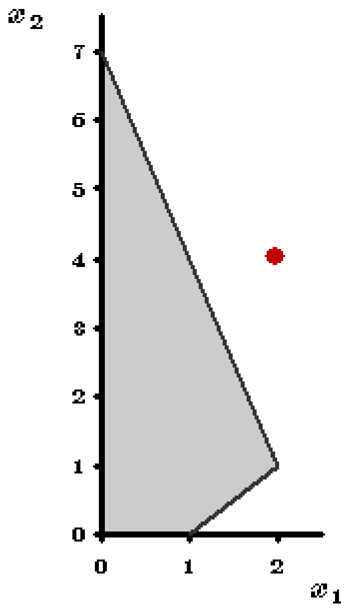
Frank-Wolfe Algorithm Demonstration

1. Introduction

To illustrate the Frank-Wolfe algorithm, consider the demonstration example for the simplex method with a new objective function:

$$
\begin{array}{llcccl}
\text{Maximize } f(X) = 32x_1 - 1x_1^4 + 8x_2 - 1x_2^2 \\
\text{subject to} & x_1 & - & x_2 & \le 1 \\
 & 3x_1 & + & x_2 & \le 7 \\
\text{and} & x_1 & \ge 0, & x_2 & \ge 0.
\end{array}
$$



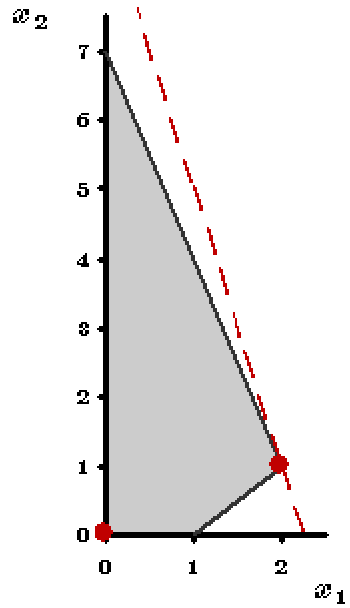
The feasible region is shown to the right. Without these constraints the maximum of $f(X)$ is easily found (by setting the partial derivatives equal to zero) to be $X = (x_1, x_2) = (2, 4)$. But what is the constrained maximum? Let us use as the initial trial solution $X = (0, 0)$ for the Frank-Wolfe algorithm.

2. Objective Approximation

Initial trial solution:

$$X = (x_1, x_2) = (0, 0)$$

$$f(X) = 32x_1 - 1x_1^4 + 8x_2 - 1x_2^2$$



We begin the first iteration by developing a "linear approximation" for $f(X)$ near this trial solution. This is done by evaluating the partial derivatives at (0, 0):

$$\frac{\partial f}{\partial x_1} = 32 - 4x_1^3 = 32$$

$$\frac{\partial f}{\partial x_2} = 8 - 2x_2 = 8$$

So this approximation is $g(X) = 32x_1 + 8x_2$. Maximizing $g(X)$ subject to the original constraints yields the solution $X = (x_1, x_2) = (2, 1)$ with $g(2, 1) = 72$. However since (2, 1) is not near (0, 0) the approximation may not be a good one at (2, 1). [Note that $f(2, 1) = 55$ is well under $g(2, 1) = 72$.] Rather than just accepting (2, 1) as the next trial solution, let us check the line segment between (0, 0) and (2, 1) and choose the X with the largest $f(X)$.

3. Changing Solution

$$f(X) = 32x_1 - 1x_1^4 + 8x_2 - 1x_2^2$$

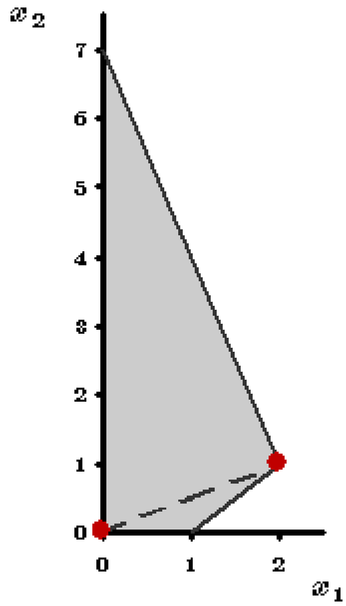
The equation for the line segment between (0, 0) and (2, 1) is

$$(0,0) + t[(2,1) - (0,0)] = (2t, t), \quad 0 \le t \le 1$$

Since $x_1 = 2t$ and $x_2 = t$, the values of $f(X)$ on the line are

$$\begin{aligned} h(t) = f(2t, t) &= 32(2t) - (2t)^4 + 8(t) - (t)^2 \\ &= 72t - t^2 - 16t^4 \end{aligned}$$

The point $X = (2t, t)$ on this line segment having the largest $f(X)$ is found by maximizing $f(2t, t)$ over $0 \le t \le 1$ by the one-dimensional search procedure, which yields $t^* = 1$. Therefore the new trial solution is $X = (2t^*, t^*) = (2, 1)$.



