Mechanics of Materials Theory

1 Thin-Walled Structures

Before we venture further into the depths of mechanics of materials, we first need to discuss some basic ideas. The first idea we will discuss is that of thin-walled structures. Let's consider the I-beam cross-section shown in figure 1. Such a cross-section is often used in constructions for reasons we will discover in later chapters.

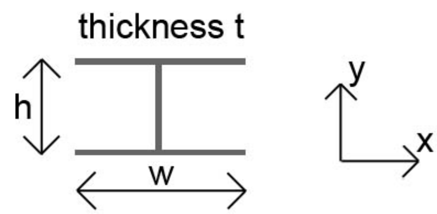

Figure 1: Another example of a cross-sectional shape: an I-beam.

We can try to calculate the area of this I-beam. We see a horizontal part of the beam with width w, a vertical part with height h and another horizontal part with height w. So we might initially think that

$$A = wt + ht + wt = 2wt + ht. \tag{1.1}$$

However, now we've counted the overlapping parts twice. If we keep those into account, we will get

$$A = wt + (h - 2t)t + wt = 2wt + ht - 2t^2. \tag{1.2}$$

The second way of calculating things is more exact. However, as cross-sectional shapes get more complicated, this second way will become rather difficult.

Now let's look at the differences. The difference between the two methods is in this case $2t^2$. Usually the thickness t is small compared to the width w and height h. The quantity t^2 is then very small. If this is the case, we say we have a **thin-walled structure**. We then may neglect any factors with t^2 and such.

So the key idea is as follows: In thin-walled structures you do not have to consider the very small parts (with high powers of t). There will be no terms like $(h - 2t)$ and such. These terms all simplify to just h. By using this assumption, the analysis of many cross-sections is simplified drastically.

2 Center of Gravity - Method 1

In the next chapters we will be analyzing forces in beams. The way in which the beams behave, strongly depends on the shape of its cross-section. So the coming couple of paragraphs we will examine the cross-sections of beams.

A first thing which we need to find is the position of the **center of gravity** of a cross-section. This position has coordinates $(\overline{x}, \overline{y})$. The values of $\overline{x}$ and $\overline{y}$ can be found using

$$\overline{x} = \frac{1}{A}\int_A x\,dA \qquad \text{and} \qquad \overline{y} = \frac{1}{A}\int_A y\,dA. \tag{2.1}$$

To explain how to use these equations, we use an example. Let's consider the rectangle shown in figure 2.

Figure 2: An example of a cross-sectional shape: a square.

The value of $\overline{x}$ can be found using

$$\overline{x} = \frac{1}{A}\int_A x\,dA = \frac{1}{hw}\int_0^w h\,x\,dx = \frac{1}{hw}\frac{1}{2}hw^2 = \frac{1}{2}w. \tag{2.2}$$

Identically we can find that $\overline{y} = \frac{1}{2}h$. So the center of gravity is exactly in the middle of the rectangle, as was expected.

3 Center of Gravity - Method 2

The method of the previous paragraph has a few downsides. For complicated cross-sections it is often very hard to evaluate the integral. Cross-sectional shapes often consist of an amount of sub-shapes, of which you already know the position of the center of gravity. It would be much easier to use that fact to find the position of the center of gravity.

Now we will do exactly that. Let's define A_{tot} as the total area of the cross-section. We can now find $\overline{x}$ and $\overline{y}$ using

$$\overline{x} = \frac{1}{A_{tot}}\sum x_i A_i \qquad \text{and} \qquad \overline{y} = \frac{1}{A_{tot}}\sum y_i A_i, \tag{3.1}$$

where x_i, y_i is the position of the center of gravity of every subpart i and A_i the corresponding area. For the shape of figure 1 (the I-beam) we will thus get

$$\overline{y} = \frac{1}{A_{tot}}\sum y_i A_i = \frac{1}{2wt+ht}\left((0)(wt) + \left(\frac{1}{2}h\right)(ht) + (h)(wt)\right) = \frac{\frac{1}{2}h^2t + wht}{2wt+ht} = \frac{1}{2}h. \tag{3.2}$$

The center of gravity is exactly in the middle. We could have expected that, since the structure is symmetric. Instead of having to integrate things, we only need to add up things with this method. That's why this method is mostly used to find the center of gravity of a cross-section.

3.1 Moment of Inertia - Method 1

A quantity that often occurs when analyzing bending moments is

$$I_x = \int_A (y - y_r)^2\,dA \qquad \text{and} \qquad I_y = \int_A (x - x_r)^2\,dA. \tag{3.3}$$

Here the point x_r, y_r is a certain reference point. The parameters I_x and I_y are called the **second moment of inertia** about the x-axis/y-axis. The second moment of inertia is also called the **area moment of inertia**, or shortened it is just **moment of inertia**. The moment of inertia is always minimal if you calculate it with respect to the center of gravity (so if $x_r = \overline{x}$ and $y_r = \overline{y}$).

