AE4304 Examination April 2011

Question 1

The frequency response of the first order system $H_{\bar{w}\bar{y}}(\omega) = \bar{Y}(\omega)/\bar{W}(\omega)$ has a "low pass filter" characteristic, see Figure 1. Its asymptotes equal '1' from zero frequency until $\omega = 1/\gamma$ and then go down with a -20 dB/decade slope.

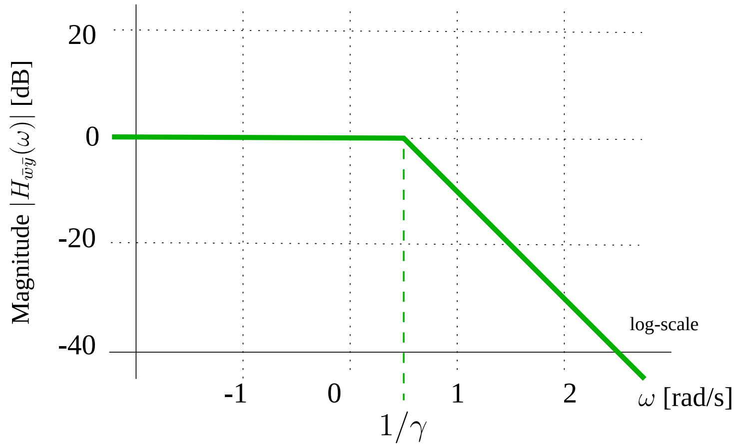

Figure 1: Bode plot (magnitude only) of first order system dynamics $H_{\bar{w}\bar{y}}(\omega)$

The input Gaussian white noise has equal power over all frequencies; its auto-covariance function is a Dirac pulse. The system dynamics "filter out" the higher frequencies of the input signal, with the filter bandwidth approximately at $1/\gamma$ [rad/s]. When the system allows more higher frequencies to pass through, i.e., $1/\gamma$ is higher (so when γ decreases), the more the output will look like the white noise input. The auto-covariance function of the output $\bar{y}$ will then more closely resemble a Dirac pulse.

Hence, the lowest γ corresponds to Figure (a) and the highest γ corresponds to Figure (c).

Answer **B**

Question 2

Proof:

$$\int_{t=-\infty}^{\infty} x^2(t)\mathrm{d}t = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} |X(\omega)|^2 \mathrm{d}\omega$$

To prove Parceval's theorem we will first use the inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}\mathrm{d}\omega$$

Starting from the left side, we can substitute the inverse Fourier transform for $x(t)$:

$$\int_{t=-\infty}^{\infty} x^2(t)\mathrm{d}t = \int_{t=-\infty}^{\infty} x(t)\cdot\left[\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}\mathrm{d}\omega\right]\mathrm{d}t$$

The time-related function $x(t)$ can be brought into the inner ω integral:

$$= \frac{1}{2\pi} \int_{t=-\infty}^{\infty} \int_{\omega=-\infty}^{\infty} x(t)X(\omega)e^{j\omega t}\mathrm{d}\omega\mathrm{d}t$$

We change the order of integration:

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \int_{t=-\infty}^{\infty} x(t)X(\omega)e^{j\omega t}\mathrm{d}t\mathrm{d}\omega$$

Now the frequency-related function $X(\omega)$ can be taken out of the inner t integral:

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega) \underbrace{\int_{t=-\infty}^{\infty} x(t)e^{j\omega t}\mathrm{d}t}_{=X(-\omega)} \mathrm{d}\omega$$

We recognize that the inner integral equals $X(-\omega)$ and we quite miraculously obtain:

$$\int_{t=-\infty}^{\infty} x^2(t)\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)X(-\omega)\mathrm{d}\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\mathrm{d}\omega \qquad \textbf{qed}$$

Question 3

The figure shows that the probability density function is perfectly symmetric about $x = 0$. The total area always equals 1.

