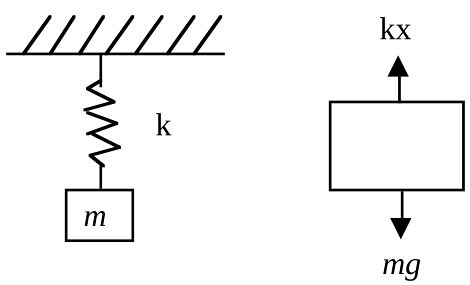
Problems and Solutions Section 1.1 (1.1 through 1.19)

1.1 The spring of Figure 1.2 is successively loaded with mass and the corresponding (static) displacement is recorded below. Plot the data and calculate the spring's stiffness. Note that the data contain some error. Also calculate the standard deviation.

m(kg)	10	11	12	13	14	15	16
x(m)	1.14	1.25	1.37	1.48	1.59	1.71	1.82

Solution:

Free-body diagram:

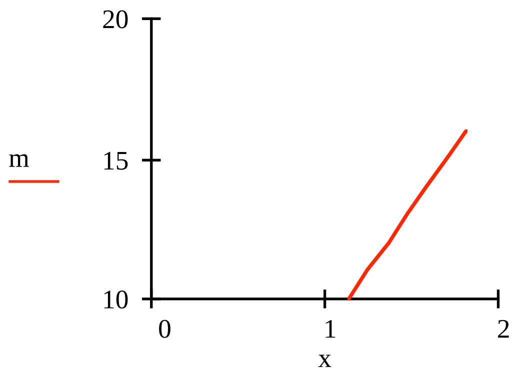


From the free-body diagram and static equilibrium:

$$kx = mg \quad (g = 9.81 m/s^2)$$

$$k = mg/x$$

$$\mu = \frac{\Sigma k_i}{n} = 86.164$$



Plot of mass in kg versus displacement in m

The sample standard deviation in computed stiffness is:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(k_i - \mu)^2}{n-1}} = \mathbf{0.164}$$

Computation of slope from *mg*/*x*

m(kg)	*x*(m)	*k*(N/m)
10	1.14	86.05
11	1.25	86.33
12	1.37	85.93
13	1.48	86.17
14	1.59	86.38
15	1.71	86.05
16	1.82	86.24

1.2 Derive the solution of $m\ddot{x} + kx = 0$ and plot the result for at least two periods for the case with ω_n = 2 rad/s, x_0 = 1 mm, and $v_0 = \sqrt{5}$ mm/s.

Solution:

Given:

$$m\ddot{x} + kx = 0 \qquad (1)$$

Assume: $x(t) = ae^{rt}$. Then: $\dot{x} = are^{rt}$ and $\ddot{x} = ar^2e^{rt}$. Substitute into equation (1) to get:

$$mar^2e^{rt} + kae^{rt} = 0$$

$$mr^2 + k = 0$$

$$r = \pm\sqrt{\frac{k}{m}}\ i$$

Thus there are two solutions:

$$x_1 = c_1e^{\left(\sqrt{\frac{k}{m}}i\right)t}, \text{ and } x_2 = c_2e^{\left(-\sqrt{\frac{k}{m}}i\right)t}$$

$$\text{where } \omega_n = \sqrt{\frac{k}{m}} = 2 \text{ rad/s}$$

The sum of x_1 and x_2 is also a solution so that the total solution is:

$$x = x_1 + x_2 = c_1e^{2it} + c_2e^{-2it}$$

Substitute initial conditions: x_0 = 1 mm, $v_0 = \sqrt{5}$ mm/s

$$x(0) = c_1 + c_2 = x_0 = 1 \Rightarrow \underline{c_2 = 1 - c_1}, \text{ and } v(0) = \dot{x}(0) = 2ic_1 - 2ic_2 = v_0 = \sqrt{5} \text{ mm/s}$$

