Partial Differential Equations Summary

1. The heat equation

Many physical processes are governed by partial differential equations. One such phenomenon is the temperature of a rod. In this chapter, we will examine exactly that.

1.1 Deriving the heat equation

1.1.1 What is a partial differential equation?

In physical problems, many variables depend on multiple other variables. For example, the temperature $u(x,t)$ $[K]$ can depend on both position and time. Such variables don't have normal derivates like du/dt. Instead, they have **partial derivates**, like $\partial u/\partial x$ and $\partial u/\partial t$.

We can set up an equation with multiple partial derivatives. We would then get a **partial differential equation** (PDE). So a partial differential equation is an equation containing partial derivatives. If a differential equation does not contain partial derivatives, it's only an **ordinary differential equation** (ODE).

1.1.2 Conservation of energy for a one-dimensional rod

Let's consider a one-dimensional rod of length L. We define the **thermal energy density** $e(x,t)$ $[J/m^3]$ as the energy per unit volume. It depends not only on the position in the rod, but also on time. This is because it can change as time passes by.

There are two reasons why e can vary in time. First, there is the **heat flux** $\phi(x,t)$ $[J/m^2 s]$. This is the heat flowing to the right through a unit cross-section per unit time. Second, there can be internal heat creation due to **heat sources**. The amount of heat created is denoted by $Q(x,t)$ $[J/m^3 s]$.

We can now apply the law of conservation of energy to our rod. Let's examine a thin slice. To be more specific, we examine the rate of change of energy in it. This must be equal to the heat created, plus the heat flowing in, minus the heat flowing out. This gives us

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q. \tag{1.1.1}$$

This equation is called the **integral conservation law**.

1.1.3 Deriving the heat equation for a one-dimensional rod

Let's define $u(x,t)$ $[K]$ as the temperature in the rod. There is a relation between this temperature u, and the variable e. But to find it, we first have to define two other variables.

Let's define the **mass density** $\rho(x)$ $[kg/m^3]$ as the mass per unit volume. (Usually ρ varies with temperature, and thus also with time. However, these variations are usually small. We thus neglect them.) We also define the **specific heat** $c(x)$ $[j/kg\,K]$ as the heat necessary to raise the temperature of a $1kg$-mass by 1 Kelvin. (We assume it to be constant in time for the same reasons as for the mass density.)

Using the above definitions, we can derive that

$$e(x,t) = c(x)\rho(x)u(x,t). \tag{1.1.2}$$

This transforms the integral conservation law into

$$c\rho \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q. \tag{1.1.3}$$

Usually Q is known. But u and ϕ are not. So we still have two unknowns. But you may be wondering, doesn't the heat flow depend on the temperature as well? In fact, it does. The heat flow depends on temperature differences. The higher these differences, the higher the heat flow. So we can say that

$$\phi(x,t) = -K_0 \frac{\partial u}{\partial x}. \tag{1.1.4}$$

This equation is called **Fourier's law of heat conduction**. The variable $K_0(x)$ $[J/Kms]$ is called the **thermal conductivity**. (It is also assumed to be constant for varying temperatures.) The above law now reduces our differential equation into

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q. \tag{1.1.5}$$

If there are no heat sources (and thus $Q = 0$), we can rewrite this to

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad \text{where} \qquad k = \frac{K_0}{c\rho}. \tag{1.1.6}$$

The important equation above is called the **heat equation**. By the way, k $[m^2/s]$ is called the **thermal diffusivity**.

1.1.4 Initial and boundary conditions

When solving a partial differential equation, we will need initial and boundary conditions. But what conditions do we exactly need?

If we look at the heat equation, we see that there is only a first time-derivative of u. So we need only one **initial condition** (IC). (An initial condition is a condition at $t = 0$.) Usually such a condition takes the form $u(x, 0) = f(x)$.

However, the heat equation contains a second derivative with respect to x. So we will need two **boundary conditions** (BC). (A boundary condition is a condition at a specified position.) These boundary conditions are usually the temperatures at the edges of the rod. So, $u(0,t) = T_1(t)$ and $u(L,t) = T_2(t)$.

However, it is also possible to set the heat flow ϕ (or equivalently $\partial u/\partial x$) at the edges of the rod. We would then have values given for $\partial u(0,t)/\partial x$ and $\partial u(L,t)/\partial x$. If the rod is perfectly insulated at its edges, then $\phi = 0$ and thus also $\partial u/\partial x = 0$ at the edges.

It is of course also possible to combine the two possibilities above. In that case, we deal with **Newton's law of cooling**. The heat flow then depends on the difference in temperature with respect to a certain reference temperature $u_B(t)$. This would give us

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = -H\left(u(0,t) - u_B(t)\right) \qquad \text{and} \qquad -K_0(L)\frac{\partial u}{\partial x}(L,t) = H\left(u(L,t) - u_B(t)\right). \tag{1.1.7}$$

Here H is the **heat transfer coefficient**. Note that if $H = 0$, we are dealing with an insulated edge. On the other hand, if we have $H = \infty$, we would have a constant temperature at the edge.

1.2 Special cases of the heat equation

1.2.1 Perfect thermal contact

Let's suppose we have two rods of length L. We can connect them to each other, such that their edges are in contact. So one rod goes from $x = 0$ to $x = L$, while the second goes from $x = L$ to $x = 2L$. We can make this connection in such a way that there is **perfect thermal contact**. You may wonder, what does that mean? Well, it means two things.