Hence: $Pr\{\bar{x} \leq 0\} = Pr\{\bar{x} \geq 0\} = 0.5$. The probability is equal to the area under the probability density function.

$$Pr\{-1 \leq \bar{x} \leq 0\} = 0.15 \cdot 1 + \frac{1}{2} \cdot 0.15 \cdot 1 = 0.225$$

$$Pr\{\bar{x} \geq -1\} = Pr\{-1 \leq \bar{x} \leq 0\} + Pr\{\bar{x} \geq 0\} = 0.225 + 0.5 = 0.725$$

Answer **C**

Question 4

Remember that for an even periodical function ($x(-t) = x(t)$), its Fourier series only has cosine components. In a Fourier transform, the cosines correspond to the real part.

For an odd periodical function ($x(-t) = -x(t)$), its Fourier series only has sine components. In a Fourier transform, the sines correspond to the imaginary part.

Here, the Fourier transform of a signal $x(t)$, $X(\omega)$, is purely imaginary. Clearly then, $x(t)$ must be odd.

Answer **A**

Basically, this is sufficient for answering this (meant to be simple) question. If you want, feel free to also prove it mathematically; no additional points are given, but it will lead to a very big smile on the lecturer's face.

<u>Proof</u>:
We start our proof by elaborating on the definition of the Fourier transform:

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t, \qquad \text{using Euler } e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t): \\
&= \int_{-\infty}^{\infty} x(t)(\cos(\omega t) - j\sin(\omega t))\mathrm{d}t \\
&= \int_{-\infty}^{0} x(t)(\cos(\omega t) - j\sin(\omega t))\mathrm{d}t + \int_{0}^{\infty} x(t)(\cos(\omega t) - j\sin(\omega t))\mathrm{d}t
\end{aligned}$$

Substitute $\sigma = -t$ in the first integral on the right-hand side (then: $t = -\sigma$, so $\mathrm{d}t = -\mathrm{d}\sigma$), and the integral limits change from (t from $-\infty$ to 0) to (σ from ∞ to 0):

$$= \int_{\sigma=\infty}^{0} x(-\sigma)(\cos(-\omega\sigma) - j\sin(-\omega\sigma))\mathrm{d}(-\sigma) + \int_{0}^{\infty} x(t)(\cos(\omega t) - j\sin(\omega t))\mathrm{d}t$$

Now $\cos(-u) = \cos(u)$ (cosine is even) and $\sin(-u) = -\sin(u)$ (sine is odd); further, we change the integral limits on the first integral on the right-hand side, which yields a minus sign that, multiplied with $-\mathrm{d}\sigma$ yields $\mathrm{d}\sigma$:

$$= \int_{0}^{\infty} x(-\sigma)(\cos(\omega\sigma) + j\sin(\omega\sigma))\mathrm{d}\sigma + \int_{0}^{\infty} x(t)(\cos(\omega t) - j\sin(\omega t))\mathrm{d}t$$

Substitute $\sigma = t$ in the integral on the left hand side and re-arrange both integrals:

$$= \int_{0}^{\infty} (x(t) + x(-t))\cos(\omega t)\mathrm{d}t - j\int_{0}^{\infty} (x(t) - x(-t))\sin(\omega t)\mathrm{d}t$$

Clearly, when $x(t)$ is odd, i.e., $x(-t) = -x(t)$, the first integral becomes zero and the result is:

$$X(\omega) = -2j\int_{0}^{\infty} x(t)\sin(\omega t)dt$$

which is an imaginary and odd function of ω. **qed**

Question 5

The relationship between all input signals ($\bar{u}$, $\bar{n}_1$, $\bar{n}_2$) and the output signal $\bar{y}$ can be found using two methods. The first method is by analyzing directly the closed loop relationship between the inputs and outputs from the block diagram. The second is by calculating the relationships mathematically step by step.

Method 1 First, rearrange the block diagram by moving the $\bar{n}_1$ input and H_3 block after the input $\bar{u}$, see Figure 2. This makes it easier to find the open and closed loop paths. Don't forget the minus sign!!

