Aircraft modes of vibration

It is finally time to look at the dynamics of an aircraft. How will an aircraft behave, when given elevator, rudder and aileron deflections? What are its modes of vibration? That's what we will look at in this chapter.

1 Eigenvalue theory

1.1 Solving the system of equations

To examine the dynamic stability of the aircraft, we examine the full longitudinal equations of motion. The symmetric part of these equations were

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) D_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1.1}$$

In this system of equations, we have assumed stick-fixed conditions. All inputs are zero. We could try to find solutions for the above system of equations. A common way to do this, is to assume a solution of the form

$$\mathbf{x}(t) = \mathbf{A} e^{\lambda_c s_c}. \tag{1.2}$$

In this equation, $\mathbf{x}(t)$ is our solution. $s_c = \frac{V}{\bar{c}} t$ is the dimensionless time. From this form follows that $D_c \mathbf{x} = \lambda_c \mathbf{x}$. If we insert this into the equations of motion, we find

$$\begin{bmatrix} C_{X_u} - 2\mu_c \lambda_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) \lambda_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -\lambda_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} \lambda_c & 0 & C_{m_q} - 2\mu_c K_Y^2 \lambda_c \end{bmatrix} \begin{bmatrix} A_u \\ A_\alpha \\ A_\theta \\ A_q \end{bmatrix} e^{\lambda_c s_c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1.3}$$

We can write this matrix equation as $[\Delta]\, \mathbf{A} e^{\lambda_c s_c} = \mathbf{0}$. The exponential in this equation can't be zero, so we can get rid of it. We thus need to solve $[\Delta]\, \mathbf{A} = \mathbf{0}$. One solution of this equation is $\mathbf{A} = 0$. However, this is a rather trivial solution, in which we are not interested. So we need to find non-trivial solutions. This is where our knowledge on linear algebra comes in. There can only be non-trivial solutions, if $\det [\Delta] = 0$. Applying this will give us an equation of the form

$$A\lambda_c^4 + B\lambda_c^3 + C\lambda_c^2 + D\lambda_c + E = 0. \tag{1.4}$$

This equation is called the **characteristic polynomial**. Solving it will give four **eigenvalues** λ_{c_1}, λ_{c_2}, λ_{c_3} and λ_{c_4}. Corresponding to these four eigenvalues are four **eigenvectors** $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ and $\mathbf{A_4}$. The final solution of the system of equations now is

$$\mathbf{x} = c_1 \mathbf{A_1} e^{\lambda_{c_1} s_c} + c_2 \mathbf{A_2} e^{\lambda_{c_2} s_c} + c_3 \mathbf{A_3} e^{\lambda_{c_3} s_c} + c_4 \mathbf{A_4} e^{\lambda_{c_4} s_c}. \tag{1.5}$$

The constants c_1, c_2, c_3 and c_4 depend on the four initial conditions.

1.2 The eigenvalues

Let's examine the eigenvalues of the system of equations. Each eigenvalue can be either real and complex. If one of the eigenvalues is complex, then its complex conjugate is also an eigenvalue. Complex eigenvalues therefore always come in pairs.

A **mode of vibration** is a characteristic way in which an object (our aircraft) can vibrate. The number of modes depends on the eigenvalues. In fact, it is equal to the number of different eigenvalues. (When performing the counting, a pair of complex conjugate eigenvalues is counted as one.) For example, an aircraft with two real eigenvalues and two complex eigenvalues has three modes of vibration.

The eigenvalues λ_c are very important for the stability of the system. To examine stability, we look at the limit

$$\lim_{s_c \to \infty} c_1 \mathbf{A_1} e^{\lambda_{c_1} s_c} + c_2 \mathbf{A_2} e^{\lambda_{c_2} s_c} + c_3 \mathbf{A_3} e^{\lambda_{c_3} s_c} + c_4 \mathbf{A_4} e^{\lambda_{c_4} s_c}. \tag{1.6}$$

If only one of the eigenvalues has a positive real part, then this limit will diverge. This means that our aircraft is **unstable**. If, however, all eigenvalues have negative real parts, then the system is **stable**.

