Atmospheric Flight Dynamics
Example Exam 1 – Solutions

1 Question

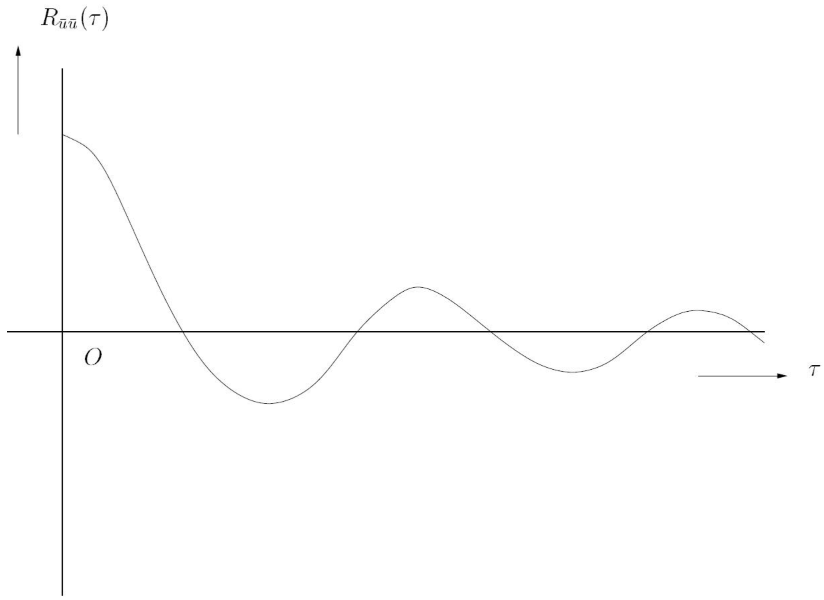

Figure 1: Product function $R_{\bar{u}\bar{u}}(\tau)$

In figure 1 the product function $R_{\bar{u}\bar{u}}(\tau)$ of the stationary stochastic process $\bar{u}$ is given. What can be said about the properties of the stochastic variable $\bar{u}$?

(a) It is white noise.

(b) It is noise with a small bandwidth.

(c) It is white noise plus a sinus.

(d) It is a sinus.

1 Solution

It can be noted that $R_{\bar{u}\bar{u}}(\tau)$ is a sinc function. If you transform a sinc function, you will get a block function. The PSD function $S_{\bar{u}\bar{u}}(\omega)$ is thus a block function. If this block function would be infinitely wide, then the PSD function would be constant. The result would thus be white noise. However, the block function is not infinitely wide. The bandwidth is thus limited. We therefore deal with white noise with a small bandwidth. The correct answer is (b).

2 Question

The random variable $\bar{x}$ has a probability density function $f_{\bar{x}}(x)$ as depicted in figure 2.

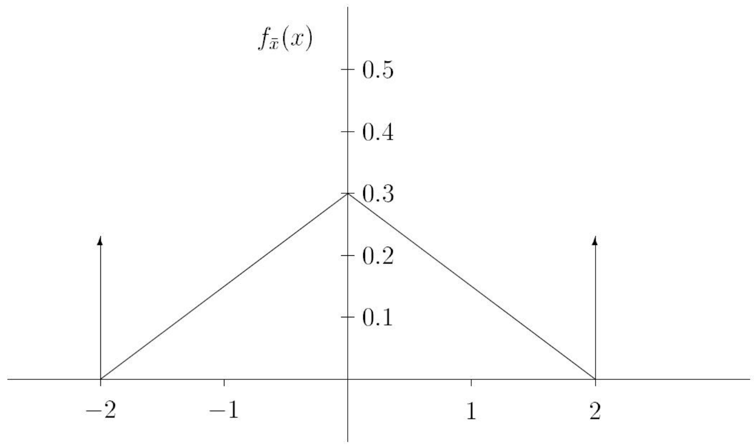

Figure 2: Probability density function $f_{\bar{x}}(x)$

What is the probability of $P(\bar{x} \geq -1)$?

(a) 0.125

(b) 0.275

(c) 0.725

(d) 0.750

(e) 0.875

(f) Not enough data available

2 Solution

We can note that the probability density function is symmetric about $x = 0$. Thus,

$$P(\bar{x} \leq 0) = P(\bar{x} \geq 0) = 0.5. \tag{2.1}$$

To find $P(-1 \leq \bar{x} \leq 0)$, we simply find the area under the function $f_{\bar{x}}(x)$ in this interval. This gives us

$$P(-1 \leq \bar{x} \leq 0) = 1 \cdot \frac{0.15 + 0.3}{2} = 0.225. \tag{2.2}$$

We thus have

$$P(\bar{x} \geq -1) = P(-1 \leq \bar{x} \leq 0) + P(\bar{x} \geq 0) = 0.725. \tag{2.3}$$

The correct answer is therefore (c).