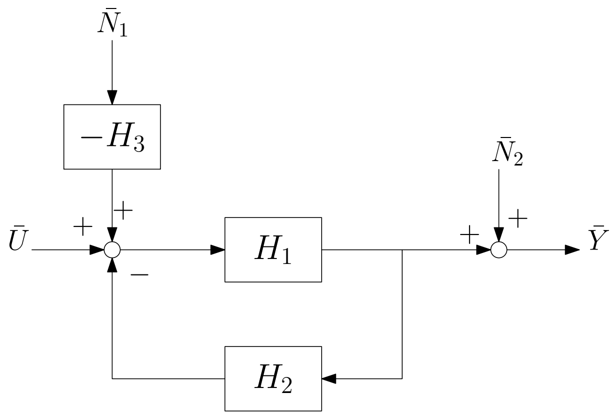

Figure 2: Block diagram re-arranged

Remember that the closed loop transfer function between an input signal and an output signal can be easily found using:

$$H_{cl}(\omega) = \frac{DC}{1 + OL},$$

where DC is the 'direct connection' between the input and output signals that you consider, and OL is the 'open' loop. Here, $OL = H_1 \cdot H_2$.

Note that in our case, Figure 2, $\bar{n}_2$ is added at the very end, *not* in the loop, so $\bar{y}$ is simply the result of adding $\bar{n}_2$ to what comes out of the closed loop before the addition.

Find the relationship between input $\bar{u}$ and output $\bar{y}$:

$$H_{\bar{u}\bar{y}}(\omega) = \frac{\bar{Y}(\omega)}{\bar{U}(\omega)} = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

Find the relationship between input $\bar{n}_1$ and output $\bar{y}$. Note the minus sign, because H_3 is moved after the input $\bar{u}$:

$$H_{\bar{n}_1\bar{y}}(\omega) = \frac{\bar{Y}(\omega)}{\bar{N}_1(\omega)} = \frac{-H_3(\omega)H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

Now the complete relationship between in- and outputs can be described as:

$$\bar{Y}(\omega) = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}\bar{U}(\omega) - \frac{H_1(\omega)H_3(\omega)}{1 + H_1(\omega)H_2(\omega)}\bar{N}_1(\omega) + \bar{N}_2(\omega)$$

Method 2 From the block diagram, the relations between the signals can also be found as follows, using $\bar{x}_1$ and $\bar{x}_2$ as defined in Figure 3.

$$\bar{Y}(\omega) = \bar{N}_2(\omega) + \bar{X}_1(\omega) \qquad (*)$$

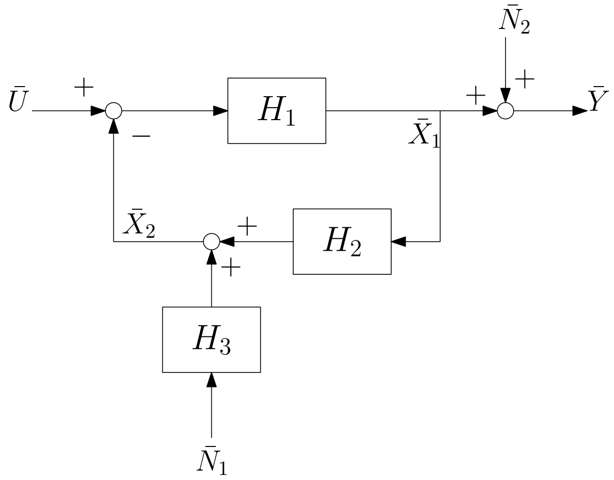

Figure 3: Block diagram definitions

$$\bar{X}_2(\omega) = H_2(\omega)\bar{X}_1(\omega) + H_3(\omega)\bar{N}_1(\omega) \qquad \text{(**)}$$

$$\bar{X}_1(\omega) = H_1(\omega)(\bar{U}(\omega) - \bar{X}_2(\omega)) \qquad \text{(***)}$$

