Equations of motion simplifications in general

From equations of motion to eigenvalues

During most exercises you will be asked to simplify the equations of motion, get to the characteristic equation, then find the eigenvalues, and finaly to find the period, damping ratio, etc., from this motion. The sections on the following pages explain how to simplify the equations of motion for some standard eigenmotions.

After that is done, the characteristic equation needs to be found. For this you insert the unknown λ_c in the resulting equation in place of the differential operator D_c, and then use your linear algebra skills to get to the characteristic equation (usually a linear or quadratic equation). Solve the equation for λ_c to get the eigenvalues.

Why should you replace D_c with λ_c? Because it is assumed that the final equation for a variable (airspeed for example) has the form $f(t) = Ae^{\lambda_c \frac{V}{\bar{c}} t}$. Furthermore $D_c = \frac{\bar{c}}{V}\frac{d}{dt}$, and so: $D_c f(t) = \frac{\bar{c}}{V}\frac{d}{dt}\left(Ae^{\lambda_c \frac{V}{\bar{c}} t}\right) = \lambda_c f(t)$. *See section 4.1, page 101 in the reader for more details.*

Once you have gotten solutions for the eigenvalues λ_c, you can use the equations below here to get to calculate all properties of the motion you need.

Period, damping factor, frequency

The following equations are used for the calculation of the eigenmotion responses such as period and damping factor. These equations use the dimensional value $\bar{c}/V$. This is the value used for the symmetric equations of motion. For the assymetric equations, replace this with b/V (so using the wing span instead of the MAC).

Refer to page 109 till 113 and page 130 in the Flight Dynamics reader for more details.

$$\begin{aligned}
\text{Eigenvalue: } & \lambda_c = \xi_c \pm \eta_c i \\
\text{Eigenvalue absolute value: } & |\lambda_c| = \sqrt{\xi_c^2 + \eta_c^2} \\
\text{Period: } & P = \frac{2\pi}{\eta_c}\frac{\bar{c}}{V} \\
\text{Time to half amplitude: } & T_{1/2} = \frac{\ln\left(1/2\right)}{\xi_c}\frac{\bar{c}}{V} \\
\text{Damping factor: } & \zeta = \frac{-\xi_c}{|\lambda_c|} \\
\text{Undamped natural frequency: } & \omega_0 = |\lambda_c|\frac{V}{\bar{c}} \\
\text{Damped natural frequency: } & \omega_n = \omega_0\sqrt{1-\zeta^2} \\
\text{Logarithmic decrement: } & \delta = \xi_c \frac{V}{\bar{c}} P = 2\pi\frac{\xi_c}{\eta_c}
\end{aligned}$$

Short period motion (a)

Reference to section 4.4 (page 123) in the Flight dynamics reader for more info.

Description of the motion

The lateral periodic motion in pitch after a step input on the elevator. The aircraft quickly pitches up and down, however airspeed hardly varies during this motion. If stable, the motion damps out very quickly.

Assumptions

- The airspeed remains constant: V = constant (which means: $\hat{u} = 0$)
- The initial flight condition is level, so $\gamma_0 = 0$ and $C_{X_0} = 0$.

Symmetric equations of motion

$$\begin{bmatrix} \cancel{C_{X_u} - 2\mu_c D_c} & \cancel{C_{X_\alpha}} & \cancel{C_{Z_0}} & \cancel{0} \\ \cancel{C_{Z_u}} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c)\, D_c & \cancel{C_{X_0}} & C_{Z_q} + 2\mu_c \\ \cancel{0} & \cancel{0} & \cancel{-D_c} & \cancel{1} \\ \cancel{C_{m_u}} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & \cancel{0} & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \cancel{\hat{u}} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \vec{0}$$

Simplification steps

1. $\hat{u} = 0$, so all terms in the first column are zero (as these are multiplied by $\hat{u}$).
2. V = constant so the forces in X_B-direction must remain constant. The equation for the force in X_B-direction (the first row) can be omitted.
3. It was assumed that $C_{X_0} = 0$ (The initial flight condition was steady).
4. As $C_{X_0} = 0$, no term with the angle of pitch (θ) appears in the Z-equation (2nd row) or M-equation (4th row) anymore. This means the angle of pitch, and with it the kinematic relation (3rd row), can be omitted. (This does not mean θ is equal to zero.)

