Chapter 1

Basics and Statistics

1.1 Models

Observations $\bar{y}$ can be related to state parameters $\bar{x}$ by a design matrix H. The design matrix contains the information of the relation between the two vectors. The result is a model:

$$\bar{y} = H\bar{x} \tag{1.1}$$

If for example, the y coordinates of a ballistic trajectory described by a second order polynomial are observed at n time instants. The model will be:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \quad ; \quad y = x_0 + x_1 t + x_2 t^2 \tag{1.2}$$

Or:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{bmatrix} 1 & \cos\omega t_1 & \sin\omega t_1 \\ \vdots & \vdots & \vdots \\ 1 & \cos\omega t_n & \sin\omega t_n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \quad ; \quad y = x_0 + x_1 \cos\omega t + x_2 \sin\omega t \tag{1.3}$$

The normal matrix $H^T H$ is used in solving methods. The following properties exist for this matrix, where the columns of U are called *left singular vectors* of H and the columns of V are called *right singular vectors* of H and Λ is a diagonal matrix with singular values, where the square root of the singular values are the eigenvalues of the normal matrix $H^T H$ and the matrix HH^T. This is a so called singular value decomposition.

$$H = U\Lambda V^T \quad ; \quad U^T U = I \quad ; \quad V^T V = I \tag{1.4}$$

$$HH^T = U\Lambda^2 U^T \quad ; \quad H^T H = V\Lambda^2 V^T \tag{1.5}$$

A decomposition can also be performed with Eigenvectors, resulting in a so called *Eigendecomposition*. Where the columns of the matrix Q contain the eigenvectors of the matrix H and the matrix E is a diagonal matrix containing the corresponding eigenvalues.

$$H = QEQ^{-1} \quad ; \quad H\boldsymbol{u} = \lambda\boldsymbol{u} \tag{1.6}$$

As a special case, for every $N \times N$ real symmetric matrix H, the eigenvectors can be chosen such that they are real, orthogonal to each other and have norm one. In this case the matrix Q is orthonormal and the inverse of the eigenvector is equal to its transpose: $Q^T = Q^{-1}$. This is the case for the normal matrix. Therefore, the matrices U and V contain the corresponding eigenvectors.

1.2 Statistics

The expectance $E[X]$ and the k^{th} order moment λ_k around zero are defined by:

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx \quad ; \quad E\left[X^k\right] = \lambda_k = \int_{-\infty}^{+\infty} x^k f(x) dx \tag{1.7}$$

The first order moment around zero is equal to the mean value. The k^{th} order moment μ_k around the mean is given below. Where the second order moment about the mean is equal to the variance μ_2.

$$E\left[(X-\lambda_1)^k\right] = \mu_k = \int_{-\infty}^{+\infty} (x-\lambda_1)^k f(x) dx \quad ; \quad E\left[(X-\lambda_1)^2\right] = \mu_2 = \int_{-\infty}^{+\infty} (x-\lambda_1)^2 f(x) dx \tag{1.8}$$

The mean can also be written as μ and the variance can also be written as the square of the standard deviation σ.

$$Variance: \ \sigma^2 = \mu_2 \quad ; \quad Mean: \ \mu = \lambda_1 \tag{1.9}$$

The covariance matrix P is a matrix which contains the standard deviations of multiple variables and the correlation between them. A property of the covariance matrix is that the transpose is equal to the matrix itself.

$$P = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1}\sigma_1\sigma_n & \rho_{n2}\sigma_2\sigma_n & \cdots & \sigma_n^2 \end{bmatrix} \quad ; \quad P^T = P \tag{1.10}$$

The covariance matrix is also related to the normal matrix $H^T H$ by its eigenvectors U and eigenvalues (matrix Λ).

$$PU = U\Lambda \quad ; \quad P = U\Lambda U^T = U^T \Lambda U \tag{1.11}$$

Using a model relating parameters x to y, the covariance matrix of y can be derived:

$$y = Hx \quad \rightarrow \quad P_y = H P_x H^T \quad \Rightarrow \quad H^{-1} P_y (H^T)^{-1} = P_x \quad \rightarrow \quad P_x = (H^T P_y^{-1} H)^{-1} \tag{1.12}$$

1.3 Linearization

Linearization is conducted when the equations that relate the observations with the state are non-linear. The state will be solved using the linearized equations in an iterative scheme.