3 Question

Proof that the Fourier transform of the signal $y(t) \left(= \frac{dy(t)}{dt}\right)$ equals,

$$\mathcal{F}\left\{\dot{y}(t)\right\} = j\omega Y(\omega) \tag{3.1}$$

with $Y(\omega)$ the Fourier transform of $y(t)$.

3 Solution

We can simply use the definition of the Fourier transform. This is

$$\mathcal{F}\left\{\dot{y}(t)\right\} = \int_{-\infty}^{+\infty} \dot{y}(t)e^{-j\omega t}\,dt. \tag{3.2}$$

Integration by parts now yields

$$\mathcal{F}\left\{\dot{y}(t)\right\} = \left[y(t)e^{-j\omega t}\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -j\omega y(t)e^{-j\omega t}\,dt = \left[y(t)e^{-j\omega t}\right]_{-\infty}^{+\infty} + j\omega\mathcal{F}\left\{y(t)\right\}. \tag{3.3}$$

Assuming that $y(\infty)e^{-j\omega\infty} = y(-\infty)e^{j\omega\infty}$ then gives the desired result.

4 Question

Proof that the Fourier transform of the product of two functions,

$$\mathcal{F}\left\{x(t)y(t)\right\} = \frac{1}{2\pi}X(\omega) * Y(\omega) \tag{4.1}$$

Note: the symbol "$*$" represents the convolution operator.

4 Solution

Before we begin the proof, we mention an important equation which we'll use. Let's say that we take the Fourier transform of a parameter $x(t)$ and then apply the inverse Fourier transform. We then again wind up with $x(t)$. So, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\alpha)e^{-j\theta\alpha}\,d\alpha\, e^{j\theta t}\,d\theta. \tag{4.2}$$

We start our proof with the left hand side of the equation. Per definition, this equals

$$\mathcal{F}\left\{x(t)y(t)\right\} = \int_{-\infty}^{+\infty} x(t)y(t)e^{-j\omega t}\,dt. \tag{4.3}$$

Applying the theorem above then turns this into

$$\mathcal{F}\left\{x(t)y(t)\right\} = \int_{-\infty}^{+\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\alpha)e^{-j\theta\alpha}\,d\alpha\, e^{j\theta t}\,d\theta\, y(t)e^{-j\omega t}\,dt. \tag{4.4}$$

We now pull all the exponents and $y(t)$ inside the integral. We also pull $1/2\pi$ outside the integral. This gives

$$\mathcal{F}\left\{x(t)y(t)\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\alpha)y(t)e^{-j(\theta\alpha-\theta t+\omega t)}\,d\alpha\,d\theta\,dt. \tag{4.5}$$

We can rewrite the part inside the integral. If we also change the order of integration slightly, we get

$$\mathcal{F}\left\{x(t)y(t)\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(x(\alpha)e^{-j\theta\alpha}\right)\left(y(t)e^{-j(\omega-\theta)t}\right)\,d\alpha\,dt\,d\theta. \tag{4.6}$$

Now note that the part with $x(\alpha)$ does not depend on t, while the part with $y(t)$ does not depend on α. So, we can split the integral up, resulting in

$$\mathcal{F}\left\{x(t)y(t)\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} x(\alpha)e^{-j\theta\alpha}\,d\alpha\right)\left(\int_{-\infty}^{+\infty} y(t)e^{-j(\omega-\theta)t}\,dt\right)\,d\theta. \tag{4.7}$$

Using the definition of the Fourier transform, this can now be rewritten as

$$\mathcal{F}\{x(t)y(t)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\theta)Y(\omega-\theta)\,d\theta = \frac{1}{2\pi}\left(X(\omega)*Y(\omega)\right). \tag{4.8}$$

And this is exactly the relation which we were supposed to proof.

5 Question

Proof that the periodogram $I_{\bar{y}\bar{y}}[k]$ of the signal $y[n] = ax[n] + b$ equals,

$$I_{\bar{y}\bar{y}}[k] = a^2 I_{\bar{x}\bar{x}}[k] + (2a\text{Re}\{X[k]\} + bN)b\delta[k] \tag{5.1}$$

with,

$$I_{\bar{x}\bar{x}}[k] = X^*[k]X[k]/N \tag{5.2}$$

and $\text{Re}\{X[k]\}$ the real part of the Fourier transform of $x[n]$.