Resulting simplified equations of motion

$$\begin{bmatrix} C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c)\, D_c & C_{Z_q} + 2\mu_c \\ C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = \vec{0}$$

Short period motion (b)

Reference to section 4.4 (page 124) in the Flight dynamics reader for more info.
This is an even further simplification of the short period equations of motion.

Assumptions

- The airspeed remains constant: V = constant (which means: $\hat{u} = 0$)
- The initial flight condition is level, so $\gamma_0 = 0$ and $C_{X_0} = 0$.
- The c.g. of the aircraft travels in a straight line during the motion. (This means that we only have rotations in pitch left and that the flight path angle is constant)

Symmetric equations of motion

$$\begin{bmatrix} \cancel{C_{X_u} - 2\mu_c D_c} & \cancel{C_{X_\alpha}} & \cancel{C_{Z_0}} & \cancel{0} \\ \cancel{C_{Z_u}} & \cancel{C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) D_c} & \cancel{C_{X_0}} & \cancel{C_{Z_q} + 2\mu_c} \\ \cancel{0} & \cancel{0} & \cancel{-D_c} & \cancel{1} \\ \cancel{C_{m_u}} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & \cancel{0} & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \cancel{\hat{u}} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \vec{0}$$

Simplification steps

1. $\hat{u} = 0$, so all terms in the first column are zero (as these are multiplied by $\hat{u}$).
2. V = constant so the forces in X_B-direction must remain constant. The equation for the force in X_B-direction (1st row) can be omitted.
3. It was assumed that $C_{X_0} = 0$ (The initial flight condition was steady).
4. As $C_{X_0} = 0$, no term with the angle of pitch (θ) appears in the Z-equation (2nd row) or M-equation (4th row) anymore. This means the angle of pitch, and with it the kinematic relation (3rd row), can be omitted.
5. The c.g. travels in a straight line, so the forces in Z_B-direction are zero. The Z-equation (2nd row) can be dropped. α can be replaced by θ as $\gamma = \theta - \alpha = 0$ (straight initial flight).

Resulting simplified equations of motion

$$\begin{bmatrix} C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = \vec{0}$$

$D_c \frac{q\bar{c}}{V} = D_c^2 \theta$ and $\theta = \alpha$, thus:

$$\left(C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c + C_{m_q} D_c - 2\mu_c K_Y^2 D_c^2 \right) \theta = 0$$

Phugoid (long period motion) (a)

Reference to section 4.4 (page 125) in the Flight dynamics reader for more info.

Description of the motion

The phugoid motion is a lateral, periodical osscilation resulting from a step input on the elevator. First the airplane pitches up. It starts to climb, losing speed and thus lift. Because of that it pitches down again, builds up speed, lift increases and it pitches up again, starting all over.

The most important parameters that vary are airspeed and pitch. The phugoid, if stable, damps out only after quite a long time ($T_{1/2} \approx 81s$ for the Cessna). The period is also rather long ($P \approx 32s$ for the Cessna).

Assumptions

- Angle of attack remains constant. So $\alpha = 0$ and $\dot{\alpha} = 0$.
- There is no acceleration around the y-axis, i.e. no change in pitch, so $\dot{q} = 0$. (This does not mean $q = 0$.)
- We started out from steady flight conditions, so $C_{X_0} = 0$.
- $2\mu_c >> C_{Z_q}$, so C_{Z_q} can be neglected.

Symmetric equations of motion

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & \cancel{C_{X_\alpha}} & C_{Z_0} & 0 \\ C_{Z_u} & \cancel{C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) D_c} & \cancel{C_{X_0}} & \cancel{C_{Z_q} +} 2\mu_c \\ 0 & \cancel{0} & -D_c & 1 \\ \cancel{C_{m_u}} & \cancel{C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c} & \cancel{0} & \cancel{C_{m_q} - 2\mu_c K_Y^2 D_c} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \cancel{\alpha} \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \vec{0}$$

Simplification steps

1. As $\alpha = 0$ and $\dot{\alpha} = 0$ all values in column 2 will be zero (as they would be multiplied by α).
2. It was assumed that there is no acceleration around the y-axis, so there is no moment. The moment equation has thus become irrelevant and can be dropped.
3. It was assumed that $C_{X_0} = 0$.
4. It was assumed that C_{Z_q} can be neglected w.r.t. $2\mu_c$.