1.3 Real eigenvalue properties

We can derive some interesting properties from the eigenvalues. First, let's examine a real eigenvalue λ_c. This eigenvalue has its own mode of vibration $\mathbf{x} = \mathbf{A} e^{\lambda_c s_c}$. The **half time** $T_{\frac{1}{2}}$ is defined as the time it takes to reduce the amplitude of the motion to half of its former magnitude. In other words, $\mathbf{x}(t + T_{\frac{1}{2}}) = \frac{1}{2}\mathbf{x}(t)$. Solving this equation will give

$$T_{\frac{1}{2}} = \frac{\ln \frac{1}{2}}{\lambda_c} \frac{\bar{c}}{V}. \tag{1.7}$$

Similarly, we define the time constant τ as the time it takes for the amplitude to become $1/e$ of its former magnitude. Solving $\mathbf{x}(t + \tau) = \frac{1}{e}\mathbf{x}(t)$ gives

$$\tau = -\frac{1}{\lambda_c} \frac{\bar{c}}{V}. \tag{1.8}$$

These two parameters of course only exist if λ_c is negative. If it is positive, then the magnitude will only grow. In this case, the **doubling time** T_2 is an important parameter. It is given by $T_2 = -T_{\frac{1}{2}}$.

1.4 Complex eigenvalue properties

Now let's examine a complex eigenvalue pair. We can write it as $\lambda_{c_{1,2}} = \xi_c \pm \eta_c i$, where $i = \sqrt{-1}$ is the complex number. This eigenvalue will cause an oscillation. The period and frequency of the oscillation only depend on η_c. In fact, the **period** P, the **frequency** f and the **angular frequency** ω_n are given by

$$P = \frac{2\pi}{\eta_c} \frac{\bar{c}}{V}, \qquad f = \frac{1}{P} = \frac{\eta_c}{2\pi} \frac{V}{\bar{c}} \qquad \text{and} \qquad \omega_n = \frac{2\pi}{P} = \eta_c \frac{V}{\bar{c}}. \tag{1.9}$$

The damping of this oscillation is caused by the real part ξ_c. Again, the **half time** $T_{\frac{1}{2}}$ is defined as the time it takes for the amplitude to reduce to half its size. It is still given by

$$T_{\frac{1}{2}} = \frac{\ln \frac{1}{2}}{\xi_c} \frac{\bar{c}}{V}. \tag{1.10}$$

Another important parameter is the **logarithmic decrement** δ. It is defined as the natural logarithm of the ratio of the magnitude of two successive peaks. In other words, it is defined as

$$\delta = \ln \left(\frac{e^{\xi_c \frac{V}{\bar{c}} (t+P)}}{e^{\xi_c \frac{V}{\bar{c}} t}} \right) = \xi_c \frac{V}{\bar{c}} P. \tag{1.11}$$

Finally, there are the **damping ratio** ζ and the **undamped angular frequency** ω_0. They are defined such that

$$\lambda_{c_{1,2}} = \left(-\zeta\omega_0 \pm i\omega_0 \sqrt{1 - \zeta^2} \right) \frac{\bar{c}}{V}. \tag{1.12}$$

Solving for ζ and ω_0 will give

$$\zeta = \frac{-\xi_c}{\sqrt{\xi_c^2 + \eta_c^2}} \qquad \text{and} \qquad \omega_0 = \sqrt{\xi_c^2 + \eta_c^2}\frac{V}{\bar{c}}. \tag{1.13}$$

1.5 Getting stable eigenvalues

Let's take a look at the characteristic equation (equation (1.4)). We usually set up the equation, such that $A > 0$. To obtain four eigenvalues with negative real parts, we must have

$$B > 0, \quad C > 0, \quad D > 0 \quad \text{and} \quad E > 0. \tag{1.14}$$

But these aren't the only conditions to ensure that we have stable eigenvalues. We must also have

$$R = BCD - AD^2 - B^2E > 0. \tag{1.15}$$

These criteria are known as the **Routh-Hurwitz Stability Criteria**. The coefficient R is called **Routh's discriminant**. These criteria hold for both the symmetric and the asymmetric modes of vibration.

