AE2-E02 / Introduction to Earth Observation

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Contents

1 Observation Platforms

- Airborne (Distortion, Cheap, High Res, Local)
- Spaceborne (Low Res, Expensive, Global)
- Marine (Sea, Underwater)
- Land (Local, Very Cheap)

The Earth is non-perfect. Principally the orbit of satellite is governed by Kepler's Equations. Additional effects (non-uniform mass distribution, bulging, pear-shape) are modelled by employing spherical harmonics.

The three main disturbances are:

- Drag (Atmospheric, Solar, ...)
- Orbital Period Increases
- Orbital Plane Precesses wrt Polar Axis
- Orbital Ellipse Rotates in own Plane

The Earth's rotational velocity at the surface is about 0.5 km/s.

Per 24h the Earth rotates more than once (due to moving around the sun). This is called sideral day and is given by:

$$d_s = 24 \times \frac{1}{1 + \frac{1}{365.24}} \tag{1}$$

A spacecraft in sun-synchronous orbit is at the same latitude at the same solar time every day. This implies an inclination > 90 °(thus, retrograde orbit).

Sun-synchronous repeating orbits are governed by:

$$\frac{P_n}{P_e} = \frac{n_1}{n_2} \tag{2}$$

Where n_1 is the time interval between successive observations of the same location. n_2 is the number of ascending and decending passes (thus, density of pass-lines). P_n (90 - 120 minutes for LEO) and P_e (24h = 1 solar day) are the satellites' and Earth's orbital periods.

$$\frac{360^\circ}{n_2} = \frac{360^\circ}{233^\circ} = 1.55^\circ \tag{3}$$

For the above n_1, the satellite tracks are 1.55 °seperated.

Altimetric Orbits are designed such that pass-lines will intersect each other at 90 °angles so orthogonal slopes measurements are possible.

Aliasing refers to the difference between observation frequency and frequency of the occurence of an event. Tides measurements are a good example. Observation (f_0) and occurence (f_1) frequency are related to the apparent frequency f_a:

$$f_a = f_1 - f_p \left(\frac{f_1}{f_0} + \frac{1}{2} \right) \tag{4}$$

2 Reference Systems, Time Scales, Geoid

Sorted in decreasing order. Transformation (scaling, distortion and rotation) is done via matrices.

1. Celestial Reference System (sun / earth centered, azimuth, declination, vertical equinox, ...) (CRS)
2. Earth Fixed Reference System (meridian based, latitude and longitude)
3. Obervation Point Fixed
4. Platform Fixed

The Earth is not static. Changes include (usually as result of mass transfer):

- Change of Polar Axis
- Length of Day
- Variance of Magnet Field

Angular Velocity can be measured by exploited the Sagnac Effect (ring-LASERs, counter travelling photons, different delays, interference pattern).

Time scales include:

- UT1 - Time Fixed with Earth's Rotation
- TAI - International Atomic Time
- UTC - Differs from TAI by Number of Integer Seconds + Leap Seconds
- GPS - TAI - 19s

The geoid can be defined by the mean sea level. It is an equipotential (gravity) surface around the Earth. GPS measures geoid height, which is different from the ellipsoid model of the Earth. The sum/difference provides the topographic height.

3 Gravity Field

Gravity potential is the sum of gravitational potential V (dependent on Earth's Density Distribution) and centrifugal potential Φ (dependent on Earth's Rotation):

$$W = V + \Phi \tag{5}$$

While Φ can be calculate reasonable easy. The gravitational potential, however, requires integration of Earth's unknown density distribution. Modelling is thus provided by a sum of spherical harmonic expansions where the coefficents (zonal, tesseral) are deduced from observations.

Furthermore, it holds:

$$V \sim \frac{1}{r^2} \tag{6}$$

$$V \sim \rho \tag{7}$$

The gravitational acceleration is given by:

$$\mathbf{g} = \nabla \cdot W \tag{8}$$

If U is given as the gravity potential of a elliptical model Earth, the gravity disturbance potential T is given as

$$T = V - U \tag{9}$$

and the gravity disturbance γ

$$\gamma = \frac{GM}{r^2} \tag{10}$$

then N is the difference between the ellipitcal model Earth and the approximate mean sea level (the geoid):

$$N = \frac{T}{\gamma} \tag{11}$$

The gravity disturbance of a point P is:

$$\delta g = g(P) - \gamma(P) \tag{12}$$

And the gravity anomaly between point P and Q:

$$\Delta g = g(P) - \gamma(Q) \tag{13}$$

Missions to determine the gravity field are underway. Mission concepts are, among others:

1. Satellite Satellite Tracking (Range-Rate)
2. High-Low Satellite Tracking
3. Gravity Gradient Measured by Array of Accelerometers

Gravity accelerometers measure the second derivatives of the gravity potential V:

$$\ddot{V} = \begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{bmatrix} \tag{14}$$

Global gravity describes the long wavelength components of the spherical expansions and is taken into account for satellite orbits, etc. Local gravity are the short-wavelength components and can be used to find resources, caves, etc.

4 Magnetic Field

The magnet field is modelled in a similar manner as the graviational field. It's main source is the dynamo effect within the Earth's core and mass transport in the crust (but this is not yet fully understood or modelled). It protects Earth from solar radition. Over the past decennia it has been weakening and it might reverse at some point in the future.

It is modelled in a smiliar fashion as the gravitational field - employing spherical harmonics expansions with coefficents determined by observations.

5 Electromagnetic Waves

Any EM-Wave must satisfy Maxwell' Equations. Those are given by:

$$\nabla \cdot \mathbf{E} = 0 \tag{15}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{16}$$

$$\nabla \times \mathbf{E} = \dot{\mathbf{B}} \tag{17}$$

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \dot{\mathbf{E}} \tag{18}$$

Where

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \tag{19}$$

and

$$\nabla \times \mathbf{F} = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix} \tag{20}$$

A wave propagating in z direction and satisfying Maxwells Equations is given by:

$$\mathbf{E} = \begin{bmatrix} E_0 cos(\omega t - kz) \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ \frac{E_0}{c} cos(\omega t - kz) \\ 0 \end{bmatrix} \tag{22}$$

Where ω is the angular frequency and k the wave number as given by:

$$\omega = 2\pi f \tag{23}$$

$$k = \frac{2\pi}{\lambda} \tag{24}$$

Furthermore, frequency f and wavelength λ are related.

$$f = \frac{c}{\lambda} \tag{25}$$

The flux density is:

$$F = \frac{E_0^2}{2Z_0} \tag{26}$$

With the free-space impedance:

$$Z_0 = \sqrt{\frac{\mu_o}{\epsilon_0}} \tag{27}$$

5.1 Polarization

Rotating the wave about the z-axis and combining those two waves gives:

$$\mathbf{E} = \begin{bmatrix} E_{0x} cos(\omega t - kz - \phi_x) \\ E_{0y} cos(\omega t - kz - \phi_y) \\ 0 \end{bmatrix} \tag{28}$$

Plane polarization (**E** pointing same way):

$$\phi_y - \phi_x = 0, \pi, -\pi \tag{29}$$

Circular polarization for ($E_{0x} = E_{0y}$), otherwise elliptical.

RHC (in direction of propagating wave):

$$\phi_y - \phi_x = \frac{\pi}{2} \tag{30}$$

LHC:

$$\phi_y - \phi_x = -\frac{\pi}{2} \tag{31}$$

All other combinations are said to be randomly polarized.

The polarizations can be characterized in terms of the Stokes Vector. The components are given as:

$$S_0 = \langle E_{0x}^2 \rangle + \langle E_{0y}^2 \rangle \tag{32}$$

$$S_1 = \langle E_{0x}^2 \rangle - \langle E_{0y}^2 \rangle \tag{33}$$

$$S_2 = \langle 2E_{0x}E_{0y}cos(\phi_y - \phi_x) \rangle \tag{34}$$

$$S_3 = \langle 2E_{0x}E_{0y}sin(\phi_y - \phi_x) \rangle \tag{35}$$

The $\langle$ and $\rangle$ indicate time averages.

The degree of polarization is defined as:

$$\sqrt{\frac{S_1^2 + S_2^2 + S_3^2}{S_0}} \tag{36}$$

Since the flux density F is proportional to the first Stokes Vector, the flux density can be rewritten as:

$$F = \frac{S_0}{2Z_0} \tag{37}$$

Furthermore, is the radiation is randomly polarized, a linear combination of the Stokes Vector can be created that decomposes the polarization into elementary polarizations (page 14).

