Probability Theory
Exam July 2007 - Problems

Question 1

Let A and B be events in a probability space. The probability of event A is $P(A) = 1/3$ and the conditional probability of B given A^c is $P(B|A^c) = 1/4$. The probability that A or B occurs, i.e. $P(A \cup B)$, equals...

Question 2

Let E and F be two events for which the probability that at least one of them occurs is 3/4. The probability that neither E nor F occurs is...

Question 3

One tosses a fair coin twice. The two events of interest are: $A = \{\text{first toss is a head}\}$ and $B = \{\text{second toss is a head}\}$. Are A and B independent? And are they disjoint?

Question 4

On January 28, 1986 the space shuttle Challenger exploded about one minute after the launch. The cause of the disaster was explosion of the main fuel tank, caused by flames of hot gas erupting from one of the solid rocket boosters. These rocket boosters are manufactured in segments, joint together with O-rings. Each rocket booster has three O-rings and per launch two rocket boosters are used, so in total six O-rings each time. Based on data on the number of failed O-rings, available from previous launches, it was found that the probability p that an individual O-ring fails depends on the launch temperature t (in degrees Fahrenheit) according to

$$p = \frac{\exp(a + bt)}{1 + \exp(a + bt)}, \tag{1}$$

with $a = 5.085$ and $b = 0.1156$. Hence, p increases with decreasing launch temperature. At the time of the fatal launch of the Challenger, t was extremely low: 31 degrees Fahrenheit. Although the above formula is based on data for which $t > 50$ degrees Fahrenheit, let us use this formula also for $t = 31$ degrees Fahrenheit. Then, the probability of at least one O-ring failing during the 1986 Challenger launch equals...

Question 5

A candidate pilot is declared suited for the job if his length lies in between certain boundaries. Now, the length of candidate pilots is normally distributed with mean 175 cm and standard deviation 8.5 cm. The goal is to declare 10% of the candidate pilots unfit based on their lengths, in such a way that the number of too short candidates equals the number of too long candidates. Which boundaries should one choose?

Question 6

A machine fastens plastic screw-on caps onto containers of motor oil. If the machine applies more torque than the cap can withstand, the cap will break. Both the applied torque and the strength of the caps vary. The capping machine torque is a normally distributed random variable with mean $0.79Nm$ and standard deviation $0.10Nm$. The cap strength, being the torque that would break the cap, is also a normally distributed random variable with mean $1.13Nm$ and standard deviation $0.14Nm$. Assume that the cap strength and the applied torque are independent. The probability that a cap will break while being fastened by the capping machine equals...

Question 7

Let $\underline{x}_1$ and $\underline{x}_2$ be random variables. The mean and standard deviation of $\underline{x}_1$ are 5.0 and 2.9, whereas the mean and standard deviation of $\underline{x}_2$ are 13.2 and 17.6. The correlation coefficient of $\underline{x}_1$ and $\underline{x}_2$ is -0.11. The standard deviation of the random variable $\underline{y} = 0.2\underline{x}_1 + 0.8\underline{x}_2$ equals...

Question 8

We want to determine the volume of a cylinder by means of measuring the cylinder length L and radius r once. Before the actual measurement, we would like to assess the precision with which the cylinder volume can be determined. We know that the cylinder length is approximately $10cm$ and that the cylinder radius is approximately $2cm$. Assuming that the length and radius measurement are independent and have a standard deviation of $1mm$, what is the standard deviation of the volume, to a first-order approximation?

Question 9

In a certain country it is established that 0.5% of the population suffers from a certain disease. For this disease there exists a test that gives the correct diagnosis for 80% of healthy persons and for 98% of sick persons. A person is tested and found sick. The probability that the diagnosis is wrong, i.e. that the person is actually healthy, equals...

Question 10

Let $\underline{x}_1$ and $\underline{x}_2$ be independent and normally distributed random variables. The mean of both $\underline{x}_1$ and $\underline{x}_2$ is 0. The variance of both $\underline{x}_1$ and $\underline{x}_2$ equals 3. Consider the linear transformation

$$\underline{y}_1 = \underline{x}_1 + \underline{x}_2 \qquad \text{and} \qquad \underline{y}_2 = \underline{x}_1 - 2\underline{x}_2. \tag{2}$$

Then the joint PDF of $\underline{y}_1$ and $\underline{y}_2$ reads...

Question 11

Let $\underline{x}$ be an exponentially distributed random variable with parameter $\lambda = 1/5$. The conditional probability $P(x < 5|3 < x < 6)$ equals...

Question 12

The random variable $\underline{x}$ is uniformly distributed on the interval $(0, 1)$. Then the PDF of the random variable $\underline{y} = -\ln \underline{x}$ reads...

Question 13

The random variable $\underline{w}$ is uniformly distributed on the interval $(\pi, 2\pi)$. What can we say about $E(\sin(\underline{w}))$ and $\sin(E(\underline{w}))$? Which one is bigger? And if they are equal, are they also equal to zero?