The linearization is performed by determining the partial derivatives of the model. This means the partial derivatives of each parameter y w.r.t. each parameter x. This results in a new matrix $\tilde{H}$, which contains the partial derivatives. As a result a new linearized model is obtained which gives the relation between a small change of the parameters.

$$\boldsymbol{y} = H\boldsymbol{x} \quad ; \quad \Delta\boldsymbol{y} = \tilde{H}\Delta\boldsymbol{x} \quad with \quad \tilde{H} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} \tag{1.13}$$

1.4 Iterative solvers

To solve non-linear equations iterative solvers are used. The first step is to linearized the model. From this the following equations are obtained, where $G(\boldsymbol{x})$ is the original non-linear equation. The vector $\Delta\boldsymbol{y}$ contains the difference between the observed values of $\boldsymbol{y}$ and the computed value of $\boldsymbol{y}$, $G(\boldsymbol{x})$. Therefore, the solved value for $\Delta\boldsymbol{x}$ contains an estimate of the difference between the guess for $\boldsymbol{x}$ and the true value of $\boldsymbol{x}$.

$$\Delta\boldsymbol{y} = \tilde{H}\Delta\boldsymbol{x} \quad with \quad \boldsymbol{y} = G(\boldsymbol{x}) \quad and \quad \Delta\boldsymbol{y} = \boldsymbol{y} - G(\boldsymbol{x}) \tag{1.14}$$

Because the equations are non-linear, the vector $\Delta \boldsymbol{x}$ is generally not exact. Therefore, iterations are performed to obtain the true value of $\boldsymbol{x}$. The first step is to assume an initial guess for the value of $\boldsymbol{x}$. With this value the vector $\Delta \boldsymbol{x}$ is calculated to update the initial guess. This new value is then used to calculate a new $\Delta \boldsymbol{x}$. This is performed repeatedly until the value of $\boldsymbol{x}$ converges. The iterative scheme representing this method is given below.

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \Delta \boldsymbol{x}_i \tag{1.15}$$

Chapter 2

The Orbit Problem

The equation of motion for a satellite orbiting a body with mass M can be written as:

$$\boldsymbol{a} = -\mu \frac{1}{|r|^3} \boldsymbol{r} \tag{2.1}$$

2.1 Differential equations

The equation of motion can be rewritten as 6 first order differential equations. Without perturbations the set of differential equations become:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{\mu}{|r|^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu}{|r|^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu}{|r|^3} & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} \quad ; \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \\ V_x \\ V_y \\ V_z \end{pmatrix} \quad ; \quad \dot{\boldsymbol{x}} = \begin{pmatrix} V_x \\ V_y \\ V_z \\ a_x \\ a_y \\ a_z \end{pmatrix} \tag{2.2}$$

Integrating the differential equations gives the state of the satellite.

2.2 State Transition Matrix

The state transition matrix Ψ can also be used to obtain the state of the satellite after a time interval Δt. This is a matrix which transforms the state of the satellite to a new state after a time interval Δt. The state transition matrix is determined by performing an Euler integration on the differential equations of motion. The second equation below shows this mathematically. Where the A matrix is similar to the differential equations as described in paragraph 2.1.

$$\Psi = \frac{\partial \boldsymbol{x}(t_1)}{\boldsymbol{x}(t_0)} \quad ; \quad \Psi = A\Delta t + I \quad ; \quad A = \frac{\partial \dot{\boldsymbol{x}}(t_1)}{\boldsymbol{x}(t_0)} \quad ; \quad A = \begin{bmatrix} \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{V}} \\ \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{V}} \end{bmatrix} \tag{2.3}$$

For the orbit problem $\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{r}}$ can be written as:

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{r}} = \frac{\partial}{\partial \boldsymbol{r}} \left(-\mu \frac{1}{|r|^3} \boldsymbol{r} \right) = -\mu \left(\frac{1}{|r|^3} I_{3x3} - 3\boldsymbol{r} \frac{\boldsymbol{r}^T}{|r|^5} \right) \tag{2.4}$$

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{r}} = \frac{\mu}{|r|^5} \begin{bmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3xy & 3y^2 - r^2 & 3yz \\ 3xz & 3yz & 3z^3 - r^2 \end{bmatrix} \tag{2.5}$$

The component $\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{r}}$ is a zero matrix, because the velocity is not dependent on the coordinates. The component $\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{V}}$ is an idententity matrix. The component $\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{V}}$ can be used to include drag. Without drag this is a zero matrix.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial a_x}{\partial x} & \frac{\partial a_x}{\partial y} & \frac{\partial a_x}{\partial z} & \frac{\partial a_x}{\partial V_x} & \frac{\partial a_x}{\partial V_y} & \frac{\partial a_x}{\partial V_z} \\ \frac{\partial a_y}{\partial x} & \frac{\partial a_y}{\partial y} & \frac{\partial a_y}{\partial z} & \frac{\partial a_y}{\partial V_x} & \frac{\partial a_y}{\partial y} & \frac{\partial a_y}{\partial V_z} \\ \frac{\partial a_z}{\partial x} & \frac{\partial a_z}{\partial y} & \frac{\partial a_z}{\partial z} & \frac{\partial a_z}{\partial V_x} & \frac{\partial a_z}{\partial V_y} & \frac{\partial a_z}{\partial V_z} \end{bmatrix} \tag{2.6}$$

