v} \frac{1}{a(\bar{v})} d\bar{v} + t_0 \tag{1.10}$$

2 Circular Motion

2.1 Definitions

r = Radius - Distance from origin (s)
θ = Counterclockwise angle from x-axis (rad)
$\mathbf{e_r}$ = Unit vector in radial direction
$\mathbf{e_\theta}$ = Unit vector in tangential direction
v = Velocity (m/s)
a = Acceleration (m/s^2)
$\dot{x}$ = Time-derivative of any variable x

2.2 Unit Vector Derivatives

The unit vectors change as follows due to a change in r or θ:

$$\frac{d\mathbf{e_r}}{d\theta} = \mathbf{e_\theta} \qquad \frac{d\mathbf{e_r}}{dr} = 0 \tag{2.1}$$

$$\frac{d\mathbf{e_\theta}}{d\theta} = -\mathbf{e_r} \qquad \frac{d\mathbf{e_\theta}}{dr} = 0 \tag{2.2}$$

2.3 Polar Coordinate Equations

The following equations are the basic equations for polar coordinates and should also be known by heart:

$$\mathbf{r} = r\mathbf{e_r} \tag{2.3}$$

$$\mathbf{v} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e_\theta} \tag{2.4}$$

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e_r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\mathbf{e_\theta} \tag{2.5}$$

3 Motion Equations

3.1 Definitions

f_x = Force in x-direction (N)
f_y = Force in y-direction (N)
f_r = Force in radial direction (N)
f_θ = Force in angular direction (N)
f_t = Force in tangential direction (N)
f_n = Force in normal direction (N)
ρ = Radius of curvature (m)

3.2 General / Linear Motion

For general motion, the following equations often come in handy:

$$f_x = m\ddot{x} \tag{3.1}$$

$$f_y = m\ddot{y} \tag{3.2}$$

3.3 (Near-)Circular Motion

For (near-)circular motion, the following equations can often be easily solved:

$$f_r = m(\ddot{r} - r\dot{\theta}^2) \tag{3.3}$$

$$f_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) \tag{3.4}$$

3.4 Path Known in Advance

If the path a particle travels is known in advance, the following equations often provide a solution:

$$f_t = m\frac{dv}{dt} \tag{3.5}$$

$$f_n = m\frac{v^2}{\rho} \tag{3.6}$$

3.5 Relative Velocity / Acceleration

If $\mathbf{v_{A/C}}$ is the velocity of particle A with respect to any particle C, $\mathbf{v_{B/C}}$ is the velocity of particle B with respect to the same particle C and $\mathbf{v_{A/B}}$ is the velocity of particle A with respect to particle B, then the following equation applies:

$$\mathbf{v_{A/B}} = \mathbf{v_{A/C}} - \mathbf{v_{B/C}} = -\mathbf{v_{B/A}} \tag{3.7}$$

Identically for the acceleration:

$$\mathbf{a_{A/B}} = \mathbf{a_{A/C}} - \mathbf{a_{B/C}} = -\mathbf{a_{B/A}} \tag{3.8}$$

4 Friction

4.1 Definitions

N = Normal force (N)
F_w = Friction force (N)
μ_k = Coefficient of kinematic friction
μ_s = Coefficient of static friction

4.2 A Moving Particle

If a particle is moving on a surface and is acted on by a normal force N, then the friction force is directed opposite to the motion, and has magnitude:

$$F_w = \mu_k N \tag{4.1}$$

4.3 A Particle About To Move

If a particle is standing still on a surface, then the magnitude of the frictional force satisfies the following equation:

$$F_w \leq \mu_s N \tag{4.2}$$

Where the force is directed in such a way that the resultant force is zero. Equality holds if the particle is about to move.