4. Next Iteration

Second trial solution:

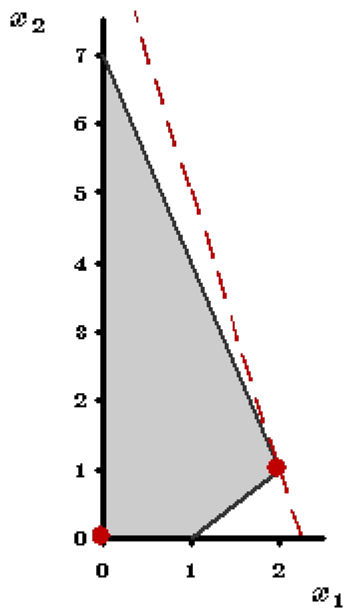
$$X = (x_1, x_2) = (2, 1)$$

$$f(X) = 32x_1 - 1x_1^4 + 8x_2 - 1x_2^2$$

We begin the second iteration by evaluating the partial derivatives of $f(X)$ at (2, 1):

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 32 - 4x_1^3 = 0 \\ \frac{\partial f}{\partial x_2} &= 8 - 2x_2 = 6 \end{aligned}$$

So the approximated objective function is $g(X) = 0x_1 + 6x_2$. Maximizing $g(X) = 0x_1 + 6x_2$ subject to the original constraints yields the solution $X = (x_1, x_2) = (0, 7)$ with $g(0, 7) = 42$. [Note that $f(0, 7) = 7$.] The line between (2, 1) and (0, 7) is



$$(2,1) + t[(0,7) - (2,1)] = (2 - 2t, 1 + 6t), \quad 0 \le t \le 1$$

$$h(t) = f(2 - 2t, 1 + 6t) = 55 + 36t - 132t^2 + 64t^3 - 16t^4$$

which is maximized at $t^* = 0.1524$. The resulting new trial solution is

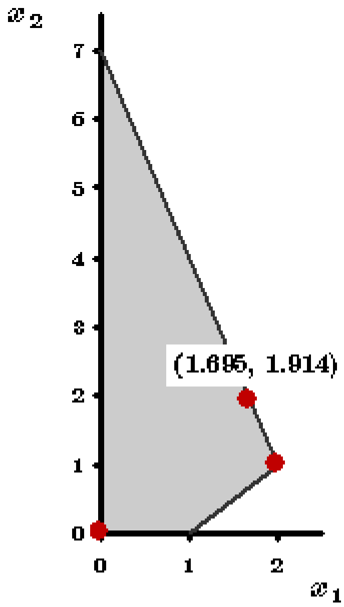
$$X = (2 - 2t^*, 1 + 6t^*) = (1.695, 1.914)$$

5. Final Iteration

Third trial solution:

$$X = (x_1, x_2) = (1.695, 1.914)$$

$$f(X) = 32x_1 - 1x_1^4 + 8x_2 - 1x_2^2$$



Evaluating the partial derivatives at (1.695, 1.914):

$$\frac{\partial f}{\partial x_1} = 32 - 4x_1^3 = 12.51$$
$$\frac{\partial f}{\partial x_2} = 8 - 2x_2 = 4.17$$

so the approximating objective function is $g(X) = 12.51x_1 + 4.17x_2$. Maximizing $g(X) = 12.51x_1 + 4.17x_2$, or equivalently (after dividing by 4.17) $g(X) = 3x_1 + x_2$, results in **every** solution on the line between (2, 1) and (0, 7) being optimal for this linear programming problem. Regardless of which solution is used as the other endpoint of the line from (1.695, 1.914), we already know from the preceding iteration that $f(X)$ is maximized along this line at (1.695, 1.914). Since the trial solution did not move, this verifies that the optimal solution for our convex programming problem is

$$X = (x_1, x_2) = (1.695, 1.914)$$