First of all the temperature u at the edges of both rods must be equal. We write this as $u(L-,t) = u(L+,t)$. In words, if we approach the point $x = L$ from the left (negative) side, we find the same temperature as if we approach it from the right (positive) side.

But, perfect thermal contact also means that no heat is lost. The energy that exits the first rod enters the second rod. In an equation this means

$$\phi(L-,t) = \phi(L+,t), \qquad \text{or, equivalently,} \qquad K_0(L-)\frac{\partial u}{\partial x}(L-,t) = K_0(L+)\frac{\partial u}{\partial x}(L+,t). \tag{1.2.1}$$

1.2.2 Finding the steady-state solution

Let's suppose we have a heat problem where $Q = 0$ and $u(x,0) = f(x)$. Also suppose that our boundary conditions are constant. (They don't change in time.) Then we can expect that, after a while, the temperature $u(x,t)$ will not change in time anymore. The corresponding solution for $u(x,t)$ is called the **equilibrium** or **steady-state solution**.

How can we find this solution? Well, we know that $\partial u/\partial t = 0$. So also $\partial^2 u/\partial x^2 = 0$. This means that the temperature is given by $u(x,t) = C_1 x + C_2$. Using the boundary conditions, we can often find C_1 and C_2. Only if both rods have an insulated edge, we can't find both constants yet. In that case we would have to find C_2 using the initial condition. This goes according to

$$C_2 = \frac{1}{L}\int_0^L f(x)\,dx. \tag{1.2.2}$$

1.2.3 The heat equation in 3D

What happens when we don't have a one-dimensional rod, but a three-dimensional body? In that case we can also derive a heat equation. There are some small differences though.

This time the temperature $u(x,y,z,t)$ depends on a lot more variables, as does the heat flow ϕ. Also, the heat flow has a direction, so it is a vector (written as ϕ). It thus has a divergence $\nabla \cdot \phi$. The relation between ϕ and u is now given by $\phi = -K_0 \nabla u$. Using this data, we can derive that the **3-dimensional heat equation** becomes

$$c\rho\frac{\partial u}{\partial t} = K_0 \nabla^2 u + Q. \tag{1.2.3}$$

Here ∇^2 is the **Laplacian operator**, defined as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{1.2.4}$$

Let's look at the conditions. The intial condition takes the simple form of $u(x,y,z,0) = f(x,y,z)$. However, the boundary conditions are slightly more difficult. We can set the temperature $u(x,y,z,t)$ at the edge of our body at a certain value. We could also set the heat flow at the edge of our body at a certain value. We would thus set $\nabla u \cdot \hat{\mathbf{n}}$. (Here, $\hat{\mathbf{n}}$ is the unit vector at the edge, pointing outward.) We also have a 3-dimensional version of **Newton's law of cooling**. This would be

$$-K_0 \nabla u \cdot \hat{\mathbf{n}} = H\left(u - u_B\right). \tag{1.2.5}$$

What would happen if we try to find the steady-state solution in 3D? In that case, we would have to solve the equation $\nabla^2 u = -Q/K_0$. This equation is called **Poisson's equation**. If we also have $Q = 0$, we would have to solve $\nabla^2 u = 0$. This equation is called **Laplace's equation**. We will solve this later in this chapter.

1.3 Basic concepts needed to solve the heat equation

It is almost time for us to solve the heat equation. However, before we do that, we will have to look at some other things first.

1.3.1 Linear operators and linear equations

A linear operator is some operator L for which

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2), \tag{1.3.1}$$

where c_1 and c_2 are constants. For example, the **heat operator**

$$\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} \tag{1.3.2}$$

is a linear operator. A **linear equation** for u is an equation of the form $L(u) = f$, with the function f known. If $f = 0$, we have a **linear homogeneous equation**. Linear homogeneous equations have a certain advantage. We can apply the principle of superposition to it. Suppose we would have two solutions u_1 and u_2. Then also $c_1u_1 + c_2u_2$ is a solution, for any constants c_1 and c_2.

1.3.2 Orthogonality

The property of orthogonality comes in very handy when solving heat equations. So let's examine it. We say that two function $f(x)$ and $g(x)$ are **orthogonal** on the interval $[0, L]$ if

$$\int_0^L f(x)g(x)dx = 0. \tag{1.3.3}$$

It can be shown that the functions $\sin(n\pi x/L)$ and $\sin(m\pi x/L)$ (with n and m positive integers) are orthogonal if $n \neq m$. And the same goes for cosines. That comes in handy! In fact, the general rules for the interval $[0, L]$ are

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n, \\ L/2 & \text{if } m = n. \end{cases} \tag{1.3.4}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n, \\ L/2 & \text{if } m = n \neq 0, \\ L & \text{if } m = n = 0. \end{cases} \tag{1.3.5}$$

Sometimes, however, we are examining the interval $[-L, L]$. In this case all values double. So,

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n. \end{cases} \tag{1.3.6}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n \neq 0, \\ 2L & \text{if } m = n = 0. \end{cases} \tag{1.3.7}$$

By the way, the functions $\sin(n\pi x/L)$ and $\cos(m\pi x/L)$ are always orthogonal on the interval $[-L, L]$. So for every n and m we have

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0. \tag{1.3.8}$$

1.4 Solving method for the heat equation

In this part we will present a basic method to solve the heat equation.