2.2.1 Examples of the State Transition Matrix

In general a differential equation can be propagated by using the State Transition Matrix, which is equivalent to Euler integration.

$$\Psi = \frac{\partial \boldsymbol{x}(t_1)}{\boldsymbol{x}(t_0)} \quad ; \quad \Psi = A\Delta t + I \quad ; \quad \Delta t = t_1 - t_0 \quad ; \quad A = \frac{\partial \dot{\boldsymbol{x}}(t_1)}{\boldsymbol{x}(t_0)} \tag{2.7}$$

Some examples are given for a better understanding of the State Transition Matrix.

Example 1
The following differential equation is integrated:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.8}$$

The derivative matrix A then becomes:

$$\Psi = A(t_1 - t_0) + I \quad ; \quad A = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{bmatrix}_{x_0} \quad ; \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.9}$$

And the state transition matrix:

$$\Psi = \begin{bmatrix} \Delta t + 1 & 2\Delta t & 0 \\ 0 & 3\Delta t + 1 & 4\Delta t \\ 0 & 0 & 1\Delta t + 1 \end{bmatrix} \tag{2.10}$$

Example 2
The following differential equation is integrated:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x^2 \\ y^3 \\ z \end{pmatrix} \tag{2.11}$$

The derivative matrix A then becomes:

$$A = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{bmatrix}_{x_0} \quad ; \quad A = \begin{bmatrix} 2x_0 & 6y_0^2 & 0 \\ 0 & 9y_0^2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.12}$$

And the state transition matrix:

$$\Psi = \begin{bmatrix} 2x_0\Delta t + 1 & 6y_0^2\Delta t & 0 \\ 0 & 9y_0^2\Delta t + 1 & 4\Delta t \\ 0 & 0 & 1\Delta t + 1 \end{bmatrix} \tag{2.13}$$

Chapter 3

Least Squares Methods

3.1 Least Squares

When the model is solved for x it is possible that a residual ϵ remains.

$$\bar{y} = H\bar{x} + \bar{\epsilon} \quad ; \quad \bar{\epsilon} = \bar{y} - H\bar{x} \tag{3.1}$$

The Least Squares method provides a solution for x by minimizing the square sum of the residuals. The Least Squares problem is defined by:

$$\min \bar{\epsilon}^T \bar{\epsilon} \quad ; \quad \min J \quad ; \quad J = (\bar{y} - H\bar{x})^T (\bar{y} - H\bar{x}) \tag{3.2}$$

The cost function J is minimized if the derivative with respect to the variable x is equal to zero. The derivative can be determined by using the chain rule and minding the Transpose.

$$\frac{dJ}{d\bar{x}} = 0 \quad ; \quad 0 = 2(\bar{y} - H\bar{x})^T(-H) = H^T(\bar{y} - H\bar{x}) \tag{3.3}$$

This result in the normal equations, which can be solved to determine the best estimate of $\bar{x}$. This vector will be called $\hat{x}$. The matrix $H^T H$ is also called the *normal matrix*.

$$H^T \bar{y} = H^T H \bar{x} \quad ; \quad \hat{x} = (H^T H)^{-1} H^T \bar{y} \tag{3.4}$$

The covariance of the state is determined to be the inverse of the normal matrix:

$$P_x = (H^T H)^{-1} \tag{3.5}$$

3.2 Weighted Least Squares

The Weighted Least Squares method puts more weight on the observations for which the variance is lower. So the more accurate observations. The weighting is obtained by adding a weight matrix P_y^{-1}. The new Least Squares problem becomes:

$$\min \bar{\epsilon}^T P_y^{-1} \bar{\epsilon} \quad ; \quad \min J \quad ; \quad J = (\bar{y} - H\bar{x})^T P_y^{-1} (\bar{y} - H\bar{x}) \tag{3.6}$$

The same method is applied to derive the normal equations.

$$\frac{dJ}{d\bar{x}} = 0 \quad ; \quad 0 = 2(\bar{y} - H\bar{x})^T P_y^{-1}(-H) = H^T P_y^{-1}(\bar{y} - H\bar{x}) \tag{3.7}$$

This result in the normal equations, which can be solved to determine the best estimate of $\bar{x}$. This vector will be called $\hat{x}$.

$$H^T P_y^{-1} \bar{y} = H^T P_y^{-1} H \bar{x} \quad ; \quad \hat{x} = (H^T P_y^{-1} H)^{-1} H^T P_y^{-1} \bar{y} \tag{3.8}$$

If the variance of the observations is a diagonal matrix with constant variance $P_y = \lambda I$, the covariance matrix of the estimate becomes:

$$P_x = \lambda (H^T H)^{-1} \tag{3.9}$$

3.3 Rank defficiency

A matrix is Rank defficient when there are less independent equations than unknowns. The columns are then linear dependent. The normal matrix is by definition symmetric, therefore the eigenvalues are always equal or greater than zero. The normal matrix is rank defficient if at least one eigenvalue is zero. This can be solved by adding constrains.