2 The symmetric modes of vibration

2.1 Example eigenvalues

Let's suppose we know all the parameters in the matrix equation that was described earlier. In this case, we can find the four eigenvalues. An example solution of these eigenvalues is given by

$$\lambda_{c_{1,2}} = -0.04 \pm 0.04i \qquad \text{and} \qquad \lambda_{c_{3,4}} = -0.0003 \pm 0.006i. \tag{2.1}$$

Of course these values will be different for different aircraft. But most types of aircraft, having the standard wing-fuselage-tailplane set-up, will have similar eigenvalues.

Let's study these eigenvalues. There are two pairs of complex conjugate eigenvalues. Both pairs of eigenvalues have negative real parts. This means that the aircraft is stable. Since there are only two pairs of complex conjugate eigenvalues, there are two modes of vibration. We will now examine these modes.

2.2 The short period oscillation

Let's look at the first pair of eigenvalues. It has a relatively big real part ξ_c The damping is therefore big. The complex part η_c is relatively big as well. So the frequency is high. In other words, we have a highly damped high-frequency oscillation. This motion is known as the **short period oscillation**.

Let's take a look at what actually happens in the aircraft. We start the short period oscillation by applying a step input to the elevator deflection. (We deflect it, and keep that deflection.) We can, for example, deflect it upward. This causes the lift on the horizontal tailplane to decrease. This, in turn, causes the pitch rate to increase. An increase in pitch rate will, however, increase the effective angle of attack of the horizontal tailplane. This then reduces the pitch rate. And the whole cycle starts over again. However, the oscillation is highly damped. After less than one period, the effects are hardly noticable anymore.

Now let's try to derive some equations for the short period motion. The short period motion is rather fast. So we assume the aircraft hasn't had time yet to change its velocity in X or Z direction. This means that $\hat{u} = 0$ and $\gamma = 0$. Therefore $\alpha = \theta$. This reduces the equations of motion to

$$\begin{bmatrix} C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c)\lambda_c & 2\mu_c + C_{Z_q} \\ C_{m_\alpha} + C_{m_{\dot{\alpha}}}\lambda_c & C_{m_q} - 2\mu_c K_Y^2 \lambda_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.2}$$

We can now find the eigenvalues for this matrix. This will still give us a rather complicated equation. If we neglect $C_{Z_{\dot{\alpha}}}$ and C_{Z_q}, then this complicated equation reduces to

$$\lambda_{c_{1,2}} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{2.3}$$

In this equation, the coefficients A, B and C are given by

$$A = 4\mu_c^2 K_Y^2, \qquad B = -2\mu_c(K_Y^2 C_{Z_\alpha} + C_{m_{\dot{\alpha}}} + C_{m_q}) \quad \text{and} \quad C = C_{Z_\alpha} C_{m_q} - 2\mu_c C_{m_\alpha}. \tag{2.4}$$

To have stability, we should have $-\frac{B}{2A}$ negative. We know that A is positive. This means that B has to be positive as well.

2.3 The phugoid

Now we look at the second pair of eigenvalues. It has a small real part ξ_c, and therefore a small damping. The complex part η_c is small as well, so the frequency is low. In other words, we have a lightly damped low-frequency oscillation. This motion is known as the **phugoid**.

Again, we look at what happens with the aircraft. This time, we apply an impulse deflection on the elevator. (We only deflect it briefly.) This will cause our pitch angle to increase. (That is, after the short period motion has more or less damped out.) We will therefore go upward. This causes our velocity to decrease. Because of this, the lift is reduced. Slowly, the pitch angle will decrease again, and we will go downward. This causes the velocity to increase. This, in turn, increases the lift. The pitch angle will again increase, and we will again go upward.