1.4.1 Introducing the method of separation of variables

Let's try to solve the homogeneous heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.4.1}$$

Of course, there are also an initial condition $u(x, 0) = f(x)$ and two boundary conditions. To solve this problem, we use the **method of separation of variables**. According to this method, we assume that we can write $u(x, t)$ as $X(x)T(t)$. Here, the function $X(x)$ only depends on x and $T(t)$ only depends on t. We can now rewrite the above equation to

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dX^2}. \tag{1.4.2}$$

We have reduced the PDE to an ODE! It can also be shown that both sides of the above equation equal a certain constant $-\lambda$. Here, λ is called the **separation constant**. We thus get two ordinary differential equations, being

$$\frac{d^2X}{dx^2} = -\lambda X \qquad \text{and} \qquad \frac{dT}{dt} = -\lambda kT. \tag{1.4.3}$$

Solving the latter one is easy. The solution is

$$T(t) = ce^{-\lambda kT}, \tag{1.4.4}$$

where c is a constant. (It depends on the initial conditions.) However, solving the equation for X is a bit more difficult. In fact, we will find that it can only be solved for certain values of λ. These values are called the **eigenvalues** of the equation. The corresponding solutions for $X(x)$ are the **eigenfunctions**. Let's take a look at how we can find them.

1.4.2 Finding the eigenvalues and the eigenfunctions

We want to solve the ODE

$$\frac{d^2X}{dx^2} = -\lambda X. \tag{1.4.5}$$

We see that $X = 0$ is a solution. We call this the **trivial solution**, in which we are not interested. If we ignore this solution, we can distinguish three cases:

- $\lambda < 0$. In this case the general solution for $X(x)$ is

  $$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}. \tag{1.4.6}$$

  When applying boundary conditions, we usually only find the trivial solution. Only in certain special cases will there be eigenvalues $\lambda < 0$.

- $\lambda = 0$. Now we would find the solution $X(x) = c_1 x + c_2$. Using the boundary conditions, we can solve for c_1 and c_2. Sometimes it turns out that $c_1 = c_2 = 0$. In this case $X(x)$ is the trivial solution, and $\lambda = 0$ is not an eigenvalue. Sometimes, however, $X(x)$ is not the trivial solution. In this case $\lambda = 0$ actually is an eigenvalue, with the corresponding eigenfunction $X(x)$.

- $\lambda > 0$. In this case the general solution for $X(x)$ is

$$X(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x. \tag{1.4.7}$$

We can apply the boundary conditions to the above equation. We should then look for a solution for which $X(x) \neq 0$. (We don't want the trivial solution.) It usually turns out that there are only solutions for certain λ. These λ are the eigenvalues. The corresponding solutions for $X(x)$ are the eigenfunctions.

The above method to find the eigenfunctions might seem a bit odd initially. However, they will become more clear after the examples that will be treated in the upcoming part.

1.4.3 Putting together the eigenfunctions

Now several eigenvalues λ_n and eigenfunctions $X_n(x)$ are known. We can see that every function $X_n(x)T_n(t)$ is a solution to the PDE. So, the general solution is then given by all linear combinations of these solutions. In an equation this becomes

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t). \tag{1.4.8}$$

To find the coefficients c_n, we must use the initial condition. Just insert $t = 0$ in the above equation. Then, using orthogonality, you can find expressions for the coefficients c_n. Although this is also slightly more difficult than it seems, we will demonstrate it using the following examples.

1.5 Example solutions of the heat equation

1.5.1 First example: Both edges having $u = 0$

Let's suppose our rod has both its sides kept at a constant temperature $u = 0$. So our boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$. From this follows that $X(0) = 0$ and $X(L) = 0$.

If $\lambda < 0$, then we can show that $X(x) = 0$ as well. So we only find the trivial solution.

What happens if $\lambda = 0$? In this case $X(x) = c_1 x + c_2$. Applying the boundary conditions will give $c_1 = c_2 = 0$. So, $\lambda = 0$ is not an eigenvalue.

However, if $\lambda > 0$, we will find some eigenvalues. We can apply the boundary condition $X(0) = 0$ in equation 1.4.7. We will then find $c_1 = 0$. If we also apply $X(L) = 0$, we will find $c_2 \sin \sqrt{\lambda} L = 0$. If we also have $c_2 = 0$, we would only find the trivial solution. So, instead, we must have $\sin \sqrt{\lambda} L = 0$. This can only be true if $\sqrt{\lambda} L = n\pi$, with $n = 1, 2, 3, \ldots$. So our eigenvalues λ_n and eigenfunctions $X_n(x)$ are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \text{and} \qquad X_n(x) = \sin\left(\frac{n\pi x}{l}\right). \tag{1.5.1}$$

Note that we have dropped the constant at $X_n(x)$. We are allowed to do this because constants don't really matter with eigenfunctions. If $X_n(x)$ is an eigenfunction, then so is any multiple of it.