5.2 Spectral Analysis

Euler's Formula:

$$e^{i\omega t} = cos(\omega t) + isin(\omega t) \tag{38}$$

With this, the equations describing an EM wave can be rewritten as follows (A - amplitude, ω - angular frequency):

$$Ae^{i\omega t} \tag{39}$$

And with a given frequency density function $a(\omega)$ any combintion of sinusoidal signals can be transferred from the time domain into the frequency domain (and vice versa) by means of Fourier Transformation:

$$f(t) = \int_{-\infty}^{\infty} a(\omega)e^{-i\omega t}d\omega \tag{40}$$

$$a(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{41}$$

5.3 Doppler Effect

The Doppler Effect shifts the frequency depending on the velocity of the sender / receiver.

5.4 Angular Radiation Distribution

Solide Angle Ω.

Power on area dA.

$$dP = Lcos(\theta)dAd\Omega \tag{42}$$

L is the radiance of incident radiation.

The irradiance E is the total incident power per area:

$$E = \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{2\pi} L_{incident}cos(\theta)d\Omega \tag{43}$$

Radiant exitance M is equal to E, but outward. If L is isotropic it moves out of the integral and M becomes equal to πL.

A black body is a perfect emitter of thermal radiation in all wavelengths. Emission and absorbtion coefficent are equal to 1. The spectral radiance per unit wavelength.

$$L_\lambda = \frac{\partial L}{\partial \lambda} = \frac{c}{\lambda^2} \tag{44}$$

From quamtum mechanics, the spectral radiance is calculated and yields the emission curve (page 23). The right hand side is approximated by Rayleigh-Jones to be approximatly linear descending.

Integrating the equation of L over all wavelength yields the total radiant exitance M:

$$M = \sigma T^4 \tag{45}$$

The maximal wavelength for a given radiant temperature is given by Wien's Law:

$$\lambda_{max} = \frac{A}{T} \tag{46}$$

For a grey body, the brightness temperature (temperature a black body with the same emission pattern would have), is given by:

$$T_b = \epsilon T \tag{47}$$

5.4.1 Solar Radiation

Suns exit radiance in all wavelength is given by

$$M = \epsilon \sigma T^4 \tag{48}$$

Power P is M over the surface of the sun's sphere. The irradiance E at a distance D is given by dividing power P by the area that sphere would have. Since the Earth only sees a solid angle $d\Omega$ of the sun, the irradiance E is dividing by this angle, yielding the mean exoathmospheric radiance. Page 27-28 for details.

5.5 Frauenhoffer Diffraction

Writing the wave equation as a wave of complex amplitude, integrating over a slit of height **w** (seperated a distance z from a screen) and multiplying with a aperture function f(y) (1 if y inside w, 0 otherwise) yields the Fraunhoffer diffraction integral (which is a Fourier Transform):

$$a(\theta) = \int_{-\infty}^{\infty} f(y) e^{ikysin(\theta)} dy \tag{49}$$

This shows the dependance of deflection angle to wavelength:

$$\delta\theta \approx \frac{\lambda}{w} \tag{50}$$

Furthermore, this implies that for an aperture with inlet width w the following equation must be satisfied:

$$z > \frac{w^2}{2\lambda} \tag{51}$$

5.6 Dispersive Media

If the medium is dispersive (wave propagation velocity depends on medium), the wave equations satisfying the Maxwell Equations change as follows:

$$\mathbf{E} = \begin{bmatrix} E_0 cos(\omega t - kz) \\ 0 \\ 0 \end{bmatrix} \tag{52}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ \frac{E_0\sqrt{\mu_r \epsilon_r}}{c} cos(\omega t - kz) \\ 0 \end{bmatrix} \tag{53}$$

Thus:

$$\frac{\omega}{k} = \frac{c}{\sqrt{\mu_r \epsilon_r}} \tag{54}$$

The refractive index of the medium:

$$n = \frac{c}{v} = \sqrt{\mu_r \epsilon_r} \tag{55}$$

The flux densiy is again given by:

$$F = \frac{S_0}{2Z} \tag{56}$$

With the impedance Z:

$$Z = Z_0 \sqrt{\frac{\mu_r}{\epsilon_r}} \tag{57}$$

Assuming $\mu_r = 1$ and the possibility that the relative dielectric constant ϵ_r can be imaginary, the change of flux density as a function of angular frequency ω and penetration distance z is then given as:

$$F = F_0 e^{-2\frac{\omega k z}{c}} \tag{58}$$

Where the absorbtion length follows as:

$$l_a = \frac{c}{2\omega k} \tag{59}$$

5.7 Phase and Group Velocity

Since the propagation velocity of the wave changes with frequency, the phase (wave) velocity is defined as:

$$v_p = \frac{w}{k} \tag{60}$$

The group velocity (propagation of information) is given by.