3.4 Constrained Least Squares

Priori information can be used to fix the rank defficieny. A constraint model is used to fix the problem. This means that equations are added to the model that solve the rank defficiency. This is modelled by the equation below. With the vector c containing the constraints and the matrix B containing the relation of the constraints with the vector x.

$$\bar{y} = H\bar{x} \quad ; \quad P_y = E(\epsilon_y \epsilon_y^T) \tag{3.10}$$

$$\bar{c} = B\bar{x} \quad ; \quad P_c = E(\epsilon_c \epsilon_c^T) \tag{3.11}$$

The cost function becomes:

$$J(\bar{x}) = \epsilon_y^T P_y^{-1} \epsilon_y + \epsilon_c^T P_c^{-1} \epsilon_c + \tag{3.12}$$

The least squares solution becomes:

$$\hat{x} = (H^T P_y^{-1} H + B^T P_c^{-1} B)^{-1} (H^T P_y^{-1} \bar{y} + B^T P_c^{-1} \bar{c}) \tag{3.13}$$

The minimum constraints solution is obtained when only the eigenvalues that are equal to zero are changed.

3.5 Batch Least Squares

If for example, Least Squares is used to obtain the position of a satellite for multiple epochs at the same time, matrices are combined in a batch. The H matrix is combined in a total A matrix which is used in the Least Squares method. The same equations hold, however the H matrix should be changed in an A matrix. The vectors y and x then contain the observations and states for all epochs sequentially added.

$$\bar{y}_{t1} = H\bar{x}_{t1} \quad ; \quad \bar{y} = A\bar{x} \quad ; \quad A = \begin{bmatrix} H & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & H \end{bmatrix} \tag{3.14}$$

This leads to a sparse matrix (matrix with a lot of zeros) and needs to be solved with special solvers.

Chapter 4

Kalman Filter

The Kalman Filter is a sequential computation algoritm that combines observations and a physical model to obtain a better estimate of the state.

Observations from a GPS satellite result in pseudoranges ρ. The pseudorange can be calculated using:

$$\rho = G(\boldsymbol{x}_k) = \sqrt{(x - x_{gps})^2 + (y - y_{gps})^2 + (z - z_{gps})^2} \tag{4.1}$$

A linearized model is used that relates the observations with the state. The observations ($\rho = \boldsymbol{Y}_k$) are related with the state through the following linearized model:

$$\boldsymbol{y}_k = \tilde{H}\boldsymbol{x}_k + \epsilon_k \quad ; \quad \boldsymbol{y}_k = \boldsymbol{Y}_k - G(\bar{\boldsymbol{x}}_k) \quad ; \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \\ V_x \\ V_y \\ V_z \end{pmatrix} \quad ; \quad \tilde{H} = \frac{\partial G(\bar{\boldsymbol{x}}_k, t_k)}{\partial \boldsymbol{x}} \tag{4.2}$$

The observations ($\rho = \boldsymbol{Y}_k$) also have a covariance matrix which is determined by the accuracy of the measurements. This matrix is called R_k.

$$P_y = R_k = E(\epsilon\epsilon^T) \tag{4.3}$$

At first the Kalman filter is initialized with an initial guess of the state $\boldsymbol{x}_0$. The first step is to integrate or propagate the equations of motion to a new state $\bar{\boldsymbol{x}}_k$ using the state of the last time step. This is conducted using the State Transition Matrix Ψ or another integration procedure. The new propagated state is used to compute the range with respect to the gps satellite $G(\bar{\boldsymbol{x}}_k)$. The difference between the computed value and the observed value is then computed, which will be used in the Kalman filter update.

$$\boldsymbol{y}_k = \boldsymbol{Y}_k - G(\bar{\boldsymbol{x}}_k) \tag{4.4}$$

The covariance matrix of the state $\bar{P}_k$ is propagated from the covariance matrix of the previous timestep P_j, with the use of the state transition matrix.

$$\bar{P}_k = \Psi_{k,j} P_j \Psi_{k,j}^T \tag{4.5}$$

The propagated estimation of the state $\bar{\boldsymbol{x}}_k$ and the observations Y_k are used for the Kalman filter update, which is conducted by the following equations. The result is a value for the *Kalman Gain* K_k, the updated state $\hat{x}_k$ and an updated covariance matrix P_k.

$$\hat{x} = \bar{x}_k + K_k y_k \quad ; \quad K_k = \bar{P}_k \tilde{H}_k^T (\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k)^{-1} \quad ; \quad P_k = (I - K_k \tilde{H}_k)\bar{P}_k \tag{4.6}$$

4.1 Noise Compensation

Because the variances in the covariance matrix of the state P_k reduce with eacht timestep, the filter becomes more convinced of the propagated state. Therefore, the covariance matrix P_k, which acts as some kind of weight will become to strong compared to the covariance matrix of the observations R_k. This results in the fact that after a certain time, the Kalman filter doesn't correct the state propagation anymore with the use of the observations. An increasing error will be the result.