Note: the Discrete Fourier Transform (FFT) of a constant b equals,

$$FFT\{b\} = \left(\sum_{n=0}^{N-1} be^{-j\frac{2\pi k}{N}n}\right) = bN\delta[k] \tag{5.3}$$

with $\delta[k]$ the Kronecker delta function. Use the result $FFT\{b\} = bN\delta[k]$ in your proof. Remember that $\delta[k]$ equals 0 for $k \neq 0$ and $\delta[k]$ equals 1 for $k = 0$.

5 Solution

First, we'll find an expression for $Y[k]$. We have

$$Y[k] = FFT\{y[n]\} = FFT\{ax[n] + b\} = a\,FFT\{x[n]\} + FFT\{b\} = aX[k] + bN\delta[k]. \tag{5.4}$$

Since a and b are real constants, we also have

$$Y^*[k] = aX^*[k] + bN\delta[k]. \tag{5.5}$$

The periodogram of $y[n]$ is now given by

$$I_{\bar{y}\bar{y}}[k] = Y^*[k]Y[k]/N = \left(aX^*[k] + bN\delta[k]\right)\left(aX[k] + bN\delta[k]\right)/N. \tag{5.6}$$

Working out brackets gives

$$I_{\bar{y}\bar{y}}[k] = a^2X^*[k]X[k]/N + ab\left(X^*[k] + X[k]\right)\delta[k] + b^2N\delta[k]^2. \tag{5.7}$$

In the discrete domain, we have $\delta[k]^2 = \delta[k]$. Also, adding a complex number to its complex conjugate gives twice its real part. (That is, $X^*[k] + X[k] = 2\text{Re}\{X[k]\}$.) And, if we also apply the definition for $I_{\bar{x}\bar{x}}[k]$, we will find that

$$I_{\bar{y}\bar{y}}[k] = a^2 I_{\bar{x}\bar{x}}[k] + \left(2a\text{Re}\{X[k]\} + bN\right)b\delta[k]. \tag{5.8}$$

And this is exactly what we needed to show.

6 Question

Make a qualitative sketch for the auto power spectral density functions of the following signals,

(a) $y_1(t) = \sin(\omega_0 t)$

(b) $y_2(t) = \cos(\omega_1 t) + 1$

(c) $y_3(t) = \sin(\omega_1 t) + 1$

6 Solution

(a) First, let's determine the covariance matrix. We have

$$C_{\bar{y}_1\bar{y}_1}(\tau) = \int_{-\infty}^{+\infty} \sin(t)\sin(t+\tau)\,dt. \tag{6.1}$$

If $\tau = 0 + 2\pi k$, then the integral will become $1/2$. However, if $\tau = \pi + 2\pi k$, then the integral will become $-1/2$. This kind of hints at the solution of the integral. Using a goniometric integral table could confirm this. We thus have

$$C_{\bar{y}_1\bar{y}_1}(\tau) = \frac{1}{2}\cos(\tau). \tag{6.2}$$

To find the PSD function, we need to apply the Fourier transform. This gives us

$$S_{\bar{y}_1\bar{y}_1}(\omega) = \mathcal{F}\left\{C_{\bar{y}_1\bar{y}_1}(\tau)\right\} = \mathcal{F}\left\{\frac{1}{2}\cos(\tau)\right\} = \frac{1}{2}\mathcal{F}\left\{\cos(\tau)\right\} = \frac{\pi}{2}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right). \tag{6.3}$$

(Note that the Fourier transform of a cosine function is something you should know by heart.) A sketch of this function would look like one peak at $\omega = \omega_0$ and another peak at $\omega = -\omega_0$.

(b) This function is not a zero-mean function. It has mean $\mu_{\bar{y}_2} = 1$. So the covariance function is now given by

$$C_{\bar{y}_2\bar{y}_2}(\tau) = \int_{-\infty}^{+\infty} (\sin(t) + 1 - \mu_{\bar{y}_2})(\sin(t+\tau) + 1 - \mu_{\bar{y}_2})\,dt. \tag{6.4}$$

It is now quite trivial that we again have

$$C_{\bar{y}_2\bar{y}_2}(\tau) = \frac{1}{2}\cos(\tau). \tag{6.5}$$

However, to find the PSD function, we need to know the product function $R_{\bar{y}_2\bar{y}_2}(\tau)$. It is given by

$$R_{\bar{y}_2\bar{y}_2}(\tau) = C_{\bar{y}_2\bar{y}_2}(\tau) + \mu_{\bar{y}_2}^2 = \frac{1}{2}\cos(\tau) + 1. \tag{6.6}$$

The PSD function can now be found using the Fourier transform. So,

$$S_{\bar{y}_2\bar{y}_2}(\omega) = \mathcal{F}\left\{R_{\bar{y}_2\bar{y}_2}(\tau)\right\} = \frac{1}{2}\mathcal{F}\left\{\cos(\tau)\right\} + \mathcal{F}\left\{1\right\} = \frac{\pi}{2}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right) + 2\pi\delta(\omega). \tag{6.7}$$

If we want to sketch this, then we should place one peak at $\omega = \omega_0$ and another peak at $\omega = -\omega_0$. Also, there should be a peak four times as big as the previous ones on $\omega = 0$.

