AE 2106 - Vibrations

A brief summary: How to solve the equations of motion

The purpose of this document is to summarize the method of solving (in-)homogeneous second order linear differential equations with constant coefficients in the context of vibrations. A sound understanding of differential equations is assumed.

1 Definitions

A discretized system has the following **general equation of motion**, where the scalar factors are the *equivalent* mass m, moment of inertia J, damping c and stiffness k. Depending on the choice of the degree of freedom, we get a linear form (1) or a polar form (2).

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{1}$$

$$J\ddot{\theta} + c\dot{\theta} + k\theta = M(t) \tag{2}$$

The **natural frequency** of a system is given by:

$$\omega_n = \sqrt{\frac{k}{m}} \tag{3}$$

The **damping ratio** of a system is given by:

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}} \tag{4}$$

The **damping frequency** of a system is given by:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{5}$$

2 Free Undamped Vibrations

In this case — $F(t) = 0$ (free of excitations) and $c = 0$ (undamped) — the general form is

$$m\ddot{x} + kx = 0. \tag{6}$$

Assume a periodic **solution**

$$x(t) = A\sin(\omega_n t + \varphi) \tag{7}$$

with time derivative

$$\dot{x}(t) = A\omega_n\cos(\omega_n t + \varphi). \tag{8}$$

Use the **initial conditions**

$$x(0) = x_0 \tag{9}$$
$$\dot{x}(0) = \dot{x}_0 \tag{10}$$

and simple algebra to determine A and φ.

3 Free Damped Vibrations

In this case — $F(t) = 0$ (free of excitations) and $c \neq 0$ (damped) — the general form is

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{11}$$

Depending on the damping ratio ζ defined in (4), three possible **solutions** can be found:

$$x(t) = \begin{cases} Ae^{-\zeta\omega_n t}\sin(\omega_d t + \varphi) & \text{for } c < c_{cr} \\ e^{-\omega_n t}(A + Bt) & \text{for } c = c_{cr} \\ e^{-\zeta\omega_n t}(Ae^{-\omega_n t\sqrt{\zeta^2-1}} + Be^{\omega_n t\sqrt{\zeta^2-1}}) & \text{for } c > c_{cr} \end{cases} \tag{12}$$

Using their time derivatives $\dot{x}$ and the initial conditions, (A, φ) or (A, B) are found.

4 Harmonically Forced Vibrations

In this case — $F(t) \neq 0$ (excitation) and c may or may not be $= 0$ — the general form is

$$m\ddot{x} + c\dot{x} + kx = F(t). \tag{13}$$

The **solution** for this equation consists of the *transient response* (homogeneous solution x^h) and the *steady-state response* (particular solution x^p).

The homogeneous solution is found as described in Section 3. However, the initial conditions should not be applied to the homogeneous solution, but only to the general solution $x(t)$. Therefore, we have to find the particular solution first:

For the particular solution, the solution depends on **how the system is excited**:

$$F(t) = \begin{cases} \hat{F}\sin\omega_e t & : \text{case (i)} \\ \hat{F}\cos\omega_e t & : \text{case (ii)} \end{cases} \tag{14}$$

Here, $\hat{F}$ can be any amplitude (dependent on whether it is a force or a base excitation). What is important here is the difference between sine and cosine. The particular solution will be of the same form.

The *complex exponential basis function* can be used to describe trigonometric functions:

$$e^{im} = \cos(m) + i\sin(m) \tag{15}$$

For $m = \omega_e t$, the following **particular solution** is used:

$$x^p = \begin{cases} \text{Im}\{\ X^p e^{i\omega_e t}\ \} & \text{for case (i)} \\ \text{Re}\{\ X^p e^{i\omega_e t}\ \} & \text{for case (ii)} \end{cases} \tag{16}$$

After obtaining the first and second time derivative of the particular solution, they can be substituted into the original equation of motion (13).

In the case of an *undamped system*, the solution is simple and straight-forward. X^p can be found in terms of the system parameters and $\hat{F}$.