$$v_g = \frac{dw}{dk} \tag{61}$$

Where

$$v_g v_p = c^2 \tag{62}$$

5.8 Scattering

For raditation with a flux density F hitting a surface dA under the angle θ_0 and bouncing back under the angle θ_1, the ratio of radiance hitting L and emitted irradiance E is given by:

$$R = \frac{L_1}{E} \tag{63}$$

The Bistatic Scattering Coefficent is given by:

$$\gamma = 4\pi R cos(\theta_1) \tag{64}$$

If $\theta_0 = \theta_1$ (in and outbound radiation follow the same path), the backscatter coefficent is given by:

$$\sigma^0 = \gamma cos(\theta_0) = 4\pi R cos^2(\theta_0) \tag{65}$$

6 Atmospheric Attenuation

Due to quamtum mechanical effects, radiation energizes atoms in the atmosphere. Since this only happens at certain wavelengths, absorbtion lines will characterize the spectrum of radiation passing the atmosphere.

Furthermore, the ionosphere influence propagation velocity of the radiation due to the electron density as follows:

$$t = \frac{z}{c} + \frac{e^2}{2\epsilon_0 m_e \omega^2 c} \int N dz \tag{66}$$

Where the second term is a function of the TEC (total electron content) and inversly proportional to the square of the frequency.

7 Eletro-Optical Imaging Systems

Scanning techniques are:

- Step Stare
- Push Broom
- Whisk Broom
- Spin Scan

The time for a CCD to record radiance is limited by the spatial resolution and the platform velocity as follows:

$$\Delta T < \frac{\Delta x}{v} \tag{67}$$

From diffraction, the angular resolution is limited by the diameter of the aperture D and the wavelength of the signal as follows:

$$\theta < \frac{\lambda}{D} \tag{68}$$

Spatial resolution follows:

$$\Delta x < \frac{H\lambda}{D} \tag{69}$$

The size of a rezel a, the focal length and the frequency determine the other limit of the detector as follows (when it is projected onto the Earth).

$$res_{rezel} = \frac{Ha}{f} \tag{70}$$

Generally, the two limiting cases are balanced.

7.1 Spectral Resolution

Spectral resolution is usually achieved with filters (high band-width signals) or prisms / diffraction grating (low band-width). Diffraction grating generates spectrum of order n with radiation lines seperated by distance d:

$$sin\theta = \frac{n\lambda}{d} \tag{71}$$

Diffraction grating achieves better line-spread than a prism.

7.2 Atmospheric Disturbances

Radiance values are received, but reflectance values are required. This is usually only function of solar geometry, but the atmospheric composition scatters radiation. This can be corrected with the following methods.

- Use Complete Model for Atmosphere (Complicated)
- Calibrate against Targets with known Brightness
- Subtract Minimum Reflectance Pixels from Image (crude!)

7.3 Thermal Inertia

The amplitude of the surface temperature variation depends on the material of the surface. The idea is, by determining the variation of temperatur, to determine the nature of the surface material.

8 Ranging Systems

The following simplifications are employed:

- Flat Earth, compensated through concept of effective height
- Coherence of Phases, real accuracy depends on height and pulse length
- Flat Surface, rough surfaces have a different power rise and drop pattern

8.1 LIDAR (LASER Pulses)

LASER ranging systems are usually used in airborne survey systems. However, also satellites are tracked and the moon is observed.

1. Send Pulse
2. Start Timer
3. Receive Pulse
4. Stop Timer
5. Calculate

Round Trip Travel Time of the signal is given as:

$$T_t = \frac{2H}{v_g} \tag{72}$$

For airborne applications, the group velocity is usually very close to the speed of light. For spaceborne applications v_g is a function of the atmospheric composition and the electron content in the ionosphere.

After a pulse is sent, the system cannot send another pulse before the old pulse returns. This is to prevent ambiguities.

