AE4304 Examination April 2012

Question 1

[1] True. This can be proved as follows:

$$R_{\bar{x}\bar{y}}(\tau) = E\{\bar{x}(t)\bar{y}(t+\tau)\}$$

Set $v = t + \tau$, then $t = v - \tau$, substitute:

$$R_{\bar{x}\bar{y}}(\tau) = E\{\bar{x}(v-\tau)\bar{y}(v)\} = E\{\bar{y}(v)\bar{x}(v-\tau)\}$$

Now, we are considering stationary processes, so v can be replaced with any other time, including t:

$$R_{\bar{x}\bar{y}}(\tau) = E\{\bar{y}(t)\bar{x}(t-\tau)\} = E\{\bar{y}(t)\bar{x}(t+(-\tau))\} = R_{\bar{y}\bar{x}}(-\tau)$$ **qed**

[2] True. The proof is very similar to **[1]**. *Note that you need to explain your answer on the exam in more elaborate terms than this!*

[3] True. All auto-functions are even functions (can be proved similar as in **[1]**).

[4] True. This can be proven as follows:

$$K_{\bar{x}\bar{x}}(0) = \frac{C_{\bar{x}\bar{x}(0)}}{\sigma_{\bar{x}}^2} = \frac{E\{(\bar{x}(t)-\mu_{\bar{x}})(\bar{x}(t+0)-\mu_{\bar{x}})\}}{\sigma_{\bar{x}}^2} = \frac{E\{(\bar{x}(t)-\mu_{\bar{x}})^2\}}{\sigma_{\bar{x}}^2} = \frac{\sigma_{\bar{x}}^2}{\sigma_{\bar{x}}^2} = 1$$

qed

Question 2

This derivation follows directly from the lecture notes, and lecture slides.

The signal $\bar{x}(t)$ can be approximated by $\hat{\bar{x}}(t)$ through a Fourier Series expansion, an array of N sine and cosine functions:

$$\hat{\bar{x}} = \sum_{k=0}^{N-1}[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)] = a_0 + \sum_{k=1}^{N-1}[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)]$$

To find the coefficients a_k and b_k a quadratic loss function J can be defined:

$$J(a_0,\ldots,a_{N-1},b_1,\ldots,b_{N-1}) = \int_{t_0}^{t_0+T}[\bar{x}(t) - \hat{\bar{x}}(t)]^2 \mathrm{d}t$$

$$= \int_{t_0}^{t_0+T}[\bar{x}(t) - (a_0 + \sum_{k=1}^{N-1}[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)])]^2 \mathrm{d}t$$

Minimizing J will yield the best fit, with necessary conditions:

$$\frac{\delta J}{\delta a_0} = \frac{\delta J}{\delta a_l} = \frac{\delta J}{\delta b_l} = 0$$

for $l = 1, 2, \ldots, N-1$.

Starting with the parameter a_0. If $\frac{\delta J}{\delta a_0} = 0$ then:

$$\int_{t_0}^{t_0+T} 2\left[\bar{x}(t) - (a_0 + \sum_{k=1}^{N-1}[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)])\right](-1)\mathrm{d}t = 0$$

$$\int_{t_0}^{t_0+T} \bar{x}(t)\mathrm{d}t = \int_{t_0}^{t_0+T} a_0\mathrm{d}t + \int_{t_0}^{t_0+T}\sum_{k=1}^{N-1}[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)]\mathrm{d}t$$

$$\int_{t_0}^{t_0+T} \bar{x}(t)\mathrm{d}t = Ta_0 + \sum_{k=1}^{N-1}\left(a_k\int_{t_0}^{t_0+T}\cos(k\omega_0 t)\mathrm{d}t + b_k\int_{t_0}^{t_0+T}\sin(k\omega_0 t)\mathrm{d}t\right)$$