So, we can now see that any function of the form

$$u_n(x,t) = X(x)T(t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda k t}, \qquad \text{with } n = 1, 2, 3, \ldots \tag{1.5.2}$$

is a solution satisfying the differential equation. In fact, any linear combination of the above solutions is a solution. So we could say that our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda k t}. \tag{1.5.3}$$

However, there is one condition we haven't satisfied yet: the initial condition. And we can use this condition to find the constants B_n. To satisfy the initial condition, we must have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right). \tag{1.5.4}$$

Now we must apply the property of orthogonality to find the coefficients B_n. To do this, we multiply by $\sin(m\pi x/L)$ and integrate from 0 to L. We then get

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right). \tag{1.5.5}$$

Now we can note that every term in the sum on the right drops out (is zero), except for the term with $n = m$. The right side of the equation thus reduces to $B_m L/2$. It follows that

$$B_m = \frac{2}{L} \int_0^L f(x) \sin\frac{m\pi x}{L} dx. \tag{1.5.6}$$

Using this equation, we can find our constants B_n, and thus also our unique solution for $u(x,t)$.

1.5.2 Second example: Both edges being insulated

In the second example we examine a rod with both its edges insulated. So our boundary conditions are $\partial u/\partial x(0,t) = 0$ and $\partial u/\partial x(L,t) = 0$.

If $\lambda < 0$, then we can again show that $X(x) = 0$ as well. So we only find the trivial solution.

Let's examine the case where $\lambda = 0$. We now do find a non-trivial solution. Once more we have $X_0(x) = c_1 x + c_2$. Both boundary conditions imply that $c_1 = 0$. However, c_2 can be anything. So we have found a non-trivial solution. Thus $\lambda = 0$ is an eigenvalue! The corresponding eigenfunction is $X_0(x) = 1$. (Remember that we were allowed to ignore constants when examining eigenfunctions.)

Now let's consider the case $\lambda > 0$. This time our solutions will be

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \text{and} \qquad X_n(x) = \cos\left(\frac{n\pi x}{l}\right). \tag{1.5.7}$$

The resulting general solution of our PDE (before applying the initial conditions) will then be

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) e^{-\lambda k t}. \tag{1.5.8}$$

The constants A_0 and A_n follow from our initial condition. This time we must multiply by $\cos(n\pi x/L)$ and integrate from 0 to L. We then find that

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx \qquad \text{and} \qquad A_n = \frac{2}{L} \int_0^L f(x) \cos\frac{n\pi x}{L} dx. \tag{1.5.9}$$

1.5.3 Laplace's equation

We can also extend the method of separation of variables to a two-dimensional plate with width L and height H. However, things get more difficult now. So to make it easier, we only want to find the steady-state solution (with $\partial u/\partial t = 0$). This turns our differential equation into Laplace's equation, being

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.5.10}$$

To demonstrate the solution method, we will show one example. Let's assume that we have boundary conditions $u(0,y)=0$, $u(L,y)=f(y)$, $u(x,0)=0$ and $u(x,H)=0$. To apply the method of separation of variables, we assume that $u(x,y)=X(x)Y(y)$. Our differential equation now turns into

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=\lambda. \tag{1.5.11}$$

So we again have two ODEs. For $Y(y)$, we have two rather easy boundary conditions, being $Y(0)=0$ and $Y(H)=0$. So let's focus on the y-part. We can solve this using methods we have seen earlier. We find as eigenvalues and eigenfunctions

$$\lambda_n=(n\pi/H)^2 \qquad \text{and} \qquad Y_n(y)=\sin\frac{n\pi y}{H}. \tag{1.5.12}$$

Now let's find $X(x)$ as well. we now have to solve

$$\frac{d^2X}{dx^2}=\lambda X, \qquad \text{or equivalently} \qquad \frac{d^2X}{dx^2}=\left(\frac{n\pi}{H}\right)^2 X. \tag{1.5.13}$$

The solution for this equation is

$$X(x)=c_1e^{\sqrt{\lambda}x}+c_2e^{-\sqrt{\lambda}x} \qquad \text{or equivalently} \qquad X(x)=c_1\cosh\sqrt{\lambda}x+c_2\sinh\sqrt{\lambda}x. \tag{1.5.14}$$

We can use either of the above relations. However, in this case the relation with sinh and cosh is more convenient, since then one term will drop out. So let's use that one.

One of our boundary conditions is $X(0)=0$. If we apply this, we find that $c_1=0$ and the part with cosh will disappear. We thus have as eigenfunctions $X_n(x)=\sinh\sqrt{\lambda}x$. Our general solution for u then becomes

$$u(x,y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi x}{H}. \tag{1.5.15}$$

However, we haven't applied one boundary condition yet, being $u(L,y)=f(y)$. We can use this to find the constants c_n. Inserting $x=L$ in the above equation gives

$$f(y)=u(L,y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}. \tag{1.5.16}$$

It can then be derived, using the porperty of orthogonality, that

$$c_n=\frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H f(y)\sin\frac{n\pi y}{H}dy. \tag{1.5.17}$$

And this completes our solution to the problem.

2. Fourier series

It is often convenient to express a function as its Fourier series. But can you do this for all functions? And can you differentiate/integrate Fourier series? That's what we will examine in this chapter.

2.1 Basic concepts

2.1.1 Definitions

Before we examine Fourier series, we must examine some definitions.