8.2 RADAR Altimetry

Use concept of scattering zone, travelling half the speed of the real pulse (ie c/2). Scattering zone moves in a cone away, yielding to a circle $\rightarrow$ annulus $\rightarrow$ nothing illumination pattern on the ground. The effective footprint (radius r_p) of the RADAR system is given by (t_p is the pulse time):

$$r_p = \sqrt{cHt_p} \tag{73}$$

With the spatial resolution given by:

$$\Delta x = H\Delta\theta \tag{74}$$

We distinguish two cases:

1. $\Delta x > 2r_p$ - Pulse Limited (Spatial Resolution)
2. $\Delta x < 2r_p$ - Beam Limited (Spatial Resolution)

8.3 Possible Problems

If the distance between two scatterer is shorter than the coherence length (l_c) / coherence width w_c, interference of the signals is possible, yielding it to be noisy.

If the surface is rough, the power receiving pattern will change. In fact, it is a function of the variance of the roughness of the surface.

Loss of Lock can occur. That is, the signal is bouncing back another way or too early when the return signal is not yet expected and the receiver if off.

Use dual-frequency systems to correct for ionospheric errors.

An application is the determination of the long term average of the mean sea level for geoi determination. This may be subject to aliasing!

9 Scattering Systems

Also use temporal structure of return signal. Primarily the backscatter is measured by sending out a pulse and recording it when it comes back. Usually the backscatter coefficent σ^0 is recorded as a function of the incidence angle θ.

LIDAR is used for vertical atmospheric profiling while RADAR mainly focuses on marine surfaces and (penetration of) solid ground features.

The RADAR equation is derived as:

$$P_r = \frac{\lambda^2 G^2 P_t^2}{(4\pi)^3 \eta R^4}\sigma^0 A \tag{75}$$

9.1 RADAR Scatterometry

As mentioned, provide measurement of $\sigma^0 = f(\theta_0, \phi_0)$.

Methods:

- Narrow Beam pointing to target (and turning as platform moves)
- Doppler Shift (transmit over beam-width, receive signals are Doppler shifted and are uniqure so the corresponding target-point can be identified)
- Send Short Pulses and Identify Time Structure

Applications:

- Wind Direction / Velocity (eg waves, need three σ^0 measurements to solve a polynomial)
- Soil, Moisture, Landmass Identification

Note: Scatterometry has good spectral resolution, but spatial resolution is inferior.

9.2 Real Aperture RADAR

Side looking microwave emitter of dimensions $L \times w$ (L - along track length of aperture, w - cross track width) Sending short pulses over narrow beams. The pulses are strong pulse compressed (modulated onto a carrier).

Angular resolutions are limited by the diffraction limit:

$$\phi = \frac{\lambda}{w} \tag{76}$$

$$\phi = \frac{\beta}{L} \tag{77}$$

With ground incidence angle θ (across track), the slant range is given by:

$$s = \frac{H}{cos(\theta)} \tag{78}$$

Yielding along track (azimuth) resolution R_a and cross-track resolution R_r limuted by pulse length:

$$R_a \approx \frac{H\lambda}{c \cdot cos\theta} \tag{79}$$

$$R_r = \frac{ct_p}{2sin\theta} \tag{80}$$

Possible Distortions are:

- Slant Range Distortion (easily corrected)
- Layover (slant points move closer → brightness, darkness)
- Shadowing (slant points move across)

9.3 Synthetic Aperture RADAR

Since the platform moves and the received power of the antenna is proportional to antenna size, a virtual antenna is created by employing the movement of the platform.

With the moving RADAR a Doppler Shift is induced in the radiation. The change in frequeny is characteristic for each point that is observed and can thus be used to identify source of the backscatter.

The virtual antenna length is limited by (x is platform coordinate, v platform velocity):

$$\frac{x}{v} - \frac{H\lambda}{2Lv} \dots \frac{x}{v} + \frac{H\lambda}{2Lv} \tag{81}$$

Coherence of the return signal is important (amplitude, phase). Furthermore, heigth differences induce speckle. Location ambiguities can be solved by employing a second SAR system (slave) next to the primary one (master).

By comparison of phases and amplitudes of the signal received at both master and slave system, the exact origin can be determined. However, difficulties with this technique arise:

- Find Pixel Correlation
- Distance Master / Slave System for Geometric Calculations

Applications:

- Generally Difficult (counter-intuitive, think of ground penetration)
- Ship Wackes (wave patterns)
- Slick (Oil Reserves)
- Density / Mass Estimations of Sea-Ice and Glaciers

9.4 Acoustic Seafloor Mapping

Largely Similar, but more intuitive interpretation than RAR and SAR data. Can either be done ship mounted or on a little fish-thing that is towed close to the ocean floor.