Again, we will try to derive some relations for the phugoid. In the phugoid, the angle of attack α is approximately constant. (γ and θ do vary a lot though.) So we have $\alpha = 0$ and $\dot{\alpha} = 0$. (Remember that we're discussing deviations from the initial position.) Since the oscillation is very slow, we also assume that $\dot{q} = 0$. If we also neglect the terms C_{Z_q} and C_{X_0}, we will find that we again have

$$\lambda_{c_{3,4}} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{2.5}$$

However, now the coefficients are given by

$$A = -4\mu_c^2, \qquad B = 2\mu_c C_{X_u} \quad \text{and} \quad C = -C_{Z_u} C_{Z_0}. \tag{2.6}$$

We can apply the approximations $C_{X_u} = -2C_D$, $C_{Z_0} = -C_L$ and $C_{Z_u} = -2C_L$. This would then give us the three parameters

$$\omega_0 = \frac{V}{\bar{c}}\sqrt{\frac{C_L^2}{2\mu_c^2}} = \frac{g}{V}\sqrt{2}, \qquad \zeta = \frac{\sqrt{2}}{2}\frac{C_D}{C_L} \quad \text{and} \quad P = \frac{2\pi}{\omega_0\sqrt{1-\zeta^2}} \approx \frac{2\pi}{\omega_0} = \sqrt{2}\pi\frac{V}{g}. \tag{2.7}$$

Note that, in the above equation for P, we have used the fact that the damping ζ is small. Although the above equations are only approximations, they can serve as quite handy tools in verifying your results.

3 The asymmetric modes of vibration

3.1 Example eigenvalues

We have just examined the symmetric equations of motion. Of course, we can do the same for the asymmetric equations of motion. These equations of motion are

$$\begin{bmatrix} C_{Y_\beta} + (C_{Y_{\dot\beta}} - 2\mu_b)D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} + C_{n_{\dot\beta}} D_b & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.1}$$

Examining them goes in more or less the same way as for the symmetric case. There is, however, one important difference. Since we are examining the asymmetric case, we don't use the chord $\bar{c}$ but we use the wing span b. The eigenvalues are thus also denoted as λ_b. Example eigenvalues for an aircraft are

$$\lambda_{b_1} = -0.4, \qquad \lambda_{b_2} = 0.01 \qquad \text{and} \qquad \lambda_{b_{3,4}} = -0.04 \pm 0.4i. \tag{3.2}$$

You might be surprised that these eigenvalues are a lot bigger than the symmetric eigenvalues. This is not very important. It's only the case, because they are based on b, instead of on $\bar{c}$. And naturally, b is a lot bigger than $\bar{c}$.

Let's examine the eigenvalues. There are two real eigenvalues and one pair of complex conjugate eigenvalues. The aircraft thus has three modes of vibration. You might also have noticed that there is a positive eigenvalue. The aircraft is thus unstable. The eigenvalue is, however, very small. This means that the aircraft state will only diverge very slowly. The pilot will have plenty of time to do something about it. So you don't have to worry: flying is still safe.

3.2 The aperiodic roll

The motion corresponding to λ_{b_1} is called the **aperiodic roll**. The eigenvalue is very negative. This motion is therefore highly damped.

The aperiodic roll is induced by applying a step input to the aileron. When this happens, the aircraft will start rolling. Let's suppose it rolls to the right. The right wing then goes down. This means that the right wing will get a higher effective angle of attack. The lift of this wing thus increases. The opposite happens for the left wing: its lift decreases. This lift difference causes a moment opposite to the rolling motion. In other words, the motion is damped out. The roll rate p will converge rather quickly to a constant value.

The aperiodic roll is a very fast motion. So there is no time for sideslip or yaw effects to appear. So we can assume that, during an aperiodic roll motion, we have $\beta = r = 0$. This reduces the equations of motion to just one equation, being

$$\left(C_{l_p} - 4\mu_b K_X^2 D_b\right) \frac{pb}{2V} = 0. \tag{3.3}$$

It directly follows that the corresponding eigenvalue is given by

$$\lambda_{b_1} = \frac{C_{l_p}}{4\mu_b K_X^2}. \tag{3.4}$$

3.3 The spiral motion

The motion corresponding to λ_{b_2} is called the **spiral motion**. The eigenvalue is positive. So the motion is unstable. However, the eigenvalue is very small. This means that divergence will occur only very slowly. We thus say that the motion is **marginally unstable**. (For some aircraft, this value is slightly negative. Such aircraft are **marginally stable**.)