Adding noise can compensate for this effect by ensuring that the covariance matrix of the state doesn't get to strong. When the state is propagated a noise vector $\bar{u}$ is added. The addition is done with the use of a matrix Γ.

The matrix Γ transforms the noise vector to a vector that is added to the propagated state. If the noise vector contains random values generated using a variance that is equal for each state, the matrix Γ can assign weights to addition of noise to each state parameter. If no weights have to be used, this can be taken as the Identity Matrix.

$$\bar{x}_{k+1} = \Psi x_k + \Gamma u_k \quad ; \quad Q_k = \lambda I \quad ; \quad \Gamma = I \tag{4.7}$$

The addition of noise is included in the update of the covariance matrix of the state by using the Q matrix. The matrix Q is a diagonal matrix, which contains the variances of the noise in the noise vector u. This matrix can also be seen as the covariance matrix of u, because the noise is uncorrelated.

$$\bar{P}_{k+1} = \Psi P_k \Psi^T + \Gamma Q_k \Gamma^T \tag{4.8}$$

Chapter 5

Reference systems, coordinates and time

5.1 Reference systems

A reference system is defined by its coordinate system and its time system.

Coordinates can be represented in different ways. The most common are:

- Geocentric coordinates (global) - this is what we normally use to represent ITRF coordinates (e.g. ITRF 2008 station coordinates). Geocentric coordinates may also be presented as pure spherical coordinates
- Geodetic coordinates (global) - the purpose is to represent the coordinates relative to a defined reference ellipsoid.
- Topocentric coordinates (local) - the purpose is to identify in the field where we find an object in the neighborhood or in space.
- Map coordinates (local and global) - the purpose is to present something on a map in an acceptable form.
  - Topographic map projection for military purposes
  - Mercator maps for the school atlas
  - UTM projections (various uses)
  - Mollweide, Lambert, cone projections etc.

5.2 Earth rotation and Time

The Earth's rotation with respect to the inertial frame yields *sidereal time*. The sidereal day length is 23h 56m 4.1s . Our 24 hour takes into account one extra rotation of the Earth in its orbit around the sun.

5.2.1 Time systems

- Solar time: Time defined by the angular distance covered by the Sun on the celestial sphere after its last crossing of the observer's celestial meridian.

- Sidereal time: Time defined by the angular distance covered by the vernal equinox on the celestial sphere after its last crossing of the observer's celestial meridian.
- Atomic time (TAI): Time based on the analysis of about 200 frequency standards (atomic clocks) maintained by several countries to keep a unit of time as close to the ideal SI second as possible (defined in terms of Caesium 133 transitions).
- Universal time (UT / UT1): A standardised mean solar time, based on a fictitious mean Sun that moves at a uniform rate eastward along the celestial equator.
- Universal time coordinated (UTC): A hybrid standard of time, of which the progression is determined by atomic time (TAI), but leap seconds are introduced, when needed, to keep up with Universal Time (UT1).

5.2.2 Earth Rotation

Because of the gravitation of the Moon and the Sun on the Earth, the Earth experiences a gravitational torque. A dominant characteristic of the changes in inertial orientation is a conical motion of the Earth's angular velocity vector in inertial space, reffered to as *precession* with a period of 26000 years. As a consequence the Earths rotation axis is not directed perpendicular to the ecliptic but under a different angle. Superimposed on this conical motion are small oscillations, reffered to as *nutation* with a period of 18.6? years.

The Earths rotation also has an anual variation of ± 1 $msec.$

Polar motion is the phenomenon where Earth's crust is shifted relative ot the rotation axis in an inertial space.

5.2.3 Clocks

Setup clock

The quality of the clock is expressed in a so called **Allen Variance** diagram. On the vertical axis df/f is displayed and on the horizontal axis the integrated time.