Beam width is limited again by diffraction limit. Repeat time is limited by travel time of signal plus signal length + time lengthening during ground boncing back. Furthermore, due to bounce-back effects between floor and ship, a factor of up to 4 can be introduced for the time.

Systems are pulse or beam limited. Pulse limited systems emit short pulses, where the data is non-ambigous and the whole of the seafloor is NEVER isonated at once (annulus), thus the return signal is only a function of the ground print.

Beam limited systems emit longer pulses and the whole ground-print is isonated (circle).

The vertical resolution depends on the pulse length:

$$\Delta z = \frac{Tc}{2} \tag{82}$$

The horizontal resolution on the beam width, thus on the frequency and the aperture size:

$$\Delta x \approx \beta H \tag{83}$$

Note:

- Objects may appear larger than they are
- Cavities on the floor might not be detected if their radius is smaller than the system's footprint due to masking effects from the reflected signal around the hole

Sidescan SONARs has large angle spread.

Multiscan SONAR has many narrow beam SONARs.

10 Data Processing & Aquiration

The time a satellite is in view of a ground station is given by the angle θ between satellite and Earth and the angle ϕ between ground station and Earth center (page 249).

$$cos(\theta + \phi) = \frac{R}{R+h} cos(\theta) \tag{84}$$

Subsequent calculations about the link budegt, compression, etc can follow. Approximate LEO orbital velocity is in the order of 7.5 km/s.

10.1 Preprocessing

- Radiometric Corrections
- Geometric Corrections (Interpolation, Geocoding)

10.2 Image Enhancement

- Histogram Stretching
- Histogram Equalization
- Averaging
- Sharpening
- Edge Detection
- Low / High-Pass Filters (accentuate low / high freuqencies)

10.3 Band Transformations

- NDVI
- PCA

10.3.1 Principal Component Analysis

Data is in directions y_i. We want a linear combination of y_i to maximize the dispersion (variance) of the data. This new direction is called θ_1. The transformation is thus:

$$\theta_1 = a_1^T y = a_{11} y_1 + a_{12} y_2 \tag{85}$$

With Variance Propagation:

$$D\{\theta_1\} = a_1^T Q_y a_1 \tag{86}$$

We want to maximize the dispersion. The RHS of the previous equation can be rewritten to (since $a^T a = 1$):

$$a_1^T Q_y a_1 = a_1^T Q_y a_1 - \lambda(a_1^T a_1 - 1) \tag{87}$$

We want to maximize this with respect to a_1. Thus, taking the derivative of the right hand side wrt to a_1 and equating it to zero should find us the maximum:

$$0 = a_1^T Q_y - \lambda a_1^T = Q_y a_1 - \lambda a_1 \quad (88)$$

This is clearly an eigenvector problem. Thus:

$$Q_y a_1 = \lambda a_1 \quad (89)$$

Furthermore, maximizing the relation $\lambda a_1^T a_1$ yields, since $a_1^T a_1 = 1$ the maximization of λ, which is the eigenvalue associated to the eigenvector a_1.

By a similar argument $\theta_2 \ldots \theta_i$ can be found, whereby the covariance between the principal components is always 0 since they are orthogoal.

Thus, the principal components can then be extracted from the original data by applying the linear transformation:

$$\theta_i = a_i^T y = a_{i1} y_1 + a_{i2} y_2 \quad (90)$$

Where a_i is the i-th eigenvector.

11 GPS

GPS receivers use 3 spatial and 1 temporal observation provided by GPS satellites. The GPS signal is subject to delay due to the atmosphere being a dispersive medium. Main error sources thus stem from ionospheric delay, tropospheric delay and signals that took a non-direct path to the receiver.

GPS satellite send signals on two frequencies. With the help of a linear combination of the received data on both frequencies it is possible to limit the error due to the aforementioned sources somewhat.

With differential GPS the error can even be reduduced further. This relies on the spatial correlation of error sources. If a user is within a certain distance of a second receiver (usually a officially run GPS station whose coordinates are very accuratly determined), the relative position of the two receivers can be determined to a very accurate degree, yielding in a very accurate overall position determination.