Substitute (**) in (***):

$$\bar{X}_1(\omega) = H_1(\omega)(\bar{U}(\omega) - H_2(\omega)\bar{X}_1(\omega) - H_3(\omega)\bar{N}_1(\omega))$$

$$= H_1(\omega)\bar{U}(\omega) - H_1(\omega)H_2(\omega)\bar{X}_1(\omega) - H_1(\omega)H_3(\omega)\bar{N}_1(\omega)$$

Rearrange:

$$(1 + H_1(\omega)H_2(\omega))\bar{X}_1(\omega) = H_1(\omega)\bar{U}(\omega) - H_1(\omega)H_3(\omega)\bar{N}_1(\omega)$$

$$\bar{X}_1(\omega) = \frac{H_1(\omega)\bar{U}(\omega) - H_1(\omega)H_3(\omega)\bar{N}_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

Substitute this result in (*):

$$\bar{Y}(\omega) = \bar{N}_2(\omega) + \frac{H_1(\omega)\bar{U}(\omega) - H_1(\omega)H_3(\omega)\bar{N}_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

$$\bar{Y}(\omega) = \underbrace{\frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}}_{\text{“}dynamics''} \underbrace{\bar{U}(\omega)}_{\text{“}signal''} - \underbrace{\frac{H_1(\omega)H_3(\omega)}{1 + H_1(\omega)H_2(\omega)}}_{\text{“}dynamics''} \underbrace{\bar{N}_1(\omega)}_{\text{“}signal''} + \bar{N}_2(\omega)$$

Methods 1 and 2 should yield the same result. Note that you should always get an equation that is the addition of terms that each consist of a “dynamics” part multiplied with a “signal”.[1] When you obtain “dynamics” parts that also consist of “signal”-components, something has gone terribly wrong somewhere! Its better to start all over again in that case.

[1]Think for yourself: where are the “dynamics” in case of $\bar{N}_2(\omega)$?

Methods 1 and 2 continued Now, the found relationship can be expressed as follows:

$$\bar{Y}(\omega) = \Diamond(\omega)\bar{U}(\omega) + \triangle(\omega)\bar{N}_1(\omega) + \bar{N}_2(\omega),$$

where:

$$\Diamond(\omega) = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)},$$

and

$$\triangle(\omega) = -\frac{H_1(\omega)H_3(\omega)}{1 + H_1(\omega)H_2(\omega)}.$$

Then:

$$\bar{Y}(-\omega) = \Diamond(-\omega)\bar{U}(-\omega) + \triangle(-\omega)\bar{N}_1 + \bar{N}_2(\omega)$$

The Power Spectral Density of the output signal $\bar{y}$ can now be derived:

$$S_{\bar{y}\bar{y}}(\omega) = E\{\bar{Y}(\omega)\bar{Y}(-\omega)\}$$

Since $\bar{u}$, $\bar{n}_1$ and $\bar{n}_2$ are all uncorrelated, the expectation of all the cross-multiplications between them will become zero and can be neglected. What remains is:

$$S_{\bar{y}\bar{y}}(\omega) = E\{\Diamond(\omega)\Diamond(-\omega)\bar{U}(\omega)\bar{U}(-\omega) + \triangle(\omega)\triangle(-\omega)\bar{N}_1(\omega)\bar{N}_1(-\omega) + \bar{N}_2(\omega)\bar{N}_2(-\omega)\}$$

The transfer functions $\Diamond(\omega)$ and $\triangle(\omega)$ are deterministic and can thus be moved outside of the brackets of the expectation:

$$S_{\bar{y}\bar{y}}(\omega) = |\Diamond(\omega)|^2 E\{\bar{U}(\omega)\bar{U}(-\omega)\} + |\triangle(\omega)|^2 E\{\bar{N}_1(\omega)\bar{N}_1(-\omega)\} + E\{\bar{N}_2(\omega)\bar{N}_2(-\omega)\}$$