- Simply said, a function $f(x)$ is **continuous** if it has no jumps, nor any places where $f(x) \to \pm\infty$ or $df/dx \to \pm\infty$.
- A function $f(x)$ is **piecewise continuous** if it can be split up into pieces, which are all continuous. This means that so-called **jump dicontinuities** are allowed for piecewise continuous functions.
- A function $f(x)$ is **smooth** if it is continuous, and its derivative df/dx is also continuous.
- A function $f(x)$ is **piecewise smooth** if it can be split up into pieces, which are all smooth.

2.1.2 Odd and even functions

A function $g(x)$ is **odd** if it satisfies $g(-x) = -g(x)$. In other words, if you rotate the graph of $g(x)$ by $180°$ about the origin and wind up with the same graph, then $g(x)$ is odd. Similarly, a function $h(x)$ is **even** if it satisfies $h(x) = h(-x)$. In other words, if you mirror the graph of $h(x)$ about the y-axis and wind up with the same graph, then $h(x)$ is even.

2.1.3 Odd and even extensions and parts

Suppose we have a function $f(x)$. Let's examine the right side of its graph (for $x > 0$). We can extend this part to the left side, such that we wind up with an odd function. As discussed before, we need to rotate this part about the origin by $180°$. This new function is called the **odd extension** of $f(x)$. Its definition is

$$f_{odd,ext}(x) = \begin{cases} f(x) & \text{if } x > 0, \\ -f(-x) & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.1.1}$$

Note that this function satisfies the definition of odd functions. Similarly, we can find the even extension of $f(x)$, being

$$f_{even,ext}(x) = \begin{cases} f(x) & \text{if } x \geq 0, \\ f(-x) & \text{if } x < 0. \end{cases} \tag{2.1.2}$$

But we can do more with a function $f(x)$. We can also split it up in parts. The odd and even parts of a function $f(x)$ are defined as

$$f_o(x) = \frac{f(x) - f(-x)}{2} \qquad \text{and} \qquad f_e(x) = \frac{f(x) + f(-x)}{2}. \tag{2.1.3}$$

Note that we have $f(x) = f_o(x) + f_e(x)$. Also, if $f(x)$ is already odd, then $f_o(x) = f(x)$ and $f_e(x) = 0$.

2.2 Fourier series and its convergence

Now it is time to examine Fourier series. What are they? And when do they actually converge?

2.2.1 Definition of the Fourier series

The **Fourier series** of a function $f(x)$ is the series satisfying

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}. \tag{2.2.1}$$

Here, a_0, a_n and b_n are the so-called **Fourier coefficients**. We can find them using the property of orthogonality. In fact, we will find that

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \tag{2.2.2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \tag{2.2.3}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{2.2.4}$$

It is important to note that this series is periodic with period $2L$. So in fact, the Fourier series is only valid for the interval $[-L, L]$.

2.2.2 Fourier series and odd and even functions

The Fourier series of odd and even functions are quite interesting. It can be shown that, for odd functions $g(x)$, we always have $a_n = 0$. On the other hand, for even functions $h(x)$, we always have $b_n = 0$. We thus find that

$$g(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad \text{and} \qquad h(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \tag{2.2.5}$$

The Fourier series of $g(x)$ is now called a **Fourier cosine series** (since it only consists of cosines). Similarly, the Fourier series of $h(x)$ is called a **Fourier sine series**.

Sometimes we only want the Fourier series of a function $f(x)$ on the interval $[0, L]$. In this case we have a certain advantage — we can choose whether we use a cosine series or a sine series. If we use a cosine series, then we actually find the Fourier series of $f_{even,ext}(x)$. Similarly, if we use a sine series, then we find the Fourier series of $f_{odd,ext}(x)$.

2.2.3 Notation for convergence of Fourier series

There is an important question mathematicians like to ask. Will the Fourier series of $f(x)$ actually converge to $f(x)$? It turns out that this is not always the case. If this is not the case, then we may not write an equality sign =. Instead, we usually write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}. \tag{2.2.6}$$

The $\sim$ sign means 'has the Fourier series'. But it doesn't imply convergence. If the series does converge, we of course can write an = sign.

2.2.4 Rules for convergence of Fourier series

Of course it would be nice to know when the Fourier series of $f(x)$ actually converges to $f(x)$. There are rules for that. First we examine the rules of normal Fourier series on the interval $[-L, L]$.

- Let's suppose $f(x)$ is piecewise smooth on the interval $[-L, L]$. Now the Fourier series of $f(x)$ converges everywhere on the interval $[-L, L]$, except at jump discontinuities. At these points the series converges to the average of the jump, being

$$\frac{f(x-) + f(x+)}{2}. \tag{2.2.7}$$

- Let's suppose $f(x)$ is both piecewise smooth and continuous on the interval $[-L, L]$. Also suppose we have $f(-L) = f(L)$. Now the Fourier series of $f(x)$ converges everywhere on the interval $[-L, L]$. (Note that the conditions simply demand that there are no jump discontinuities.)

We can state similar rules for the cosine/sine series. As you know, these series are only valid on the interval $[0, L]$.

- Let's suppose $f(x)$ is both piecewise smooth and continuous on the interval $[0, L]$. In this case, the Fourier cosine series converges everywhere on the interval $[0, L]$.
- Let's suppose $f(x)$ is both piecewise smooth and continuous on the interval $[0, L]$. Also suppose that $f(0) = f(L) = 0$. Only in this case, the Fourier sine series converges everywhere on the interval $[0, L]$.