Resulting simplified equations of motion

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{Z_0} & 0 \\ C_{Z_u} & 0 & 2\mu_c \\ 0 & -D_c & 1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \vec{0}$$

Phugoid (long period motion) (b)

Reference to section 4.4 (page 126) in the Flight dynamics reader for more info. This is a more refined approximation for the Phugoid motion than presented on the previous page.

Assumptions

- Angle of attack changes, but not very fast. This means $\dot{\alpha} = 0$ but $\alpha \neq 0$.
- There is no acceleration around the y-axis, i.e. no change in pitch, so $\dot{q} = 0$. (This does not mean $q = 0$.)
- We started out from steady flight conditions, so $C_{X_0} = 0$.

Symmetric equations of motion

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0 \\ C_{Z_u} & C_{Z_\alpha} \cancel{+ (C_{Z_{\dot{\alpha}}} - 2\mu_c) D_c} & \cancel{C_{X_0}} & C_{Z_q} + 2\mu_c \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} \cancel{+ C_{m_{\dot{\alpha}}} D_c} & 0 & C_{m_q} \cancel{- 2\mu_c K_Y^2 D_c} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \vec{0}$$

Simplification steps

1. As $\dot{\alpha} = 0$ all differential terms in column 2 will be zero ($D_c \alpha = \frac{\bar{c}}{V} \frac{d}{dt} \alpha = 0$).
2. As $\dot{q} = 0$ all differential terms in column 4 will be zero ($D_c q = \frac{\bar{c}}{V} \frac{d}{dt} q = 0$).
3. It was assumed that $C_{X_0} = 0$.

Resulting simplified equations of motion

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0 \\ C_{Z_u} & C_{Z_\alpha} & 0 & C_{Z_q} + 2\mu_c \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} & 0 & C_{m_q} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \vec{0}$$

Aperiodic roll motion

Reference to section 5.5 (page 145) in the Flight dynamics reader for more info.

Description of the motion

The aperiodic roll motion is a very heavily damped roll motion. The aircraft rolls onto its side because of an aileron deflection. The aircraft rolls but this motion stops almost immediatly as the wings provide an enormous resistance against this motion.

The only parameter really of importance is the roll motion. All other effects can usually be neglected as the aperiodic roll has damped out before any other effects really start ($T_{1/2} \approx 0.2s$ for the Cessna).

Assumptions

- The only motion is a rolling motion, which means no sideslip ($\beta = 0$) and no yaw ($r = 0$)

Symmetric equations of motion

$$\begin{bmatrix} \cancel{C_{Y_\beta} + (C_{Y_{\dot{\beta}}} - 2\mu_b) D_b} & \cancel{C_L} & \cancel{C_{Y_p}} & \cancel{C_{Y_r} - 4\mu_b} \\ \cancel{0} & \cancel{-\frac{1}{2} D_b} & \cancel{1} & \cancel{0} \\ \cancel{C_{l_\beta}} & \cancel{0} & C_{l_p} - 4\mu_b K_X^2 D_b & \cancel{C_{l_r} + 4\mu_b K_{XZ} D_b} \\ \cancel{C_{n_\beta} + C_{n_{\dot{\beta}}} D_b} & \cancel{0} & \cancel{C_{n_p} + 4\mu_b K_{XZ} D_b} & \cancel{C_{n_r} - 4\mu_b K_Z^2 D_b} \end{bmatrix} \begin{bmatrix} \cancel{\beta} \\ \cancel{\phi} \\ \frac{pb}{2V} \\ \cancel{\frac{rb}{2V}} \end{bmatrix} = \vec{0}$$

Simplification steps

1. There is no sideslip: $\beta = 0$, so all terms in the first column are zero.
2. There is no movement in the Y-direction (there is only roll), so the lateral force equation (1st row) can be omitted.
3. The yaw rate is zero: $r = 0$, so all terms in the last column are zero.
4. There is no yaw movement (only roll), so the yawing moment equation (4th row) can be omitted.
5. The rolling moment equation (3th row) does not contain a term with the roll angle ϕ. So we don't need the kinetic equation (2nd row) to calculate the roll. We can scrape column 2 and row 2.