$\Rightarrow \underline{-2c_1 + 2c_2 = \sqrt{5}\,i}$. Combining the two underlined expressions (2 eqs in 2 unkowns):

$$-2c_1 + 2 - 2c_1 = \sqrt{5}\,i \Rightarrow \underline{\underline{c_1 = \frac{1}{2} - \frac{\sqrt{5}}{4}i}}, \text{ and } \underline{\underline{c_2 = \frac{1}{2} + \frac{\sqrt{5}}{4}i}}$$

Therefore the solution is:

$$x = \left(\frac{1}{2} - \frac{\sqrt{5}}{4}i\right)e^{2it} + \left(\frac{1}{2} + \frac{\sqrt{5}}{4}i\right)e^{-2it}$$

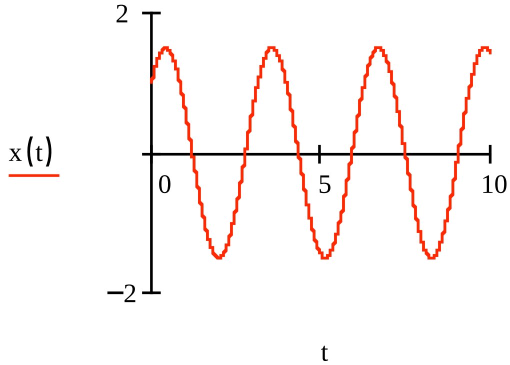
Using the Euler formula to evaluate the exponential terms yields:

$$x = \left(\frac{1}{2} - \frac{\sqrt{5}}{4}i\right)(\cos 2t + i\sin 2t) + \left(\frac{1}{2} + \frac{\sqrt{5}}{4}i\right)(\cos 2t - i\sin 2t)$$

$$\Rightarrow \underline{x(t) = \cos 2t + \frac{\sqrt{5}}{2}\sin 2t = \sqrt{\frac{3}{2}}\sin(2t + 0.7297)}$$

Using Mathcad the plot is:

$$x(t) := \cos(2\cdot t) + \frac{\sqrt{5}}{2}\cdot\sin(2\cdot t)$$



1.3 Solve $m\ddot{x} + kx = 0$ for k = 4 N/m, m = 1 kg, x_0 = 1 mm, and v_0 = 0. Plot the solution.

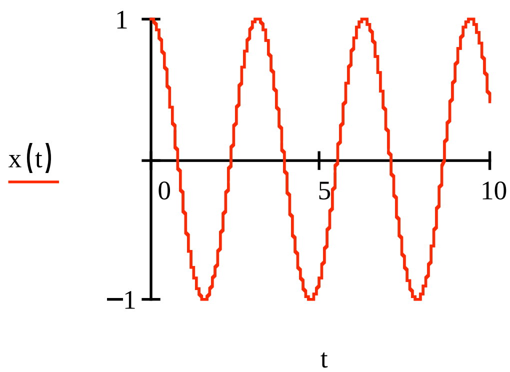
Solution:

This is identical to problem 2, except $v_0 = 0$. $\left(\omega_n = \sqrt{\dfrac{k}{m}} = 2 \text{ rad/s}\right)$. Calculating the initial conditions:

$$x(0) = c_1 + c_2 = x_0 = 1 \Rightarrow c_2 = 1 - c_1$$
$$v(0) = \dot{x}(0) = 2ic_1 - 2ic_2 = v_0 = 0 \Rightarrow c_2 = c_1$$
$$c_2 = c_1 = 0.5$$
$$x(t) = \frac{1}{2}e^{2it} + \frac{1}{2}e^{-2it} = \frac{1}{2}(\cos 2t + i\sin 2t) + \frac{1}{2}(\cos 2t - i\sin 2t)$$
$$\underline{x(t) = \cos(2t)}$$

The following plot is from Mathcad:

$$x(t) := \cos(2 \cdot t)$$



Alternately students may use equation (1.10) directly to get

$$x(t) = \frac{\sqrt{2^2(1)^2 + 0^2}}{2}\sin(2t + \tan^{-1}[\frac{2 \cdot 1}{0}])$$
$$= 1\sin(2t + \frac{\pi}{2}) = \cos 2t$$

1.4 The amplitude of vibration of an undamped system is measured to be 1 mm. The phase shift from $t = 0$ is measured to be 2 rad and the frequency is found to be 5 rad/s. Calculate the initial conditions that caused this vibration to occur. Assume the response is of the form $x(t) = A\sin(\omega_n t + \phi)$.