5 Energy

5.1 Definitions

T = Kinetic energy (J)
U = Work done by a force (J)
V^g = Potential gravitational energy (J)
$\mathbf{F_s}$ = Spring force (N)
k = Spring constant (N/m)
$\mathbf{r_0}$ = Position at which spring is not stretched (m)
V^e = Potential spring energy (J)
$U'_{1,2}$ = Work by external forces between 1 and 2

5.2 Basic Energy Equations

The following equations apply:

$$U = \int \mathbf{F} \cdot d\mathbf{s} \tag{5.1}$$

$$T = \frac{1}{2}mv^2 \tag{5.2}$$

$$V^g = mgh \tag{5.3}$$

5.3 Springs

The force caused by a spring is:

$$\mathbf{F_s} = -k(\mathbf{r} - \mathbf{r_0}) \tag{5.4}$$

The energy of a spring in a position $\mathbf{r}$ is:

$$V^e = \frac{1}{2}k(\mathbf{r} - \mathbf{r_0})^2 \tag{5.5}$$

5.4 Energy Equation

The total work done by all other external forces (which is often 0) is equal to the change in energy:

$$U'_{1,2} = \Delta T + \Delta V^g + \Delta V^e \tag{5.6}$$

6 Impulse and Momentum

6.1 Definitions

$\mathbf{G}$ = Linear Momentum (Ns)
$\mathbf{H}$ = Angular Momentum (Nms)
$\mathbf{M_0}$ = Moment about a point (Nm)

6.2 Linear Momentum

The linear momentum is defined as:

$$\mathbf{G} = m\mathbf{v} \tag{6.1}$$

The linear impulse (which doesn't have its own symbol) is:

$$\dot{\mathbf{G}} = m\dot{\mathbf{v}} = m\mathbf{a} = \Sigma\mathbf{f} \tag{6.2}$$

Note that linear momentum (in a certain direction) is conserved if there are no external forces acting on the system (in that direction).

6.3 Angular Momentum

The angular momentum about a point is defined as:

$$\mathbf{H} = m\mathbf{r} \times \mathbf{v} \tag{6.3}$$

The angular impulse (which doesn't have its own symbol) is:

$$\dot{\mathbf{H}} = m\mathbf{r} \times \dot{\mathbf{v}} = \mathbf{r} \times (m\mathbf{a}) = \mathbf{r} \times \Sigma\mathbf{f} = \Sigma\mathbf{M_0} \tag{6.4}$$

Note that angular momentum is conserved if there are no external moments acting on the system.

7 Linear Collisions

7.1 Definitions

e = Coefficient of restitution (dimensionless)
v_1 = Initial velocity of particle 1 (m/s)
v_2 = Initial velocity of particle 2 (m/s)
v_1' = Final velocity of particle 1 (m/s)
v_2' = Final velocity of particle 2 (m/s)

7.2 Coefficient of Restitution

The coefficient of restitution is defined as:

$$e = \frac{\text{Restitution impulse}}{\text{Deformation impulse}} \tag{7.1}$$

From this can be derived that:

$$e = -\frac{v1' - v2'}{v1 - v2} \tag{7.2}$$

The coefficient of restitution is usually between 0 and 1.

7.3 Collision Types

There are two special types of collisions:

$$\text{Plastic collision} \Rightarrow e = 0 \Rightarrow v_1' = v_2' \tag{7.3}$$

$$\text{Elastic collision} \Rightarrow e = 1 \Rightarrow v_1 + v_1' = v_2 + v_2' \tag{7.4}$$

In an elastic collision, kinetic energy is conserved.

8 Systems of Particles

8.1 Definitions

M = Total mass of the system (kg)
$\mathbf{r_G}$ = Position vector for the COG (m)
$\mathbf{v_G}$ = Velocity of the COG (m/s)
$\mathbf{a_G}$ = Acceleration of the COG (m/s)

8.2 Center of Gravity Properties

$$M = \Sigma M_i \tag{8.1}$$

$$\mathbf{r_G} = \frac{\Sigma m_i \mathbf{r_i}}{\Sigma m_i} = \frac{\Sigma m_i \mathbf{r_i}}{M} \tag{8.2}$$

$$M\mathbf{v_G} = M\dot{\mathbf{r}}_\mathbf{G} = \Sigma m_i \mathbf{v_i} \tag{8.3}$$

$$\Sigma M\mathbf{a_G} = M\ddot{\mathbf{r}}_\mathbf{G} = \Sigma m_i \mathbf{a_i} = \Sigma \mathbf{f_i^{ext}} \tag{8.4}$$