2.3 Differentiating and integrating Fourier series

2.3.1 Differentiating Fourier series term by term

Let's suppose we have a Fourier series of some function $f(x)$. We now want to find the Fourier series of the derivative df/dx. Can we then simply take the derivative of the Fourier series? Well, it turns out that we can only do that under certain conditions. We can only differentiate the Fourier series of $f(x)$ term by term if...

- $f(x)$ is piecewise smooth on the interval $[-L, L]$,
- $f(x)$ is continuous on the interval $[-L, L]$,
- we have $f(-L) = f(L)$.

All of the above conditions must hold. (It can be noted that the above conditions simply mean that there are no jump discontinuities in $f(x)$.)

Now let's ask ourselves, when can we differentiate a Fourier cosine series term by term? We can simply modify the above rule for that. It can be noted that cosine series always automatically have $f(-L) = f(L)$. So, we may drop that condition. We thus find that we may differentiate cosine series if $f(x)$ is both piecewise smooth and continuous on the interval $[0, L]$.

Now let's ask ourselves, when can we differentiate a Fourier sine series term by term? Sadly, we can not ignore any conditions now. In fact, there is an extra condition. We can only differentiate a Fourier sine series if $f(x)$ is both piecewise smooth and continuous on $[0, L]$ and also $f(0) = f(L) = 0$.

You may wonder, what happens if we differentiate a sine series, but $f(0) \neq f(L) \neq 0$? We then have a special case. Let's suppose we differentiate the Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}. \tag{2.3.1}$$

Our result will then be

$$\frac{df(x)}{dx} \sim \frac{f(L) - f(0)}{L} + \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} B_n + \frac{2}{L} \left((-1)^n f(L) - f(0) \right) \right) \cos \frac{n\pi x}{L}. \tag{2.3.2}$$

2.3.2 Integrating Fourier series term by term

Let's examine the Fourier series of $f(x)$, being

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{2.3.3}$$

Now we want to find the integral of $f(x)$, being

$$F(x) = \int_{-L}^{x} f(x) dx. \tag{2.3.4}$$

Are we allowed to integrate the Fourier series term by term? Well, luckily it turns out that we can. We are always allowed to integrate a Fourier series term by term. And the integral always converges. There are no special conditions attached. We can thus say that

$$F(x) = a_0(x + L) + \sum_{n=1}^{\infty} \frac{a_n}{n\pi/L} \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} \frac{b_n}{n\pi/L} \left(\cos n\pi - \cos \frac{n\pi x}{L} \right). \tag{2.3.5}$$

3. Vibrating Strings

In this chapter we will examine a vibrating string. And, surprisingly, we can also model this with a partial differential equation! Let's find out how.

3.1 What is the wave equation?

3.1.1 Deriving the wave equation

Let's suppose we have a string of length L. Its deviation from a certain position is given by $u(x,t)$ $[m]$. Here x $[m]$ denotes the horizontal position on the string, and t $[s]$ denotes time. Also, we define $\rho(x)$ $[kg/m]$ to be the mass per unit length of the string.

To derive a PDE for u, we look at a very small piece of string. This piece has length Δx and thus weight $\rho(x)\Delta x$. Now we examine all the vertical forces acting on this piece of string. First of all, there is the **tension** $T(x,t)$ $[N]$ in the string. It can be shown that T causes a vertical force on our particle of magnitude

$$F_T = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right). \tag{3.1.1}$$

By the way, the above equation is only valid for small deviations u. It lacks accuracy if the deviations/slopes of the string become very high.

Besides tension, there can also be external forces. Let's denote the **body force** $Q(x,t)$ $[m/s^2]$ as the vertical force per unit mass acting on the string. This body force then causes a force $Q(x,t)\rho(x)$ on our piece of string.

Now we have examined all the forces. The sum of the forces should of course equal to ma, or, equivalently, to $m\partial^2 u/\partial t^2$. This gives us the equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\rho(x). \tag{3.1.2}$$

This equation is still rather difficult to solve. To make things easier, we assume that the string is **perfectly elastic**. This implies that $T(x,t)$ is actually constant for the entire string. We therefore now denote it as T_0. We also assume that there are no body forces. ($Q(x,t) = 0$.) And if we then also define c $[m/s]$ such that $c^2 = T_0/\rho(x)$, then our PDE turns into

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{3.1.3}$$

The above equation is called the **one-dimensional wave equation**.

3.1.2 Boundary conditions

What kind of boundary conditions can we apply to a string? Of course, we can give the edges of the string a certain deflection $u(0,t) = f(t)$. (Or, similarly, $u(L,t) = f(t)$.) We can also give the edges a fixed slope $\partial u/\partial x(0,t) = f(t)$. In fact, if we attach the edge of the string to a (frictionless) vertical slider, then we have $\partial u/\partial x(0,t) = 0$.