Solution:

Given: $A = 1\text{mm},\ \phi = 2\,\text{rad},\ \omega = 5\,\text{rad/s}$. For an *undamped* system:

$$x(t) = A\sin(\omega_n t + \phi) = 1\sin(5t + 2) \quad \text{and}$$
$$v(t) = \dot{x}(t) = A\omega_n\cos(\omega_n t + \phi) = 5\cos(5t + 2)$$

Setting $t = 0$ in these expressions yields:

$$x(0) = 1\sin(2) = \underline{0.9093\text{ mm}}$$
$$v(0) = 5\cos(2) = \underline{-2.081\text{ mm/s}}$$

1.5 Find the equation of motion for the hanging spring-mass system of Figure P1.5, and compute the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency.

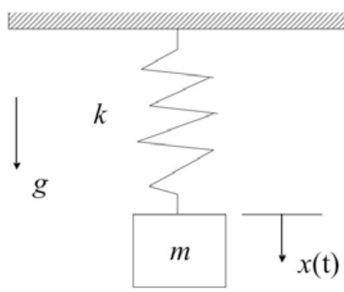
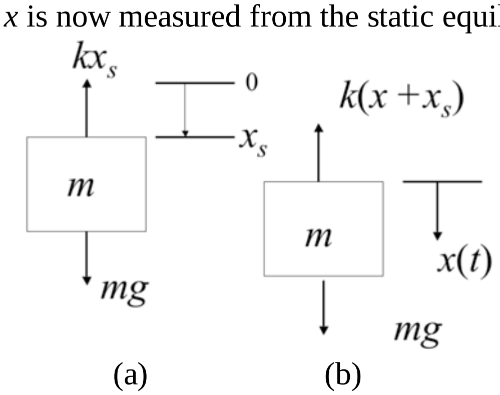


Figure P1.5

Solution:

The free-body diagram of problem system in (a) for the static case and in (b) for the dynamic case, where x is now measured from the static equilibrium position.



From a force balance in the static case (a): $mg = kx_s$, where x_s is the static deflection of the spring. Next let the spring experience a dynamic deflection $x(t)$ governed by summing the forces in (b) to get

$$m\ddot{x}(t) = mg - k(x(t) + x_s) \Rightarrow m\ddot{x}(t) + kx(t) = mg - kx_s$$

$$\Rightarrow \underline{m\ddot{x}(t) + kx(t) = 0} \Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m}}}$$

since $mg = kx_s$ from static equilibrium.

1.6 Find the equation of motion for the system of Figure P1.6, and find the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency. Assume the block slides without friction.