8.3 Total Linear Momentum

$$\mathbf{G} = \Sigma\mathbf{G_i} = \Sigma m_i \mathbf{v_i} = M\mathbf{v_G} \tag{8.5}$$

$$\Sigma\mathbf{f_i^{ext}} = \mathbf{F_{res}} = \dot{\mathbf{G}} \tag{8.6}$$

9 Rotations

9.1 Definitions

θ = Angle with respect to a reference point (rad)
ω = Angular velocity (rad/s)
α = Angular acceleration (rad/s^2)

9.2 Basic Relations

$$\omega = \frac{d\theta}{dt} \tag{9.1}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{9.2}$$

$$\alpha\, d\theta = \omega\, d\omega \tag{9.3}$$

9.3 Velocity and Acceleration

$$\mathbf{v} = \omega \times \mathbf{r} \tag{9.4}$$

$$\mathbf{a} = \alpha \times \mathbf{r} - \omega \times (\omega \times \mathbf{r}) \tag{9.5}$$

These equations are often combined with equations from paragraph 3.5.

10 Mass Moment of Inertia

10.1 Definitions

I = Mass moment of inertia ($kg\ m^2$)
ρ_A = Distance between point A and the COG (m)
k = Radius of gyration (m)
m = Mass (kg)
M = Moment (Nm)

10.2 Basic Equations

$$I_G = \int_\Omega r^2 dA \tag{10.1}$$

$$I_A = I_G + m\rho_A^2 = mk_A^2 \tag{10.2}$$

Moment of inertia for a slender bar with length l:

$$I_G = \frac{1}{12} ml^2 \tag{10.3}$$

Moment of inertia for a disc with radius r:

$$I_G = \frac{1}{2} mr^2 \tag{10.4}$$

11 Kinetic Energy

11.1 Kinetic Energy Equations

Basic equation:

$$T = \frac{1}{2} mv_G^2 + \frac{1}{2} I_G\omega^2 \tag{11.1}$$

For pure translation:

$$T = \frac{1}{2} mv^2 \tag{11.2}$$

For pure rotation about point O:

$$T = \frac{1}{2} I_0\omega^2 \tag{11.3}$$

12 Angular Momentum About Points

12.1 Definitions

h = Distance between A and the COG perpendicular to the direction of motion (m)
l = Distance between A and the COG perpendicular to the direction of acceleration (m)

12.2 Angular Momentum Equations

$$\mathbf{H_A} = I_G\omega + \mathbf{r_{AG}} \times m\mathbf{v_G} \tag{12.1}$$

$$H_A = I_G\omega + mv_G h \tag{12.2}$$

12.3 Angular Momentum Derivative Equations

$$\mathbf{M_A} = \dot{\mathbf{H}}_\mathbf{A} + \mathbf{v_A} \times m\mathbf{v_G} \tag{12.3}$$

$$\mathbf{M_A} = \dot{\mathbf{H}}_\mathbf{G} + \mathbf{r_{AG}} \times m\mathbf{a_G} \tag{12.4}$$

$$M_A = I_G\alpha + ma_G d \tag{12.5}$$

13 Table Of Useful Equations

These equations are often useful in solving problems. Notice the similarities between the left and the right column.

$$\begin{array}{cc} \Sigma\mathbf{F} = m\mathbf{a} & \Sigma\mathbf{M} = I\alpha \\ \mathbf{G} = m\mathbf{v} & \mathbf{H} = I\omega \\ \dot{\mathbf{G}} = \Sigma\mathbf{F} & \dot{\mathbf{H}} = \Sigma\mathbf{M} \\ T = \frac{1}{2}mv^2 & T = \frac{1}{2}I\omega^2 \end{array} \tag{13.1}$$