Clock errors are obtained by comparing clocks. The ideal set-up is:

1. Quartz clock against atomic clock
2. Quartz against Quartz
3. Atomic against Atomic

The estimated clock error is then expressed in:

$$\frac{\Delta f}{f} = \frac{\Delta t}{t} \tag{5.1}$$

5.3 Transformation from ECI to ECF

The transformation from the *Earth Center Inertial* to the *Earth Center Fixed* has to account for the deviations from the mean rotation of the Earth. So it has to account for Nutation N, Precession P, True sidereal time S' and Polar motion W. The transformation is conducted by multiplying the coordinates with a transformation matrix:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ECF} = T \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ECI} \qquad with \quad T = WS'NP \tag{5.2}$$

Chapter 6

Orbit Determination

The problem of Orbit Determination (OD) is to determine the state of the spacecraft at any future epoch. The ordinary differential equations that describe the motion of the spacecraft are known approximately. Therefore there will be errors as the integration progresses. As a result the spacecraft needs to be tracked to from ground stations whose positions are known exactly. A best estimate of the spacecraft is determined to predict the future state. Predicting the state of a spacecraft is referred to as generating the ephemeris of a satellite.

Inaccuracies in the estimated state vector:

- Approximations in the method of orbit improvement and in the mathematical model (difficult)
- Errors in the observations (difficult)
- Errors in the computational procedure (moderate)

Errors in the numerical integration procedure:

- Dynamical model (difficult)
- Computer truncation and roundoff effects (easy)

Chapter 7

Observations

7.1 Common observations

Common observations are the range ρ, elevation θ, range-rate$\dot{\rho}$ and the elevation rate $\dot{\theta}$. The ideal values can be calculated using the following equations:

$$\rho = |\boldsymbol{X} - \boldsymbol{X}_{GPS}| = \sqrt{(x - x_{gps_s})^2 + (y - y_{gps_s})^2 + (z - z_{gps_s})^2} \tag{7.1}$$

$$\tan\theta = (Y - Y_{gps})/(X - X_{gps}) \tag{7.2}$$

$$\dot{\rho} = \frac{\boldsymbol{\rho}\dot{\boldsymbol{\rho}}}{\rho} = \frac{1}{\rho}\left[(X - X_{gps})(\dot{X} - \dot{X}_{gps}) + (Y - Y_{gps})(\dot{Y} - \dot{Y}_{gps})\right] \tag{7.3}$$

$$\dot{\theta} = \frac{1}{\rho^2}\left[(X - X_{gps})(\dot{Y} - \dot{Y}_{gps}) - (\dot{X} - \dot{X}_{gps})(Y - Y_{gps})\right] \tag{7.4}$$

7.2 Measurements

7.2.1 One-way

A one-way measurement is when the signal is send from a transmitter and received at antoher loaction by a receiver, so the light wave travels only one way.

7.2.2 Two-way

A two-way measurement is conducted by a travelling electromagnetic wave which is send, reflected by a r̈eceiveränd received.

7.3 Realization

Observations are made using *electromagnetic waves* (light) which travel with a constant speed c. The frequency f and wavelength λ are then related by:

$$c = \lambda \times f \qquad c = 299792458\ m/s \tag{7.5}$$

The range can be determined by determining the time that the wave has travelled. This results in a pseudorange, which is not an ideal range because multiple errors result in an error in the range estimate.

$$\rho = ct_f \tag{7.6}$$

7.3.1 Atmospheric effects

Electro magnetic waves travel at the speed of light c in vacuum. The propagating speed reduces when it travels through a medium. The **group velocity** of the wave is the velocity at which a photon travels through the Electromagnetic wave. Thus this is the speed at which information will travel. The phase velocity of a wave is the rate at which the phase of the wave propagates in space, which can be larger than the speed of light. This is not in contradiction with relativity, because the information is not propagated at this rate.

Refraction
Refraction of a light beam, thus the bending of can be calculated using Snellius Law:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \tag{7.7}$$

Signal delay
The signal delay can be determined using the equations below. Where v is the group velocity and Δt the time delay.

$$s = ct + v(t + \Delta t) \quad \rightarrow \quad (t + \Delta t) = \frac{c}{v} t = nt \quad \rightarrow \quad \Delta t = (n-1)t \tag{7.8}$$

The effect on the distance can be determined by:

$$\Delta d = c\Delta t = c(n-1)t = (n-1)s \quad \rightarrow \quad \Delta d = \int_{s=0}^{l} (n-1) ds \tag{7.9}$$

Ionosphere
The ionopshere free:

$$\rho_{L1} = \rho_c + \frac{\alpha}{f_{L1}^2} \quad ; \quad \rho_{L2} = \rho_c + \frac{\alpha}{f_{L2}^2} \tag{7.10}$$

$$f_{L1}^2 \rho_{L1} - f_{L2}^2 \rho_{L2} = (f_{L1}^2 - f_{L1}^2)\rho + (\alpha - \alpha) \quad ; \quad \rho = \frac{f_{L1}^2 \rho_{L1} - f_{L2}^2 \rho_{L2}}{f_{L1}^2 - f_{L1}^2} \tag{7.11}$$