For a *damped system*, the same approach is chosen. However, the $\dot{x}$-term results in X^p having a complex number $v \pm iw$ in the denominator. By multiplying with its complex conjugate $v \mp iw$, the complex number appears in the numerator (and $v^2 - w^2$ in the denominator). By considering the complex plane, the rules below are derived.

$$v \pm iw = \sqrt{v^2 + w^2} e^{\pm i\phi} \tag{17}$$

where

$$\phi = \tan^{-1} (\frac{w}{v}) \tag{18}$$

Using (17), X^p can easily be converted into a complex exponential basis function. After both the homogeneous and the particular solution have been found, the **initial conditions** can be applied to the **general solution**

$$x(t) = x^h(t) + x^p(t). \tag{19}$$

5 Arbitrary Forced Vibrations

Arbitrary forced vibrations have the general form

$$m\ddot{x} + c\dot{x} + kx = F(t). \tag{20}$$

To solve it, the equation of motion is generally transfered into the Laplace domain with the aid of the Laplace transform:

$$\mathcal{L}[\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x] = \mathcal{L}[f(t)] \tag{21}$$

Since $f(t)$ usually only takes values on a certain inverval (e.g. a wing experiencing gusts of wind), the standard Laplace transform does not apply. Therefore, the general transform rule is used:

$$F(s) = \int_0^\infty f(t)e^{-st}\text{d}t \tag{22}$$

Combining with the left hand side, the following is found:

$$(s^2X(s) - s\dot{x}_0 - x_0) + 2\zeta\omega_n(sX(s) - x_0) + \omega_n^2 X(s) = F(s) \tag{23}$$

By rearranging the transformed equation for X(s) and then performing the inverse Laplace transform, a solution for x(t) is found. Here, the standard Laplace transform table is used.

6 Multiple Degrees of Freedom Systems

By discretizing the system and analyzing the free body diagram, a linear system of the form

$$\mathbf{M}\underline{\ddot{x}} + \mathbf{K}\underline{x} = \underline{F} \tag{24}$$

with vectors $\underline{\ddot{x}} = (\ddot{x}_1 \quad \ddot{x}_2)^T$, $\underline{x} = (x_1 \quad x_2)^T$ and $\underline{F} = (F_1 \quad F_2)^T$ is found. Further, $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, respectively. To solve this system, the *transient* (homogeneous) and the *steady-state* (particular) solution must be determined.

For the **homogeneous equation**

$$\mathbf{M}\underline{\ddot{x}} + \mathbf{K}\underline{x} = \underline{0} \tag{25}$$

we try

$$\underline{x}^h = \underline{\hat{x}}e^{i\omega_n t} \tag{26}$$

$$\underline{\ddot{x}}^h = -\underline{\hat{x}}\omega_n^2 e^{i\omega_n t} \tag{27}$$

and — by substituting into the linear system — we get

$$[-\omega_n^2\mathbf{M} + \mathbf{K}]\underline{\hat{x}} = \mathbf{A}\underline{\hat{x}} = \underline{0}. \tag{28}$$

This has the *trivial solution* $\underline{\hat{x}} = 0$, in which we are not interested. A *non-trivial solution* is found only if $\det(A) = 0$.

The values of ω_n for which this is true are the **eigenfrequencies** of the system. Negative eigenfrequencies are discarded and $\omega_n^{(m)}$ (for m=1,2,...) is then used to find the corresponding **eigenmodes** $\underline{\hat{x}}^{(m)}$ (or eigenvectors) of the system.

Then, the **solution** of the system is

$$\underline{x}^h = \sum_m A_m \sin{(\omega_n^{(m)}t + \varphi_1)}\underline{\hat{x}}^{(m)} \tag{29}$$

To determine the **particular solution** for

$$\underline{F}(t) = \begin{Bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{Bmatrix} \cos{(\omega_x t)} \tag{30}$$

we use

$$\underline{x}^p(t) = \underline{\hat{x}}^p \cos{(\omega_x t)} \tag{31}$$

$$\underline{\ddot{x}}^p(t) = -\omega_x^2\underline{\hat{x}}^p \cos{(\omega_x t)}, \tag{32}$$

which gives

$$[-\omega_x^2\mathbf{M} + \mathbf{K}]\underline{\hat{x}}^p = \underline{\hat{F}}. \tag{33}$$

Because $\mathbf{M}$ and $\mathbf{K}$ are always symmetric, $\underline{\hat{x}}^p$ can be found:

$$\underline{\hat{x}}^p = [-\omega_x^2\mathbf{M} + \mathbf{K}]^{-1}\underline{\hat{F}}. \tag{34}$$