We could make the situation even more complicated. Let's suppose we attach the edge of the string to a mass (with weight m $[kg]$) attached to a vertical spring (having stiffness k $[N/m]$). Let's examine the forces acting on this mass. There is of course the force of the spring. There is also the tension T caused by the cable. Furthermore, there can be an external force $g(t)$. The governing equation of the mass then becomes

$$m\frac{d^2u}{dt^2}(0,t) = -k(u(0,t) - u_E(t)) + T_0\frac{\partial u}{\partial x}(0,t) + g(t). \tag{3.1.4}$$

By the way, $u_E(t)$ is the equilibrium position of the spring. The above equation may look very complicated. But usually $g(t) = 0$. If we also assume the situation to be steady (everything stands still), then also $d^2u/dt^2(0,t) = 0$. We remain with

$$k(u(0,t) - u_E(t)) = T_0\frac{\partial u}{\partial x}(0,t). \tag{3.1.5}$$

This is quite interesting. If k equals zero, then we again deal with a horizontal slider. If, however, $k \to \infty$, then we have simply given the edge of the string a fixed position $u(0,t) = u_E(t)$.

3.1.3 Initial conditions

Next to boundary conditions, we also need initial conditions. The wave equation contains a second derivative w.r.t. time. So we need to initial conditions. Usually both the initial position and velocity are prescribed. So,

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{3.1.6}$$

3.2 Solving the wave equation using separation of variables

We will now solve the wave equation. We do this using the method of separation of variables, which we are familiar with.

3.2.1 The solving method

You might have noticed that the wave equation kind of looks like the Laplace equation we have seen earlier. Solving it also goes similar. We assume that we can write $u(x,t) = X(x)T(t)$. If we insert this into the heat equation, then we can derive that

$$\frac{d^2X}{dx^2} = -\lambda X \qquad \text{and} \qquad \frac{d^2T}{dt^2} = -\lambda c^2 T. \tag{3.2.1}$$

First we examine the left part. Together with the boundary conditions, we can find eigenfunctions $X_n(x)$. After that, we find a solution for the right part. We can combine it with the initial conditions to find our final solution for $u(x,t)$.

Well, that's easier said than done. So we demonstrate this solving method with an example.

3.2.2 Vibrating string example problem

Let's suppose we have a vibrating string with initial conditions given by equation (3.1.6). The boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$. We assume that $u(x,t) = X(x)T(t)$. It follows that $X(0) = 0$ and $X(L) = 0$. Using this, we can find that the eigenfunctions are

$$X_n(x) = \sin\sqrt{\lambda_n}x = \sin\frac{n\pi x}{L}, \qquad \text{with corresponding eigenvalues} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \tag{3.2.2}$$

with $n = 1, 2, 3, \ldots$. We can now use the values of λ_n to find the general solution for $T_n(t)$. This is

$$T_n(t) = A_n \cos\sqrt{\lambda_n}ct + B_n \sin\sqrt{\lambda_n}ct = A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}. \tag{3.2.3}$$

Our general solution for $u(x,t)$ thus becomes

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}. \tag{3.2.4}$$

We only haven't applied the initial conditions yet. However, we can use the property of orthogonality for that. We can then find the coefficients A_n and B_n. They are

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \tag{3.2.5}$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx. \tag{3.2.6}$$

And this concludes our solution.

3.2.3 Interpreting the vibrating string solution

There are several things we can learn from the solution that we just found. So we take a closer look at it. Our solution consisted of a summation of terms

$$\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right). \tag{3.2.7}$$

Every term represents a **normal mode of vibration**. It has its own **natural circular frequency**, given by

$$f_n = \frac{n\pi c}{L}. \tag{3.2.8}$$

This circular frequency is the amount of oscillations in 2π seconds. It also has its own **amplitude**, being

$$\text{Amplitude} = \sqrt{A_n^2 + B_n^2}. \tag{3.2.9}$$

Together, all these modes of vibration form the actual vibration of the string.

3.3 The method of characteristics

There is another way to solve the wave equation. It is called the method of characteristics. Let's examine this method.

3.3.1 Characteristics

Let's examine the wave equation. We can rewrite this equation in two ways, being

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0. \tag{3.3.1}$$

To write this in a more simple way, we define w and v as

$$w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} \qquad \text{and} \qquad v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}. \tag{3.3.2}$$

We can now rewrite the wave equation as two first-order partial differential equations. These equation are

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0. \tag{3.3.3}$$

That's great news, but we're not satisfied yet. We want to transform those equations into ordinary differential equations as well. To do that, we examine these equations along the lines $x(t) = x_0 + ct$ and $x(t) = x_0 - ct$, respectively. Now why would we do that? To find that out, we consider the derivative of $w(x,t)$ w.r.t. t along these lines. By using the chain rule, we can find that

$$\frac{d}{dt}w(x(t),t) = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x} = \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0. \tag{3.3.4}$$

What does this mean? It means that $dw/dt = 0$ along the lines $x = x_0 + ct$. Similarly, we can find that $dv/dt = 0$ along the lines $x = x_0 - ct$. So we have reduced the PDEs to ODEs. Any curve that reduces a PDE to an ODE is called a **characteristic**. Thus the lines we just examined are characteristics of the corresponding PDEs.

But, what's the use of all this? Well, now we know we can write $w(x,t)$ and $v(x,t)$ as

$$w(x,t) = P(x_0) = P(x - ct) \qquad \text{and} \qquad v(x,t) = Q(x_0) = Q(x + ct), \tag{3.3.5}$$

with $P(x)$ and $Q(x)$ any function. We have thus found the general solution to equation (3.3.3).