The two integrals on the right hand side are zero for all k which yields:

$$\boxed{a_0 = \frac{1}{T}\int_{t_o}^{t_0+T}\bar{x}(t)dt}$$

qed

We continue with deriving an expression for variables a_k. If $\frac{\delta J}{\delta a_l} = 0$ (for all ℓ) then:

$$\int_{t_0}^{t_0+T} 2\left[\bar{x}(t) - (a_0 + \sum_{k=1}^{N-1}[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)])\right](-\cos(l\omega_0 t))\mathrm{d}t = 0$$

$$\int_{t_0}^{t_0+T} \bar{x}(t)\cos(l\omega_0 t)\mathrm{d}t = \int_{t_0}^{t_0+T} a_0\cos(l\omega_0 t)\mathrm{d}t + \int_{t_0}^{t_0+T}\sum_{k=1}^{N-1}[a_k\cos(k\omega_0 t)\cos(l\omega_0 t) + b_k\sin(k\omega_0 t)\cos(l\omega_0 t)]\mathrm{d}t$$

$$\int_{t_0}^{t_0+T} \bar{x}(t)\cos(l\omega_0 t)\mathrm{d}t = a_0\int_{t_0}^{t_0+T}\cos(l\omega_0 t)\mathrm{d}t + \sum_{k=1}^{N-1}\left(a_k\int_{t_0}^{t_0+T}\cos(k\omega_0 t)\cos(l\omega_0 t)\mathrm{d}t + b_k\int_{t_0}^{t_0+T}\sin(k\omega_0 t)\cos(l\omega_0 t)\mathrm{d}t\right)$$

The first integral on the right hand-side is zero for all l. The last integral term on the right hand side is also zero for all k and l, using the fact that all basic cosine and sine functions are orthogonal. The second integral term is 0 if $k \neq l$ and equals $\frac{T}{2}$ if $k = l$. Hence:

$$\int_{t_0}^{t_0+T} \bar{x}(t)\cos(l\omega_0 t)\mathrm{d}t = a_l\frac{T}{2}$$

Substituting k for l (both are just our integer indices):

$$\boxed{a_k = \frac{2}{T}\int_{t_0}^{t_0+T}\bar{x}(t)\cos(k\omega_0 t)\mathrm{d}t}$$

qed

Similarly, for b_k we can derive:

$$\boxed{b_k = \frac{2}{T}\int\limits_{t_0}^{t_0+T} \bar{x}(t)\sin(k\omega_0 t)\mathrm{d}t}$$

qed

Note that in the 2nd year BSc course *Instrumentation & Signals* another (and perhaps simpler and more elegant) derivation is given, which you can also follow here.

Question 3

Starting from the time-domain and the definition of convolution:

$$z(t) = x(t) * y(t) = \int\limits_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)\mathrm{d}\tau$$

Fourier transforming:

$$\begin{aligned}
Z(\omega) &= F\{x(t) * y(t)\} = \int\limits_{t=-\infty}^{\infty}\left[\int\limits_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)\mathrm{d}\tau\right] e^{-j\omega t}\mathrm{d}t \\
&= \int\limits_{t=-\infty}^{\infty}\left[\int\limits_{\tau=-\infty}^{\infty} x(\tau)y(t-\tau)\mathrm{d}\tau\right] e^{-j\omega(t-\tau)}e^{-j\omega\tau}\mathrm{d}t && \text{now, change the order of integration} \\
&= \int\limits_{\tau=-\infty}^{\infty}\int\limits_{t=-\infty}^{\infty} x(\tau)y(t-\tau)e^{-j\omega(t-\tau)}e^{-j\omega\tau}\mathrm{d}t\mathrm{d}\tau && \text{take all } \tau\text{-functions out of inner } t \text{ integral} \\
&= \int\limits_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\underbrace{\int\limits_{t=-\infty}^{\infty} y(t-\tau)e^{-j\omega(t-\tau)}\mathrm{d}(t-\tau)}_{=Y(\omega)}\mathrm{d}\tau && \text{for } \sigma = t-\tau \text{ the CTFT of } y(\sigma) \text{ is found} \\
&= \int\limits_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}Y(\omega)\mathrm{d}\tau && Y(\omega) \text{ can be taken out of the } \tau\text{-integral} \\
&= \underbrace{\int\limits_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\mathrm{d}\tau}_{=X(\omega)}\cdot Y(\omega) = X(\omega)Y(\omega)
\end{aligned}$$

qed

Question 4

The input signal to the system, white noise, has equal power over all frequencies. Consider the frequency response (magnitude only) of a second order transfer function (Figure 1). If the damping ratio decreases, the peak at the natural frequency increases. When the damping ratio becomes smaller and smaller and eventually becomes zero, the output of the second order system to an input *at* the natural frequency will become an undamped sine function; the autocorrelation of a sine is a cosine function.