7.3.2 Antenna divergence

The divergence angle of an antenna can be used to determine the resolution of the measurement. The diverence angle α can be calculated and used to determine the resolution. A triangle can be drawn with an angle α and a length L to determine the resolution. The other length is then the minimum resolution r of the antenna. Objects that are smaller than this resolution r can thus not be observed.

$$\alpha = \frac{\lambda}{D} \quad ; \quad \tan \alpha = \frac{r}{L} \tag{7.12}$$

7.4 Errors

In general the errors that can occur are:

- Multipath
- Atmospheric (ionospheric dispersion & tropospheric (humidity, pressure)) (1)(2) as long as the signals travel through the same region in the ionosphere/troposphere
- Relativity
- Interference
- Instrument errors (noise & path delay)

- Orbital parameters of GPS satellites (Ephemeris)
- Clock error (receiver (1) & transmitter (2))
- Antenna offset (receiver (1) & transmitter (2))

With the following comments:

1. this error can be corrected by differencing the signals from two satellites to one ground station
2. this error can be corrected by differencing the signals from one satellite to two ground stations

7.4.1 Relativity

Relativity consists of two parts. The first part is *Time dilation*, also known as *special relativity*.

$$dt = \frac{2d}{c'} \quad ; \quad c' = \sqrt{c^2 - V^2} \quad ; \quad f' = \frac{1}{dt} = \frac{c'}{2d} = \frac{\sqrt{c^2 - V^2}}{2d} = \frac{c}{2d}\frac{\sqrt{c^2 - V^2}}{c} \tag{7.13}$$

Results in:

$$f' = f\sqrt{1 - (v/c)^2} \quad ; \quad \frac{df}{f} = \sqrt{1 - (v/c)^2} \tag{7.14}$$

The doppler effect is modelled by:

$$f + df = \left(1 + \frac{dv}{c}\right) f \quad ; \quad \frac{df}{f} = \frac{dt}{t} = \frac{dv}{c} \tag{7.15}$$

The graviational redshift or *general Relativity* is derived in the following way:

$$\frac{df}{f} = \frac{dv}{c} = \frac{gdh}{c^2} = -\frac{d\Phi}{c^2} \quad ; \quad \Phi = -\frac{\mu}{r} \quad ; \quad \frac{d\Phi}{dh} = g = \frac{\mu}{r^2} \quad ; \quad gdh = -\frac{d\Phi r^2}{\mu}\frac{\mu}{r^2} \tag{7.16}$$

The graviational redshift or *general Relativity* can be modelled by:

$$\frac{df}{f} = \frac{gdh}{c^2} \tag{7.17}$$

A clock ticks slower when it is closer to the Earth (Gravitation is higher). A clock ticks faster when it is moving slower. So when comparing a satellite flying over the Earth with a clock on the surface on the Earth. The clock of the satellite the ticks slower because the satellite has a higher speed which is due to the *time dilation effect*. However the clock of the satellite ticks faster, because it is farther from the Earth due to the *Gravitational redshift*.

7.5 Differencing

Differencing can be used to mitigate some of the errors. Differencing implies that the ranges of two satellites are subtracted so that the relative position of one of the two receivers w.r.t. the other receiver can determined.

Spatially correlated errors are removed. Noise and Multipath increase

Chapter 8

GPS

8.1 Position determination using iterative Least Squares

The coordinates of an observable are determined using the position of the available GPS satellites and the pseudoranges ρ. For each available GPS satellite s, the range equation is set up.

$$G(\boldsymbol{X}) = \rho_s = \sqrt{(x - x_{gps_s})^2 + (y - y_{gps_s})^2 + (z - z_{gps_s})^2} \quad ; \quad \boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{8.1}$$

This equation is a nonlinear equation, therefore it cannot be solved by linear algebra. The equation is linearized and solved iteratively to get a solution. The linearized equation will be:

$$\boldsymbol{l} = A\boldsymbol{\delta X} \quad ; \quad \boldsymbol{l} = \begin{pmatrix} \delta\rho_1 = \rho_1 - G(\boldsymbol{X_1}) \\ \vdots \\ \delta\rho_s = \rho_s - G(\boldsymbol{X_s}) \end{pmatrix} \quad ; \quad \boldsymbol{\delta X} = \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} \tag{8.2}$$

Equation 8.2 is solved by using least Squares if there are more satellites available than estimated parameters. An initial guess is needed to start the iteration. The vector $\boldsymbol{l}$ contains the difference between the computed pseudorange $G(X)$ and the observed pseudorange ρ. The vector $\boldsymbol{\delta X}$ contains values which are used to update the coordinates from the last iteration step. The matrix A contains the partial derivatives of the range with respect to the coordinates.

$$A = \begin{bmatrix} \frac{\partial \rho_{gps_1}}{\partial x} & \frac{\partial \rho_{gps_1}}{\partial y} & \frac{\partial \rho_{gps_1}}{\partial z} \\ \vdots & \vdots & \vdots \\ \frac{\partial \rho_{gps_s}}{\partial x} & \frac{\partial \rho_{gps_s}}{\partial y} & \frac{\partial \rho_{gps_s}}{\partial z} \end{bmatrix} \tag{8.3}$$