(c) Note that this function has exactly the same covariance function as $y_2(t)$. (In fact, $y_3(t)$ is only a time-shift of $y_2(t)$.) So, also the PSD function is the same. We thus have

$$S_{\bar{y}_3\bar{y}_3}(\omega) = \frac{\pi}{2}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right) + 2\pi\delta(\omega). \tag{6.8}$$

The sketch of this function is also the same as the sketch of $S_{\bar{y}_2\bar{y}_2}(\omega)$.

7 Question

Assume the stochastic process $\bar{x}$ which is defined as the wave-height in the North-Sea at a certain position. At a certain instant in time t_1, this stochastic process has a certain probability density function $f_{\bar{x}}(x;t_1)$, with t_1 the time during the day when hardly any wind is present. At time instant t_2, representing a time in a period with strong winds, the probability density function is written as $f_{\bar{x}}(x;t_2)$.

Make a qualitative sketch of the probability density functions $f_{\bar{x}}(x;t_1)$ and $f_{\bar{x}}(x;t_2)$.

Note: assume that $\int_{-\infty}^{+\infty} f_{\bar{x}}(x;t)\,dx = 1 \forall t$.

7 Solution

Due to the central limit theorem, it can be assumed that the wave-height is normally distributed. The mean is given by $\mu_{\bar{x}}(t)$ and the standard deviation by $\sigma_{\bar{x}}(t)$. At time t_1, the waves aren't very high, so $\mu_{\bar{x}}(t_1)$ is low. Also, the height of the waves doesn't vary a lot, so $\sigma_{\bar{x}}(t_1)$ is also low. However, at time t_2, the waves are high. Also, the height will vary quite a bit. Thus, both $\mu_{\bar{x}}(t_2)$ and $\sigma_{\bar{x}}(t_2)$ are relatively big.

Now let's examine the shape of the probability density functions. (This is what we need to sketch.) Since $\bar{x}$ is normally, distributed, the PDF has a bell-shaped curve. In theory this curve has a nonzero value for every x from $-\infty$ to ∞. However, in reality a wave height below 0 isn't really possible. So, on the left, we cut off the PDF at $x = 0$.

The PDF at time t_1 has a low mean. So its peak is relatively close to 0. Also, the standard deviation is low, so the curve's value drops quite quickly. However, the integral over the PDF does have to equal 1. This implies that the peak of the bell curve has to be quite high.

The PDF at time t_2 has a higher mean. So its peak is further away from 0. Also, the standard deviation is big, so the curve's value drops less quickly to zero. And, because the integral over the PDF has to equal 1, the peak of the bell curve is not very high.

8 Question

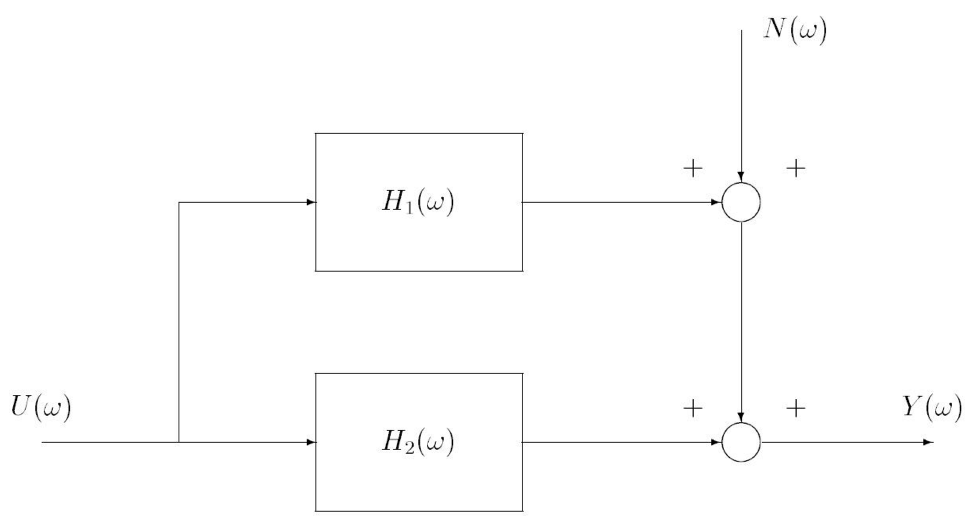

Figure 3: System description

Given the system in figure 3 of which the frequency response functions $H_1(\omega)$ and $H_2(\omega)$ are known. The input $U(\omega)$ and the noise on the output $N(\omega)$ are stochastic and their power spectral density functions are also known (both $U(\omega)$ and $N(\omega)$ are white noise).