Resulting simplified equations of motion

$$\left(C_{l_p} - 4\mu_b K_X^2 D_b\right) \frac{pb}{2V} = 0$$

Dutch roll motion (a)

Reference to section 5.5 (page 145) in the Flight dynamics reader for more info.

Description of the motion

The Dutch roll motion is a periodic motion in which the aircraft sideslips, yaws and rolls. Usually the motion is stable and will damp out after some time. The period is approximately 2s for the Cessna with a time to half amplitude of approximately 2.3s.

The motion is initiated with an input on the rudder. Say that this causes the aircraft to first swing to the right. This causes a roll moment as the left wing gets more speed in the turn than the right wing. The result of the yaw is an angle of sideslip. Because of this angle the vertical tailplane (and the increased drag on the left wing w.r.t. the right wing) push the craft back to the left. The aircraft start yawing to the left and thus also rolling to the left. The nose overshoots the flight direction, giving rise to a sideslip angle to the left. The process repeats put mirrored to push the aircraft to the right again, etcetera, etcerta.

As the roll is approximately 90° behind on the yaw in phase, the nose of the aircraft is describing a circular motion.

Assumptions

- The rolling motion is neglectable, so $\phi = 0$ and $p = 0$
- $C_{Y_{\dot{\beta}}}$ and $C_{n_{\dot{\beta}}}$ are neglectable.
- $4\mu_b >> C_{Y_r}$ so C_{Y_r} can be neglected.

Symmetric equations of motion

$$\begin{bmatrix} C_{Y_B} + \left(\cancel{C_{Y_{\dot{B}}}} - 2\mu_b\right) D_b & \cancel{C_L} & \cancel{C_{Y_p}} & \cancel{C_{Y_r}} - 4\mu_b \\ \cancel{0} & \cancel{-\frac{1}{2}D_b} & \cancel{1} & \cancel{0} \\ \cancel{C_{l_\beta}} & \cancel{0} & \cancel{C_{l_p} - 4\mu_b K_X^2 D_b} & \cancel{C_{l_r} + 4\mu_b K_{XZ} D_b} \\ C_{n_\beta} \cancel{+ C_{n_{\dot{\beta}}} D_b} & \cancel{0} & \cancel{C_{n_p} + 4\mu_b K_{XZ} D_b} & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \cancel{\phi} \\ \cancel{\frac{pb}{2V}} \\ \frac{rb}{2V} \end{bmatrix} = \vec{0}$$

Simplification steps

1. No rolling motion: $\phi = 0$, so all elements in column 2 becom zero.
2. No rolling motion: $p = 0$, so all elements in column 3 become zero.
3. All terms in the kinetic equation (2nd row) are zero, so it can be removed.
4. The rolling motion equation (3th row) has become irrelevant as we have assumed that no rolling motion will occure.
5. $C_{Y_{\dot{\beta}}}$, $C_{n_{\dot{\beta}}}$ and C_{Y_r} can all be neglected.

Resulting simplified equations of motion

$$\begin{bmatrix} C_{Y_B} - 2\mu_b D_b & -4\mu_b \\ C_{n_\beta} & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \frac{rb}{2V} \end{bmatrix} = \vec{0}$$

Dutch roll motion (b)

Reference to section 5.5 (page 146) in the Flight dynamics reader for more info. This is an even more rough approximation of the equations of motion for the Dutch roll.

Assumptions

- The rolling motion is neglectable, so $\phi = 0$ and $p = 0$
- $C_{Y_{\dot{\beta}}}$ and $C_{n_{\dot{\beta}}}$ are neglectable.
- $4\mu_b >> C_{Y_r}$ so C_{Y_r} can be neglected.
- There is only rotation in yaw, this means the airplane's c.g. moves along a straight trajectory and the course angle (χ) is constant.