3.3.2 The solution of the wave equation

How can we use the things we just found, to solve the wave equation? That's an interesting question. To answer it, we define two new functions $F(x)$ and $G(x)$ as

$$F(x) = -\frac{1}{2c}\int P(x)dx \qquad \text{and} \qquad G(x) = \frac{1}{2c}\int Q(x)dx. \tag{3.3.6}$$

We can combine these definitions with equation (3.3.2). If we do this, we will find the general solution for $u(x,t)$. This solution is

$$u(x,t) = F(x - ct) + G(x + ct). \tag{3.3.7}$$

This holds for all functions $F(x)$ and $G(x)$.

So, what does this mean? It means that we can split the solution to $u(x,t)$ up in two parts, being $F(x-ct)$ and $G(x+ct)$. Let's examine the part $F(x-ct)$. This function is constant as $x - ct$ is constant. Now let's plot $F(x-ct)$ versus x for different times t. If both the time t and the position x increase, then the function $F(x-ct)$ remains constant. In other words, the graph simply slides to the right (the positive x-direction). And it does this with a velocity c. Similarly, we can find that the graph of $G(x+ct)$ moves to the left. It also does this with a velocity c.

So, what can we conclude from this? It means that $u(x,t)$ consists of two separate 'waves'. One wave moves to the left, while the other moves to the right with. Both do this with a velocity c.

3.3.3 Initial conditions

Now let's add initial conditions to our problem. Let's suppose that

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t} = g(x). \tag{3.3.8}$$

We can insert this into equation (3.3.7). By working things out, we can then find that

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\bar{x})d\bar{x} \qquad \text{and} \qquad G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\bar{x})d\bar{x}. \tag{3.3.9}$$

So, by using the initial conditions, we can find $F(x)$ and $G(x)$. How can we derive the solution $u(x,t)$ from this? Well, one way is to simply find $F(x-ct)$ and $G(x+ct)$, and then add them up. But (depending on the situation) there might be an easier way.

First you can simply plot $F(x-ct)$ and $G(x+ct)$ for $t=0$. (You thus actually plot $F(x)$ and $G(x)$.) Now what if you want to find the graphs of $F(x-ct)$ and $G(x+ct)$ at some time t? Well, in this case you simply shift the graph of $F(x)$ a distance ct to the right. Similarly, you shift the graph of $g(x)$ a distance ct to the left. Finally, you should add up both graphs to find the graph for $u(x,t)$.

3.3.4 Boundary conditions at $x = 0$

Previously we have not considered boundary conditions. In other words, we just assumed that our string was infintely long. This is, of course, not the case. Now let's add a boundary condition at $x=0$. We then only examine the string to the right of this boundary (with $x>0$).

We now have a slight problem. The string is only present at $x>0$. So, also the initial conditions $f(x)$ and $g(x)$ are defined only for $x>0$. This means that also $F(x)$ and $G(x)$ are defined for $x>0$. In other words, we may not insert negative variables in the functions $F(x)$ and $G(x)$. For $G(x)$, this isn't a very big problem. (We only use $G(x+ct)$. And we have $x>0$, $c>0$ and $t>0$.) However, if $x<ct$, then $F(x-ct)$ is not defined. This means that we have a problem.

To solve it, we need to use the boundary condition at $x=0$. Let's suppose we give the string a fixed position. So, $u(0,t)=0$. We can insert this into our general solution. We then find that

$$u(0,t) = F(-ct) + G(ct) = 0 \qquad \text{which implies that} \qquad F(z) = -G(-z) \quad \text{for } z < 0. \tag{3.3.10}$$

So, we have now defined the right-moving wave $F(z)$ for $z<0$. This means that our problem is solved.

But what is the physical meaning of this? It means that, once the left-moving wave $G(x+ct)$ reaches the left end, it is reflected back. The new **reflected wave** takes the shape of $-G(x)$ and moves to the right. (Note the minus sign.)

You may wonder, what would happen if the boundary condition was different? For example, let's suppose that $\partial u(0,t)/\partial t = 0$. This time we can find that

$$\frac{\partial u(0,t)}{\partial t} = -c\frac{dF}{dx}(-ct) + c\frac{dG}{dx}(ct) = 0 \qquad \text{which implies that} \qquad \frac{dF}{dx}(z) = \frac{dG}{dx}(-z) \quad \text{for } z < 0. \tag{3.3.11}$$

If we integrate the result, we can find that $F(z) = G(-z) + k$ for $z<0$, with k a constant. It can be shown that this constant is zero, which implies that $F(z) = G(-z)$, for $z<0$.

Again, we examine the physical meaning of this. Once the left-moving wave $G(x+ct)$ reaches the left end, it is reflected back. This time, the reflected wave takes the shape of $G(x)$ and moves to the right. (Note that the minus sign is gone.)

3.3.5 Other boundary conditions

Of course we can also put boundary conditions at other positions. What happens if we put a boundary condition at a right edge? Physically, exactly the same happens as when the boundary condition was at the left edge.

Let's suppose the boundary condition is $u(L,t)=0$. Once a right-moving wave $F(x-ct)$ encounters this boundary, it is reflected back to the left. Its new shape is that of the function $-F(x)$. (Note the minus sign.) Things are similar if the boundary condition is $\partial u(L,t)/\partial t = 0$. But this time the reflected wave has the shape of $F(x)$. (The minus sign is gone.)

So, you only need to remember the following. A fixed position at the edge reverses the wave (with a minus sign) when bouncing it back. A fixed slope at the edge just bounces the wave back.