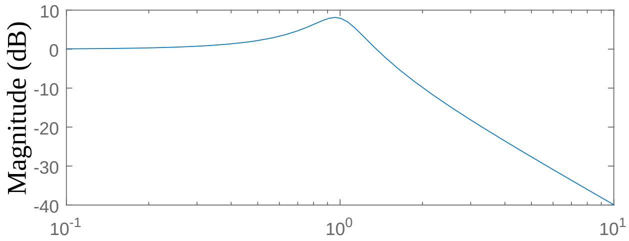

Figure 1: 2nd order system dynamics (horizontal axis is ω, in [rad/s])

The white noise signal, containing energy at *all* frequencies, will excite the system also at the natural frequency, yielding an undamped sine when ζ is extremely small.

Therefore, ζ_1 corresponds to the smallest damping ratio and ζ_3 to the largest damping ratio.

Answer **B**

Question 5

This question is the same as in the exam of April 2011, please look there.

Question 6

The Nyquist rate of this signal is 2·5=10 Hz, so it should be sampled at a frequency higher than 10 Hz.[1] Now, the current sampling frequency equals 5 Hz, which is too small. Remember that sampling does two things: (i) it creates an infinite number of copies (also called aliases) of the original spectrum, located at each integer multiple of the sampling frequency, and (ii) all these copies are *scaled* with the sampling frequency.

Here, aliasing will occur, the sampling frequency is too low causing the copies of the original spectrum located at the integer multiples of the sampling frequency to overlap, see Figure 2. The aliases are scaled with the sampling frequency: their maximum values become 10·5=50. The resulting CTFT of the sampled signal is then the summation of all these individual aliases, which becomes a straight line at $b = 50$, see Figure 3.

Question 7

Remember:

$$S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 S_{\bar{w}\bar{w}}(\omega) = |H(\omega)|^2 W$$

The value of the PSD at zero frequency ($\omega = 0$ [rad/s]) can be easily calculated:

[1] In this case, since $Y(f)$=0 at 5 Hz, sampling at 10 Hz would be just sufficient.

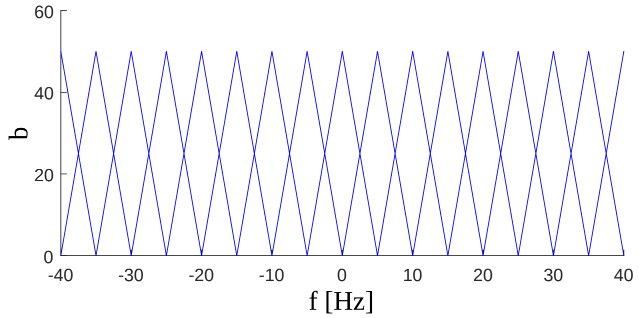

Figure 2: CTFT of the sampled signal: all individual aliases (figure shows only the positive frequencies)

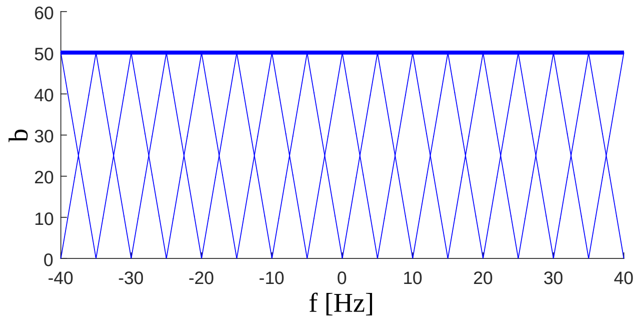

Figure 3: CTFT of the sampled signal: final result (positive frequencies only)