(a) Calculate the power spectral density function of the output $S_{yy}(\omega)$.

(b) Calculate the power spectral density function of the output for the case the input $U(\omega)$ and the noise $N(\omega)$ do not resemble white noise ánd may even be correlated.

8 Solution

The system can be described by

$$Y(\omega) = H_1(\omega)U(\omega) + H_2(\omega)U(\omega) + N(\omega). \tag{8.1}$$

If we define $H(\omega) = H_1(\omega) + H_2(\omega)$, then we have

$$Y(\omega) = H(\omega)U(\omega) + N(\omega). \tag{8.2}$$

For this system description, the PSD functions are related through

$$S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) + S_{\bar{n}\bar{n}}(\omega). \tag{8.3}$$

This is the solution to the second part of the question. In the special case that both $U(\omega)$ and $N(\omega)$ are white noise, we have $S_{\bar{u}\bar{u}}(\omega) = S_{\bar{n}\bar{n}}(\omega) = 1$ and thus

$$S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 + 1. \tag{8.4}$$

9 Question

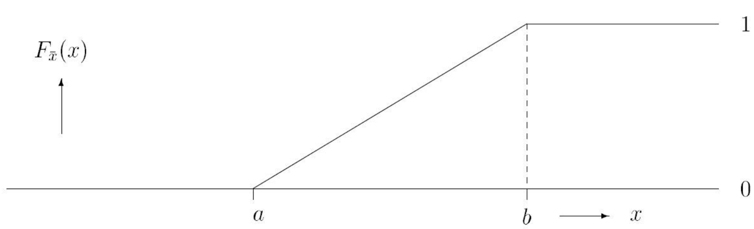

Figure 4: The probability distribution function $F_{\bar{x}}(x)$ for a uniformly distributed stochastic variable $\bar{x}$

The probability distribution function $F_{\bar{x}}(x)$ of a uniformly distributed stochastic variable $\bar{x}$ is written as (with $b > a$, see also figure 4),

$$F_{\bar{x}}(x) = \begin{cases} 0 & \text{for } x \leq a, \\ \frac{x-a}{b-a} & \text{for } a < x \leq b, \\ 1 & \text{for } x > b \end{cases} \tag{9.1}$$

Calculate the probability density function $f_{\bar{x}}(x)$ and proof that the stochastic variable's mean value and variance are respectively,

$$\mu_x = \frac{a+b}{2} \quad \text{and} \quad \sigma_x^2 = \frac{1}{12}(b-a)^2. \tag{9.2}$$

9 Solution

To find the probability density function, we simply take the derivative of the probability distribution function. This gives

$$f_{\bar{x}}(x) = \frac{dF_{\bar{x}}(x)}{dx} = \begin{cases} 0 & \text{for } x \leq a, \\ \frac{1}{b-a} & \text{for } a < x \leq b, \\ 0 & \text{for } x > b. \end{cases} \tag{9.3}$$

To find the mean, we can use

$$\mu_{\bar{x}} = \int_{-\infty}^{+\infty} x f_{\bar{x}}(x)\, dx = \left[\frac{1}{2}\frac{x^2}{b-a}\right]_a^b = \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{1}{2}\frac{(b-a)(b+a)}{b-a} = \frac{a+b}{2}. \tag{9.4}$$

To find the variance, we use

$$\sigma_{\bar{x}}^2 = \int_{-\infty}^{+\infty} (x-\mu_{\bar{x}})^2 f_{\bar{x}}(x)\, dx = \left[\frac{1}{3}\frac{(x-\mu_{\bar{x}})^3}{b-a}\right]_a^b = \frac{1}{3}\frac{\left(b-\frac{a+b}{2}\right)^3 - \left(a-\frac{a+b}{2}\right)^3}{b-a}. \tag{9.5}$$

This can be rewritten to

$$\sigma_{\bar{x}}^2 = \frac{1}{3}\frac{\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3}{b-a} = \frac{1}{24}\frac{(b-a)^3 + (b-a)^3}{b-a} = \frac{(b-a)^2}{12}. \tag{9.6}$$