And from this equation we can easily see that all expectations on the right-hand side are themselves the auto-PSDs of the input signals:

$$S_{\bar{y}\bar{y}}(\omega) = |\Diamond(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) + |\triangle(\omega)|^2 S_{\bar{n}_1\bar{n}_1}(\omega) + S_{\bar{n}_2\bar{n}_2}(\omega)$$

which is our final result.[2]

Question 6

[a] Start the proof by Discrete Fourier Transforming the (auto) circular covariance function $C_{xx}[r]$:

[2] You can further practice with this example through looking for instance at the situation where $\bar{u}$ and $\bar{n}_1$ are correlated, and $\bar{n}_2$ uncorrelated with the other two inputs.

$$I_{xx}[k] = \sum_{r=0}^{N-1} C_{xx}[r] e^{-jk\frac{2\pi}{N}r} = \sum_{r=0}^{N-1} \left[\frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n+r] \right] e^{-jk\frac{2\pi}{N}r}$$

Change the order of the summation:

$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{r=0}^{N-1} x[n+r] e^{-jk\frac{2\pi}{N}r}$$

Expand the exponential term:

$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{r=0}^{N-1} x[n+r] e^{-jk\frac{2\pi}{N}(r+n-n)}$$
$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \left[\sum_{r=0}^{N-1} x[n+r] e^{-jk\frac{2\pi}{N}(r+n)} \right] e^{jk\frac{2\pi}{N}n}$$

The part between brackets is the DFT of $x[n]$, and $x[n]$ is assumed to be *periodic* in N (circular covariance used here!), so $X[k]$ can be substituted:

$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n] X[k] e^{jk\frac{2\pi}{N}n}$$

Re-arrange:

$$= \frac{1}{N} \underbrace{\left[\sum_{n=0}^{N-1} x[n] e^{jk\frac{2\pi}{N}n} \right]}_{=X[-k]} X[k]$$

Again the DFT is found, and substitution yields the final result:

$$I_{xx}[k] = \frac{1}{N} X[-k] X[k] \qquad \textbf{qed}$$

[b] Note that the DFTs $X[k]$ and $X[-k]$ can be expressed as complex tensors:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}, \text{ where } e^{-jk\frac{2\pi}{N}n} = \cos(k\frac{2\pi}{N}n) - j\sin(k\frac{2\pi}{N}n)$$
$$X[-k] = \sum_{n=0}^{N-1} x[n] e^{jk\frac{2\pi}{N}n}, \text{ where } e^{+jk\frac{2\pi}{N}n} = \cos(k\frac{2\pi}{N}n) + j\sin(k\frac{2\pi}{N}n)$$

When multiplying these DFTs, the imaginary parts will cancel out, and the periodogram becomes real valued for all k.

[c] This can be explained similarly as in the former question. This time the imaginary parts of the DFTs $X[k]$ and $Y[k]$ will not cancel out, because the imaginary part of the complex tensors are likely to have very different values. The chance that different signals have exactly the opposite complex tensor at the same frequency index k is very small. Hence, the cross periodogram will, generally speaking, always be complex valued.

[d] The periodogram $I_{\bar{x}\bar{y}}[k]$ is an estimate of the discrete-time power spectral density $\hat{S}_{\bar{x}\bar{y}}[k]$. From the relationship between in and output with a transfer function $\bar{Y}(\omega) = H(\omega)\bar{U}(\omega)$ an expression can be found which relates the power spectral densities of the input and output.

$$\bar{Y}(\omega) = H(\omega)\bar{U}(\omega)$$

So:

$$\bar{Y}(\omega)\bar{U}(-\omega) = H(\omega)\bar{U}(\omega)\bar{U}(-\omega)$$

Take expectation (only 'works' on the stochastic elements):

$$E\{\bar{Y}(\omega)\bar{U}(-\omega)\} = H(\omega)E\{\bar{U}(\omega)\bar{U}(-\omega)\}$$