$$S_{\bar{y}\bar{y}}(0) = |H(0)|^2 \cdot 2 = 2,$$

so we see that PSD(3) is incorrect. Further, the cut-off frequency of the first order system appears at a frequency of $\frac{1}{\tau} = 0.5$ [rad/s], eliminating PSD(4) (and again PSD(3) for that matter!). Finally, the slope of a first order system is "-1" (when the Bode magnitude plot is shown on a log-log scale, like here). Hence, the slope of the output signal PSD should be "-2", eliminating PSD(1). The correct PSD is therefore PSD(2).

Answer **B**

Question 8

The relationship between all input signals (which are just two here, $\bar{u}$ and $\bar{n}$) and the output signal $\bar{y}$ can be found using two methods. The first method is by analysing directly the closed loop relationship between the inputs and outputs from the block diagram. The second is by calculating the relationships mathematically, step by step.

Method 1 First, rearrange the block diagram by moving the $\bar{N}$ input and H_3 block before the loop, see Figure 4. This makes it perhaps a little easier to find the open and closed loop paths. Remember that the closed loop transfer function between an input and output can be found by where DC is the 'direct connection' between that input and output, and OL is the 'open loop':

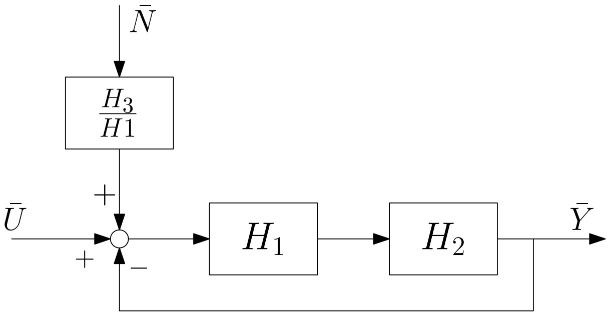

Figure 4: Block diagram re-arranged

$$H_{cl}(\omega) = \frac{DC}{1+OL}$$

Find the relationship between input $\bar{U}(\omega)$ and output $\bar{Y}(\omega)$:

$$H_{\bar{u}\bar{y}}(\omega) = \frac{\bar{Y}(\omega)}{\bar{U}(\omega)} = \frac{H_1(\omega)H_2(\omega)}{1+H_1(\omega)H_2(\omega)}$$

Find the relationship between input $\bar{N}(\omega)$ and output $\bar{Y}(\omega)$:

$$H_{\bar{n}\bar{y}}(\omega) = \frac{\bar{Y}(\omega)}{\bar{N}(\omega)} = \frac{H_2(\omega)H_3(\omega)}{1+H_1(\omega)H_2(\omega)}$$

The complete relationship between the output and all inputs can be described as:

$$\bar{Y}(\omega) = \frac{H_1(\omega)H_2(\omega)}{1+H_1(\omega)H_2(\omega)}\bar{U}(\omega) + \frac{H_2(\omega)H_3(\omega)}{1+H_1(\omega)H_2(\omega)}\bar{N}(\omega)$$

Method 2 From the block diagram the relations between the signals can be found as follows:

$$\bar{X}(\omega) = H_1(\omega)(\bar{U}(\omega) - \bar{Y}(\omega)) \qquad (*)$$
$$\bar{Y}(\omega) = H_2(\omega)(H_3(\omega)\bar{N}(\omega) + \bar{X}(\omega)) \qquad (**)$$

Substituting (*) in (**) yields:

$$\bar{Y}(\omega) = H_2(\omega)(H_3(\omega)\bar{N}(\omega) + H_1(\omega)(\bar{U}(\omega) - \bar{Y}(\omega)))$$
$$= H_2(\omega)H_3(\omega)\bar{N}(\omega) + H_1(\omega)H_2(\omega)\bar{U}(\omega) - H_1(\omega)H_2(\omega)\bar{Y}(\omega)$$