The spiral motion is induced by an initial roll angle. (An angle of 10° is sufficient.) This causes the lift vector to be tilted. The horizontal component of the lift will cause the aircraft to make a turn. In the meanwhile, the vertical component of the lift vector has slightly decreased. This causes the aircraft to lose altitude. Combining these two facts will mean that the aircraft will perform a spiral motion.

If the eigenvalue λ_{b_2} is positive, then the roll angle of the aircraft will slowly increase. The spiral motion will therefore get worse. After a couple of minutes, the roll angle might have increased to 50°. This phenomenon is, however, not dangerous. The pilot will have plenty of time to react. It is also very easy to pull the aircraft out of a spiral motion.

Let's try to derive an equation for λ_{b_2}. The spiral motion is a very slow motion. We thus neglect the derivatives of β, p and r. Also, the coefficients C_{Y_r} and C_{Y_p} are neglected. After working out some equations, we can eventually find that

$$\lambda_{b_2} = \frac{2C_L \left(C_{l_\beta} C_{n_r} - C_{n_\beta} C_{l_r}\right)}{C_{l_p} \left(C_{Y_\beta} C_{n_r} + 4\mu_b C_{n_\beta}\right) - C_{n_p} \left(C_{Y_\beta} C_{l_r} + 4\mu_b C_{l_\beta}\right)}. \tag{3.5}$$

The denominator of this relation is usually negative. We say we have **spiral stability** if $\lambda_{b_2} < 0$. This is thus the case if

$$C_{l_\beta} C_{n_r} - C_{n_\beta} C_{l_r} > 0. \tag{3.6}$$

You might remember that we've seen this equation before.

3.4 The Dutch roll

The pair of eigenvalues $\lambda_{b_{3,4}}$ has a slightly low damping and a slightly high frequency. In the mode of vibration corresponding to these eigenvalues, the aircraft alternately performs a yawing and a rolling motion. The mode of vibration is called the **Dutch roll**.

Let's take a look at what actually happens with the aircraft. To initiate the Dutch roll, an impulse input is applied to the rudder. This causes the aircraft to yaw. Let's suppose the aircraft yaws to the right. The lift on the left wing then increases, while the lift on the right wing decreases. This moment causes the aircraft to roll to the right.

When the aircraft is rolling to the right, then the lift vector of the right wing is tilted forward. Similarly, the left wing will have a lift vector that is tilted backward. This causes the aircraft to yaw to the left. (This effect is still called adverse yaw.) In this way, roll and yaw alternate each other. It is important to remember that roll and yaw are alternately present. When the roll rate is at a maximum, the yaw rate is approximately zero, and vice verse.

The Dutch roll is not very comfortable for passenger. To increase passenger comfort, a yaw damper is used. This is an automatic system, which uses rudder/aileron deflections to reduce the effects of the Dutch roll.

Let's try to find a relation for $\lambda_{b_{3,4}}$. This is rather hard, since both roll and yaw are present. However, experience has shown that we still get slightly accurate results, if we neglect the rolling part of the motion. We thus assume that $\varphi = p = 0$. This reduces the system of equations to a 2×2 matrix. From it, we can again find that

$$\lambda_{b_{3,4}} = \frac{-B \pm i\sqrt{4AC - B^2}}{2A}. \tag{3.7}$$

However, this time the coefficients A, B and C are given by

$$A = 8\mu_b^2 K_Z^2, \qquad B = -2\mu_b \left(C_{n_r} + 2K_Z^2 C_{Y_\beta}\right) \qquad \text{and} \qquad C = 4\mu_b C_{n_\beta} + C_{Y_\beta} C_{n_r}. \tag{3.8}$$

And that concludes our discussion on the modes of vibration.

3.5 Stability criteria

From the characteristic equation (equarion (1.4)) we can see which eigenmotions are stable. We have seen earlier that, if A, B, C, D, E and R are all positive, then all eigenmotions are stable. In other words, we have spiral stability and a convergent Dutch roll.

However, if some of the coefficients become negative, then there will be unstable eigenmotions. If $E < 0$, then we have spiral instability. Similarly, if $R < 0$, then we will have a divergent Dutch roll. So, to ensure stability, we'd best keep the coefficient of the characteristic equation positive.