And we see the auto- and cross-PSD appearing:

$$S_{\bar{u}\bar{y}}(\omega) = H(\omega)S_{\bar{u}\bar{u}}(\omega)$$

Hence we obtain:

$$H(\omega) = \frac{S_{\bar{u}\bar{y}}(\omega)}{S_{\bar{u}\bar{u}}(\omega)} \qquad (*)$$

Now, in discrete-time we work with the periodograms,

$$S_{\bar{u}\bar{y}}[k] = \frac{1}{N}\bar{U}[-k]\bar{Y}[k], \qquad \text{and} \qquad S_{\bar{u}\bar{u}}[k] = \frac{1}{N}\bar{U}[-k]\bar{U}[k]$$

and substituting these in (*) will yield:[3]

$$H(\omega) = H[k] = \frac{\bar{Y}[k]}{\bar{U}[k]}$$

[e] The discrete frequency array ω_k is defined as $k\frac{2\pi}{N}\frac{1}{\Delta t}$ (always remember to check the units! (here [rad/s])).

Question 7

When sampling a continuous-time signal $y(t)$ to obtain a discrete-time signal $y[k]$, the continuous-time Fourier transform of the latter simply consists of an infinite number of aliases (copies of the original CTFT spectrum $Y(f)$) that occur at integer multiples of the sampling frequency f_s, scaled with f_s. This follows directly from theory (Chapter 4 of the lecture notes).

[3]Note that the correction needed when moving from discrete-time spectra to it continuous-time counterpart – multiply with sample time $\triangle t$ – is irrelevant here since we take the quotient of two discrete-time spectra, which both need to be corrected by the same amount.

Hence, in this question we obtain at each integer multiple of 20 [Hz] (the sampling frequency) a copy of the original CTFT spectrum $Y(f)$, multiplied with 20 (the sampling frequency). Figure (b) is therefore the right answer.[4]

Answer **B**

Question 8

We start by inverse Laplace transforming the system transfer function $H(s)$ to the system governing differential equation:

$$H(s) = \frac{\bar{Y}(s)}{\bar{U}(s)} = \frac{1}{1 + \dfrac{2\zeta}{w_0}s + (\dfrac{s}{w_0})^2}$$

So:

$$\frac{1}{w_0^2}s^2\bar{Y}(s) + \frac{2\zeta}{w_0}s\bar{Y}(s) + \bar{Y}(s) = \bar{U}(s)$$

Transform to the time-domain (the system is at rest, all initial conditions are zero):

$$\frac{1}{w_0^2}\ddot{\bar{y}}(t) + \frac{2\zeta}{w_0}\dot{\bar{y}}(t) + \bar{y}(t) = \bar{u}(t)$$

Transform the differential equation to state space; define $\bar{x}_1(t) = \dot{\bar{y}}(t)$ and $\bar{x}_2(t) = \bar{y}(t)$:

$$\dot{\bar{x}}_1(t) = \ddot{\bar{y}}(t) = -2\zeta w_0\dot{\bar{y}}(t) - w_0^2\bar{y}(t) + w_0^2\bar{u}(t) = -2\zeta w_0\bar{x}_1(t) - w_0^2\bar{x}_2(t) + w_0^2\bar{u}(t)$$

$$\dot{\bar{x}}_2(t) = \dot{\bar{y}}(t) = \bar{x}_1(t)$$

And we obtain the state space description that we need:

$$\begin{bmatrix}\dot{\bar{x}}_1(t)\\ \dot{\bar{x}}_2(t)\end{bmatrix} = \begin{bmatrix}-2\zeta w_0 & -w_0^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\end{bmatrix} + \begin{bmatrix}w_0^2\\ 0\end{bmatrix}\bar{u}(t)$$
$$\bar{y}(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}\bar{u}(t)$$