Question 14

Given is the linear model $E(\underline{y}) = Ax$, $D(\underline{y}) = 4I_m$ with the $m \times 1$ matrix $A = [1, \ldots, 1]^T$. $\hat{\underline{x}}$ is the Best Linear Unbiased Estimator (BLUE) of x. There is a requirement that the variance of $\hat{\underline{x}}$ is at most equal to 0.5. How many observations m should one then at least take?

Question 15

Three points A, B and C lie on a straight line. All pairwise distances $y_1 = AB$, $y_2 = BC$ and $y_3 = AC$ are measured. Thus, the observables are $\underline{y}_i = y_i + \underline{e}_i$, whereby it may assumed that $E(\underline{e}_i) = 0$ and $D(\underline{e}_i) = 3cm^2$ for $i = 1, 2, 3$, and $C(\underline{e}_i, \underline{e}_j) = 0$ for $i \neq j$. The variance of the BLUE of y_3 is then given as...

Question 16

Given is the linear model $E(\underline{y}) = Ax$ and $D(y) = \sigma^2 I_3$, where $y = [2, 1, 1/2]^T$, $x = [x_1, x_2]^T$ and

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \tag{3}$$

Let $\hat{\underline{x}}_1$ denote the BLUE of x_1. We want the standard deviation of $\hat{\underline{x}}_1$ to be smaller than or at most equal to 1. What is the maximum allowable value of the standard deviation σ of the observations?

Question 17

Consider the Gauss-Markov model $\underline{y} \sim N_m(Ax, Q_{yy})$. Are $\hat{\underline{x}}$ and $\underline{y}$ dependent? And are $\hat{\underline{x}}$ and $\hat{\underline{e}}$ dependent?

Question 18

We want to fit a plane $y = A + Bt + Cz$ to the four points $y = 3$ at $(t, z) = (1, 1)$, $y = 6$ at $(t, z) = (0, 3)$, $y = 5$ at $(t, z) = (2, 1)$ and $y = 0$ at $(t, z) = (0, 0)$. The system of normal equations from which the least squares solution for the unknown parameters can be obtained, is...

Question 19

At $t = 0$ ESA discovers an asteroid headed directly for Earth. In the two days ($t = 1$ and $t = 2$) after the discovery they measure the distance toward the object (d) using radar. By coincidence, they also find a useable observation taken three days before the discovery ($t = 3$) and determine the distance at that moment. The measured distances are 12.6, 9.1, and 8.3 ($\times 10^6 km$) for $t = 3$, $t = 1$, and $t = 2$, respectively. Assume that the asteroid moves directly toward us with an unknown constant velocity (v) and that the position $d(0)$ at $t = 0$ is also unknown. From the least squares solution for $d(0)$ and v, we estimate the time at which the asteroid will hit the Earth. This time equals...

Question 20

For testing observable $\underline{y}$ with normal distribution, two simple hypotheses are put forward: $H_0 : \underline{y} \sim N(0, 4)$ and $H_a : \underline{y} \sim N(-3, 4)$ If the type I error probability is $\alpha = 0.05$ using a left sided critical region, which of the following values is the power of the test?

Question 21

We have four observations y_i with $i = 1, \ldots, 4$ to determine a single unknown parameter x, according to $E(y_i) = x$. The observations are $y_1 = 5.0$, $y_2 = 6.4$, $y_3 = 4.8$ and $y_4 = 7.0$. The observables, which are normally distributed, are uncorrelated and all have standard deviation $\sigma = 1.4$. Determine the squared norm of the BLUE residual vector, and check this squared norm against its nominal distribution at the 10% significance level.

The squared norm and the critical value respectively read...

Question 22

A random variable $\underline{x}$ is distributed as follows

$$\underline{x} \sim N(8, 9). \tag{4}$$

What is the probability that $x > 11$?

Question 23

The Probability Density Function of observable $\underline{y}$ is given as

$$f_y(y|x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-x)^2} \tag{5}$$

with two competing hypotheses concerning parameter x

$$H_1 : x = a_1 \qquad \text{and} \qquad H_2 : x = a_2 \tag{6}$$

with $a_2 > a_1$. The hypotheses are equally likely. One observation y is made. The best decision rule for deciding between H_1 and H_2 reads: reject H_1 (and accept H_2) if...

Question 24

Concerning an m-vector of observables $\underline{y}$ two simple hypotheses are tested against each other using the SLR-test. The hypotheses are specified as: $H_0 : \underline{y} \sim f_y(y|x = x_0)$ and $H_a : \underline{y} \sim f_y(y|x = x_a)$ with $x_a > x_0$. The observables are normally distributed, mutually uncorrelated and all have equal variance σ^2 (with σ a known value). The parameter x pertains to the location of the PDF: $E(y_i) = x$, for every $i = 1, \ldots, m$.

Let's look at the probability of incorrectly accepting H_0. What happens to it if we change the number of observations? And what if we change the standard deviation?