Symmetric equations of motion

$$\begin{bmatrix} \cancel{C_{Y_\beta} + (C_{Y_{\dot{\beta}}} - 2\mu_b) D_b} & \cancel{C_L} & \cancel{C_{Y_p}} & \cancel{C_{Y_r} - 4\mu_b} \\ \cancel{0} & \cancel{-\frac{1}{2} D_b} & \cancel{1} & \cancel{0} \\ \cancel{C_{l_\beta}} & \cancel{0} & \cancel{C_{l_p} - 4\mu_b K_X^2 D_b} & \cancel{C_{l_r} + 4\mu_b K_{XZ} D_b} \\ C_{n_\beta} \cancel{+ C_{n_{\dot{\beta}}} D_b} & \cancel{0} & \cancel{C_{n_p} + 4\mu_b K_{XZ} D_b} & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \cancel{\phi} \\ \cancel{\frac{pb}{2V}} \\ \frac{rb}{2V} \end{bmatrix} = \vec{0}$$

Simplification steps

1. No rolling motion: $\phi = 0$, so all elements in column 2 become zero.
2. No rolling motion: $p = 0$, so all elements in column 3 become zero.
3. All terms in the kinetic equation (2nd row) are zero, so it can be removed.
4. The rolling motion equation (3th row) has become unrelevant as we have assumed that no rolling motion will occure.
5. $C_{Y_{\dot{\beta}}}$, $C_{n_{\dot{\beta}}}$ and C_{Y_r} can all be neglected.
6. The lateral force equation (1st row) has become superfluous as it is assumed the aircraft's c.g. travels along a straight line.

Resulting simplified equations of motion

$$C_{n_\beta} \beta + \left(C_{n_r} - 4\mu_b K_Z^2 D_b\right) \frac{rb}{2V} = 0$$

And further: $\beta = -\psi$ (as the flight path angle is zero: $\chi = \beta + \phi = 0$) and $\frac{rb}{2V} = \frac{1}{2} D_b \psi$. This all makes:

$$\left(-C_{n_\beta} + \frac{1}{2} C_{n_r} D_b - 2\mu_b K_Z^2 D_b^2\right) \psi = 0$$

Aperiodic spiral motion

Reference to section 5.5 (page 146) in the Flight dynamics reader for more info.

Description of the motion

The aperiodic roll motion is usually a very slow motion that is often also the only unstable eigenmotion of an aircraft. The aircraft enters into a slow downwards spiral motion in which the aircraft sideslips, yaws and rolls.

Assumptions

- Due to the slow nature of this motion, all linear and angular accelerations can be negected, i.e. all terms with a differental operator (D_b) in the force and moment equations become zero.
- C_{Y_r} and C_{Y_p} are neglectable.

Symmetric equations of motion

$$\begin{bmatrix} C_{Y_B} \cancel{+ (C_{Y_{\dot{B}}} - 2\mu_b) D_b} & C_L & \cancel{C_{Y_p}} & \cancel{C_{Y_r}} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} \cancel{- 4\mu_b K_X^2 D_b} & C_{l_r} \cancel{+ 4\mu_b K_{XZ} D_b} \\ C_{n_\beta} \cancel{- C_{n_{\dot{\beta}}} D_b} & 0 & C_{n_p} \cancel{+ 4\mu_b K_{XZ} D_b} & C_{n_r} \cancel{- 4\mu_b K_Z^2 D_b} \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \vec{0}$$

Simplification steps

1. All terms with the differential operator D_b in the force and moment equations (rows 1, 3 and 4, so not in the kinetic equation (2nd row)) become zero.
2. C_{Y_r} and C_{Y_p} are neglectable.

Resulting simplified equations of motion

$$\begin{bmatrix} C_{Y_B} & C_L & 0 & -4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} & C_{l_r} \\ C_{n_\beta} & 0 & C_{n_p} & C_{n_r} \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \vec{0}$$