Substitute (A, B, W) in the Lyapunov equation and solve for the unknown steady-state covariance matrix $C_{\bar{x}\bar{x},ss}$:

$$AC_{\bar{x}\bar{x},ss} + C_{\bar{x}\bar{x},ss}A^T + BWB^T = 0, \text{ with } W = 1, \text{ so:}$$

$$\begin{bmatrix}-2\zeta w_0 & -w_0^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix} + \begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}-2\zeta w_0 & 1\\ -w_0^2 & 0\end{bmatrix} + \begin{bmatrix}w_0^2\\ 0\end{bmatrix}\begin{bmatrix}w_0^2 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Matrix multiplication:

[4] The Nyquist rate of the signal is 2·5=10 [Hz], so we are sampling at a sufficiently high frequency, no overlap of the aliases, also known as aliasing, occurs.

$$\begin{bmatrix} -2\zeta w_0 C_{11} - w_0^2 C_{21} & -2\zeta w_0 C_{12} - w_0^2 C_{22} \\ C_{11} & C_{12} \end{bmatrix} + \begin{bmatrix} -2\zeta w_0 C_{11} - w_0^2 C_{12} & C_{11} \\ -2\zeta w_0 C_{21} - w_0^2 C_{22} & C_{21} \end{bmatrix} + \begin{bmatrix} w_0^4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrix addition:

$$\begin{bmatrix} w_0^4 - 4\zeta w_0 C_{11} - w_0^2 (C_{12} + C_{21}) & C_{11} - 2\zeta w_0 C_{12} - w_0^2 C_{22} \\ C_{11} - 2\zeta w_0 C_{21} - w_0^2 C_{22} & C_{12} + C_{21} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This gives us four equations with four unknowns:

$$w_0^4 - 4\zeta w_0 C_{11} - w_0^2 (C_{12} + C_{21}) = 0 \quad (1)$$
$$C_{11} - 2\zeta w_0 C_{12} - w_0^2 C_{22} = 0 \quad (2)$$
$$C_{11} - 2\zeta w_0 C_{21} - w_0^2 C_{22} = 0 \quad (3)$$
$$C_{12} + C_{21} = 0 \quad (4)$$

Inserting (4) in (1) allows us to solve for C_{11}, adding (2) and (3) and then substituting (4) allows us to solve for C_{22}. We obtain:

$$C_{\bar{x}\bar{x},ss} = \begin{bmatrix} \frac{w_0^3}{4\zeta} & 0 \\ 0 & \frac{w_0}{4\zeta} \end{bmatrix}$$

From the earlier defined transformation $\bar{y}(t) = \bar{x}_2(t)$ the variance of the output signal is found:

$$\sigma_{\bar{y}}^2 = \sigma_{\bar{x}_2}^2 = C_{22} = \frac{w_0}{4\zeta}$$ **qed**.

Question 9

The output PSD can be calculated as:

$$S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 \cdot S_{\bar{w}\bar{w}}(\omega) = |H(\omega)|^2 \cdot W$$

The value of the output PSD at zero frequency (ω=0 [rad/s]) can be easily obtained:

$$S_{\bar{y}\bar{y}}(0) = |H(0)|^2 \cdot W = 2$$

PSD(2) has a value of 1 at the smallest frequency and can therefore be directly eliminated. The cut-off frequency of the second order system is determined by the natural frequency, ω_0=1 [rad/s]. Since the output PSD is simply equivalent to the system dynamics *squared*, its cut-off frequency also lies around this frequency of 1 [rad/s]. PSD(4) can therefore be eliminated.

Finally, we look at the slope of the curve for frequencies beyond the cut-off frequency. Note that both scales are logarithmic. Then, because $H(\omega)$ is a second order system the slope of its frequency response is “-2”. The PSD of the output signal is equivalent to $|H(\omega)|^2$ so must have a slope of “-4”. When we look at the two remaining PSDs (1 and 3), over the frequency range ω=1 to 100 [rad/s] (2 decades), the slope of “-4” should result in a magnitude change of 10^{-8}. Hence, PSD(3) is the correct spectrum.

Answer **C**

That's all Folks!