With the following partial derivatives:

$$\frac{\partial \rho}{\partial x} = \frac{1}{\rho}(x - x_{gps}) \quad ; \quad \frac{\partial \rho}{\partial y} = \frac{1}{\rho}(y - y_{gps}) \quad ; \quad \frac{\partial \rho}{\partial z} = \frac{1}{\rho}(z - z_{gps}) \tag{8.4}$$

The coordinates are solved by an iterative process.

$$\boldsymbol{X_{i+1}} = \boldsymbol{X_i} + \boldsymbol{\delta X_i} \tag{8.5}$$

If the receiver clock is not accurate enough 4 satellites are needed to estimate the clock error. The clock error can be estimated by changing the A matrix and the vector $\boldsymbol{X}$.

$$A = \begin{bmatrix} \frac{\partial \rho_{gps_1}}{\partial x} & \frac{\partial \rho_{gps_1}}{\partial y} & \frac{\partial \rho_{gps_1}}{\partial z} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \rho_{gps_s}}{\partial x} & \frac{\partial \rho_{gps_s}}{\partial y} & \frac{\partial \rho_{gps_s}}{\partial z} & 1 \end{bmatrix} \quad ; \quad \boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \\ t_{clk} \end{pmatrix} \tag{8.6}$$

8.2 Dilution of Precision

To determine the positional accuracy the constellation or geometry of the group of satellites of which the signals are being received, is very important. The Position Dilution of Precision (PDOP) only depends on the geometry of the satellites, which is determined by, the position of the satellites and how many satellites you can see. It is a value of probability for the geometrical effect on GPS accuracy. The effect of the geometry of the satellites on the accuracy is graphically shown in the figure below. As the figures illustrate, the accuracy is better when the relative position of the satellites position is optimal. This means that the satellites are not near each other, and the relative position must minimize the area of uncertainty.

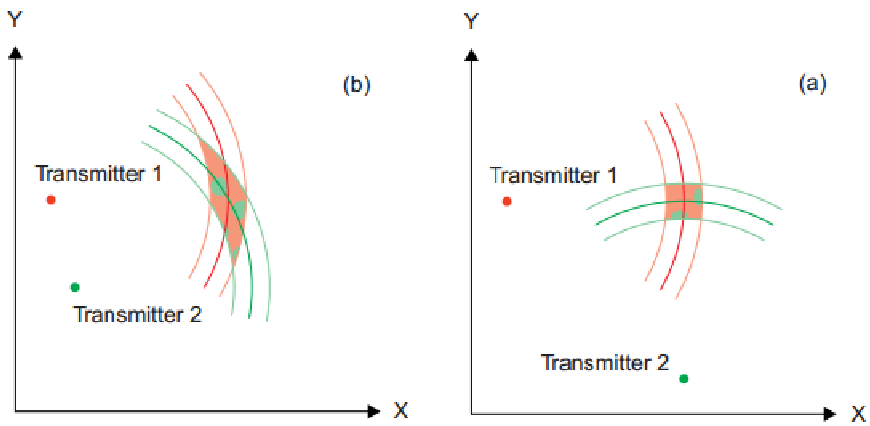

Figure 8.1: Graphical illustration of PDOP

The PDOP is determined by using the standard deviations of the state in each direction.

$$PDOP = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \tag{8.7}$$

The standard deviations can be determined from the covariance matrix of the state P_x.

$$P_x = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix} \tag{8.8}$$

Chapter 9

Applications

9.1 Cryosat-2

Cryosat 2 is an ocean altimeter. It has an advanced radar altimeter system and an optimised orbit to optimise the ground track. The satellite has a star tracker, a laser retroflector and a DORIS antenna.

The Precision Orbit Determination is performed by using 50 Radar beacons for the DORIS System. This provides Doppler data. 10 stations are used to track the satellite using Laser Ranging. The required accuracy was 2-3 cm.

Three star trackers provide the attitude of the spacecraft.

9.2 Grace

The gravity change can be fitted onto a curve using the observations y.

$$y = x_0 + x_1 t + x_2 \cos \omega t + +x_3 \cos \omega t \tag{9.1}$$

The following model is used to fit the data:

$$y = Hx \quad ; \quad \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{bmatrix} 1 & t_1 & \cos \omega t_1 & \sin \omega t_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & \cos \omega t_n & \sin \omega t_n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{9.2}$$

The amplitudes of the sin and cos can be used to derive the periodic changes in the gravity. The coefficient x_1 says something about the gravity change rate, so continous change. The factor x_0 is used to account for a constant difference.