Re-arranging:

$$(1 + H_1(\omega)H_2(\omega))\bar{Y}(\omega) = H_1(\omega)H_2(\omega)\bar{U}(\omega) + H_2(\omega)H_3(\omega)\bar{N}(\omega)$$

And the same relationship is found as in Method 1:

$$\bar{Y}(\omega) = \underbrace{\frac{H_1(\omega)H_2(\omega)}{1+H_1(\omega)H_2(\omega)}}_{\text{``}dynamics''}\underbrace{\bar{U}(\omega)}_{\text{``}signal''} + \underbrace{\frac{H_2(\omega)H_3(\omega)}{1+H_1(\omega)H_2(\omega)}}_{\text{``}dynamics''}\underbrace{\bar{N}(\omega)}_{\text{``}signal''}$$

Note that one should always end up with something in the form of summations of 'dynamics' multiplied with 'signals'; when these two mix up somewhere, something has gone terribly wrong and it is better to start all over again!

Methods 1 and 2 continued... Now, to save time, the relationship that we just derived can be expressed as follows:

$$\bar{Y}(\omega) = \square(\omega)\bar{U}(\omega) + \triangle(\omega)\bar{N}(\omega),$$

where:

$$\square(\omega) = \frac{H_1(\omega)H_2(\omega)}{1 + H_1(\omega)H_2(\omega)},$$

and:

$$\triangle(\omega) = \frac{H_2(\omega)H_3(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

The conjugate is then described by:

$$\bar{Y}(-\omega) = \square(-\omega)\bar{U}(-\omega) + \triangle(-\omega)\bar{N}(-\omega)$$

Now the PSD of the output signal can be easily calculated:

$$\begin{aligned}
S_{\bar{y}\bar{y}}(\omega) &= E\{\bar{Y}(\omega)\bar{Y}(-\omega)\} \\
&= E\{\square(\omega)\square(-\omega)\bar{U}(\omega)\bar{U}(-\omega) + \square(\omega)\triangle(-\omega)\bar{U}(\omega)\bar{N}(-\omega) + \triangle(\omega)\square(-\omega)\bar{N}(\omega)\bar{U}(-\omega) + \\
&\quad \triangle(\omega)\triangle(-\omega)\bar{N}(\omega)\bar{N}(-\omega)\} \\
&= |\square(\omega)|^2 E\{\bar{U}(\omega)\bar{U}(-\omega)\} + \square(\omega)\triangle(-\omega)E\{\bar{U}(\omega)\bar{N}(-\omega)\} + \triangle(\omega)\square(-\omega)E\{\bar{N}(\omega)\bar{U}(-\omega)\} + \\
&\quad |\triangle(\omega)|^2 E\{\bar{N}(\omega)\bar{N}(-\omega)\}
\end{aligned}$$

Filling in all auto- and cross-PSDs that we see in this equation:

$$S_{\bar{y}\bar{y}}(\omega) = |\square(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) + \square(\omega)\triangle(-\omega)S_{\bar{n}\bar{u}}(\omega) + \triangle(\omega)\square(-\omega)S_{\bar{u}\bar{n}}(\omega) + |\triangle(\omega)|^2 S_{\bar{n}\bar{n}}(\omega)$$

Note that because $\bar{u}$ and $\bar{n}$ are correlated, none of the cross-PSDs are zero. They all are maintained in the equations!

Question 9

Start by transforming the transfer function to the differential equation (note that I skipped the bars over the signal symbols to save typing effort):

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{(1 + s\tau_1)(1 + s\tau_2)}$$

$$(1 + (\tau_1 + \tau_2)s + \tau_1\tau_2 s^2)Y(s) = U(s)$$

Transform to the time-domain:

$$\tau_1\tau_2\ddot{y}(t) + (\tau_1 + \tau_2)\dot{y}(t) + y(t) = u(t)$$

Transform the differential equations to state space: $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$:

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{y}(t) = -\tfrac{\tau_1+\tau_2}{\tau_1\tau_2}\dot{y}(t) - \tfrac{1}{\tau_1\tau_2}y(t) + \tfrac{1}{\tau_1\tau_2}u(t) = -\tfrac{\tau_1+\tau_2}{\tau_1\tau_2}x_2(t) - \tfrac{1}{\tau_1\tau_2}x_1(t) + \tfrac{1}{\tau_1\tau_2}u(t)$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{\tau_1\tau_2} & -\frac{\tau_1+\tau_2}{\tau_1\tau_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{\tau_1\tau_2}\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}u(t)$$

Substitute A, B and W in the Lyapunov equation and solve for the steady-state covariance matrix $C_{xx,ss}$:

$AC_{xx,ss} + C_{xx,ss}A^T + BWB^T = 0$, with $W = 1$ so

$$\begin{bmatrix}0 & 1\\ -\frac{1}{\tau_1\tau_2} & -\frac{\tau_1+\tau_2}{\tau_1\tau_2}\end{bmatrix}\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix} + \begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}0 & -\frac{1}{\tau_1\tau_2}\\ 1 & -\frac{\tau_1+\tau_2}{\tau_1\tau_2}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{\tau_1\tau_2}\end{bmatrix}\begin{bmatrix}0 & \frac{1}{\tau_1\tau_2}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Matrix multiplication:

$$\begin{bmatrix}C_{21} & C_{22}\\ -\frac{1}{\tau_1\tau_2}C_{11} - \frac{\tau_1+\tau_2}{\tau_1\tau_2}C_{21} & -\frac{1}{\tau_1\tau_2}C_{12} - \frac{\tau_1+\tau_2}{\tau_1\tau_2}C_{22}\end{bmatrix} + \begin{bmatrix}C_{12} & -\frac{1}{\tau_1\tau_2}C_{11} - \frac{\tau_1+\tau_2}{\tau_1\tau_2}C_{12}\\ C_{22} & -\frac{1}{\tau_1\tau_2}C_{21} - \frac{\tau_1+\tau_2}{\tau_1\tau_2}C_{22}\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \frac{1}{(\tau_1\tau_2)^2}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Matrix summation:

$$\begin{bmatrix}C_{21} + C_{12} & C_{22} - \frac{1}{\tau_1\tau_2}C_{11} - \frac{\tau_1+\tau_2}{\tau_1\tau_2}C_{12}\\ C_{22} - \frac{1}{\tau_1\tau_2}C_{11} - \frac{\tau_1+\tau_2}{\tau_1\tau_2}C_{21} & \frac{1}{(\tau_1\tau_2)^2} - \frac{1}{\tau_1\tau_2}(C_{12} + C_{21}) - 2\frac{\tau_1+\tau_2}{\tau_1\tau_2}C_{22}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

This gives us four equations with four unknowns (C_{11}, C_{12}, C_{21} and C_{22}):

$$C_{21} + C_{12} = 0 \quad (1)$$
$$C_{22} - \tfrac{1}{\tau_1\tau_2}C_{11} - \tfrac{\tau_1+\tau_2}{\tau_1\tau_2}C_{12} = 0 \quad (2)$$
$$C_{22} - \tfrac{1}{\tau_1\tau_2}C_{11} - \tfrac{\tau_1+\tau_2}{\tau_1\tau_2}C_{21} = 0 \quad (3)$$
$$\tfrac{1}{(\tau_1\tau_2)^2} - \tfrac{1}{\tau_1\tau_2}(C_{12} + C_{21}) - 2\tfrac{\tau_1+\tau_2}{\tau_1\tau_2}C_{22} = 0 \quad (4)$$

Inserting (1) into (4) allows C_{22} to be computed; adding (2) and (3) and then substituting (1) allows C_{11} to be computed. This yields:

$$C_{xx,ss} = \begin{bmatrix} \frac{1}{2(\tau_1+\tau_2)} & 0 \\ 0 & \frac{1}{2\tau_1\tau_2(\tau_1+\tau_2)} \end{bmatrix}$$

From the earlier defined transformation $y(t) = x_1(t)$ the variance of the output signal is found:

$$\sigma_y^2 = \sigma_{x_1}^2 = C_{11} = \frac{1}{2(\tau_1+\tau_2)}$$ **qed**

That's all Folks!