To see how these equations work, we will find the value I_x for a rectangle, as shown in figure 2. We do this with respect to the center of gravity. We thus get

$$I_x = \int_A (y - \overline{y})^2 \, dA = \int_0^h \left(y - \frac{1}{2}h\right)^2 w \, dx = \left[\frac{1}{3}w\left(y - \frac{1}{2}h\right)^3\right]_0^h = \frac{1}{12}wh^3. \tag{3.4}$$

Identically, we find that the moment of inertia about the y-axis is $I_y = \frac{1}{12}hw^3$.

Next to I_x and I_y, there is also the **polar moment of inertia** I_p, defined as

$$I_p = \int_A \left(x^2 + y^2\right) \, dA = I_x + I_y. \tag{3.5}$$

This quantity is also often denoted as J.

4 Moment of Inertia - Method 2

Evaluating an integral every time we need to know a moment of inertia is rather annoying. There must be a more simple method. It would be great if we can evaluate a cross-section part by part. Well, why can't we do this?

There is a good reason for that. A moment of inertia is always with respect to a certain reference point. We can't add up the moment of inertias of parts with different reference points. So what we need to do, is make the reference points for every part equal. We need to move them! More specifically, we need to move them to the center of gravity of the entire cross-section. How do we do this?

There is a rule, called **Steiner's rule** (also called the **parallel axis theorem**), with which you can move the reference point. Suppose we have the moment of inertia with respect to the center of gravity and want to know it with respect to another point x_r, y_r. The equations we can then use are

$$I_{x_r} = I_{x_{cog}} + A\left(y_r - y_{cog}\right)^2 \qquad \text{and} \qquad I_{y_r} = I_{y_{cog}} + A\left(x_r - x_{cog}\right)^2. \tag{4.1}$$

So if we move the reference point away from the center of gravity by a distance d in a certain direction, then the moment of inertia for the corresponding axis increases by Ad^2. This term Ad^2 is called **Steiner's term**.

So, taking into account the Steiner's term, we can derive another expression for the moment of inertia. It will become

$$I_{x_{tot}} = \sum\left(I_{x_i} + A_i\left(y_{cog_{tot}} - y_{cog_i}\right)^2\right) \qquad \text{and} \qquad I_{y_{tot}} = \sum\left(I_{y_i} + A_i\left(x_{cog_{tot}} - x_{cog_i}\right)^2\right). \tag{4.2}$$

If we apply this to the I-shaped beam of figure 1, we find

$$I_x = 2\left(\frac{1}{12}wt^3 + (wt)\left(\frac{1}{2}h\right)^2\right) + \frac{1}{12}th^3 = \frac{1}{2}wth^2 + \frac{1}{12}th^3. \tag{4.3}$$

We don't have a Steiner's term for the vertical part of the beam (the so-called **web**). This is because the center of gravity of this part coincides with the center of gravity of the whole cross-section. Also note that we have ignored the term involving t^3 at the horizontal parts (the **flanges**). This is because we assumed we are dealing with a thin-walled structure.

5 Common Cross-Sectional Shapes

To apply the method of the previous paragraph, we need to know the moment of inertia for a couple of shapes. Some of them are given here. The following moment of inertias are with respect to the center of gravity of that cross-section. For that reason, the position of the center of gravity is also given.

- **A rectangle**, with width w and height h. The center of gravity is in its center (at height $\frac{1}{2}h$ and width $\frac{1}{2}w$). The moment of inertias are

$$I_x = \frac{1}{12} w h^3 \qquad \text{and} \qquad I_y = \frac{1}{12} h w^3. \tag{5.1}$$

- **An isosceles triangle**, with its base down and tip up. (The tip then lies centered above the base.) The base has width w, and the triangle has height h. The center of gravity now lies on height $\frac{1}{3}h$ and on width $\frac{1}{2}w$. The moment of inertias are

$$I_x = \frac{1}{36} w h^3 \qquad \text{and} \qquad I_y = \frac{1}{48} h w^3. \tag{5.2}$$

- **A circle**, with radius R. Its center of gravity lies at the center of the circle. The moment of inertias are

$$I_x = I_y = \frac{1}{4} \pi R^4 \tag{5.3}$$

- **A tube**, with inner radius R_1 and outer radius R_2. Its center of gravity lies at the center of the tube. The moment of inertias are

$$I_x = I_y = \frac{1}{4} \pi \left(R_2^4 - R_1^4 \right) \tag{5.4}$$