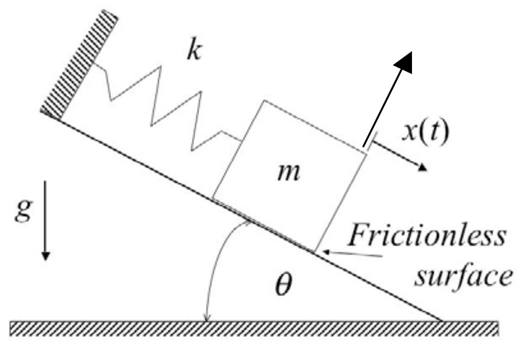
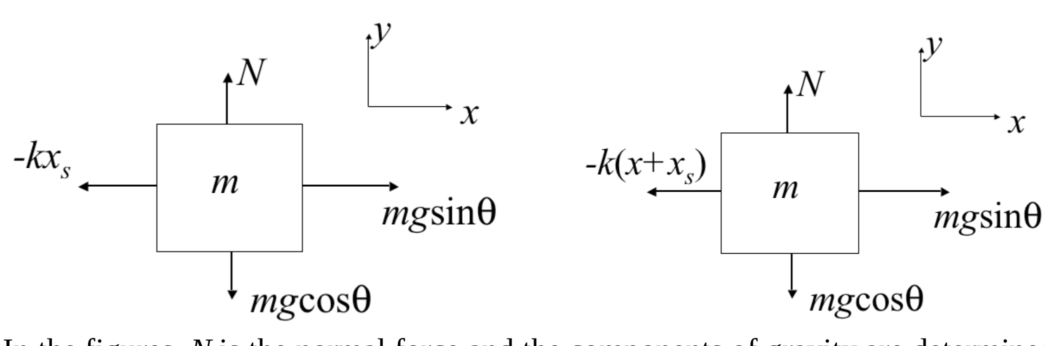


Figure P1.6

Solution:
Choosing a coordinate system along the plane with positive down the plane, the free-body diagram of the system for the static case is given and (a) and for the dynamic case in (b):



In the figures, N is the normal force and the components of gravity are determined by the angle θ as indicated. From the static equilibrium: $-kx_s + mg\sin\theta = 0$. Summing forces in (b) yields:

$$\sum F_i = m\ddot{x}(t) \Rightarrow m\ddot{x}(t) = -k(x + x_s) + mg\sin\theta$$

$$\Rightarrow m\ddot{x}(t) + kx = -kx_s + mg\sin\theta = 0$$

$$\Rightarrow \underline{m\ddot{x}(t) + kx = 0}$$

$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m}}} \text{ rad/s}$$

1.7 An undamped system vibrates with a frequency of 10 Hz and amplitude 1 mm. Calculate the maximum amplitude of the system's velocity and acceleration.

Solution:

Given: First convert Hertz to rad/s: $\boldsymbol{\omega}_n = 2\boldsymbol{\pi}f_n = 2\boldsymbol{\pi}(10) = 20\boldsymbol{\pi}$ rad/s. We also have that A= 1 mm.

For an undamped system:

$$x(t) = A\sin(\omega_n t + \phi)$$

and differentiating yields the velocity: $v(t) = A\omega_n \cos(\omega_n t + \phi)$. Realizing that both the sin and cos functions have maximum values of 1 yields:

$$v_{\max} = A\omega_n = 1(20\pi) = \mathbf{62.8\ mm/s}$$

Likewise for the acceleration: $a(t) = -A\omega_n^2 \sin(\omega_n t + \phi)$

$$a_{\max} = A\omega_n^2 = 1(20\pi)^2 = \mathbf{3948\ mm/s^2}$$

1.8 Show by calculation that $A \sin(\omega_n t + \phi)$ can be represented as $B\sin\omega_n t + C\cos\omega_n t$ and calculate C and B in terms of A and ϕ.

Solution:

This trig identity is useful: $\sin(a+b) = \sin a \cos b + \cos a \sin b$

Given: $A\sin(\omega_n t + \phi) = A\sin(\omega_n t)\cos(\phi) + A\cos(\omega_n t)\sin(\phi)$

$$= B\sin\omega_n t + C\cos\omega_n t$$

$$\text{where } B = A\cos\phi \quad and \quad C = A\sin\phi$$

1.9 Using the solution of equation (1.2) in the form $x(t) = B\sin\omega_n t + C\cos\omega_n t$ calculate the values of B and C in terms of the initial conditions x_0 and v_0.

Solution:
Using the solution of equation (1.2) in the form

$$x(t) = B\sin\omega_n t + C\cos\omega_n t$$

and differentiate to get:

$$\dot{x}(t) = \omega_n B\cos(\omega_n t) - \omega_n C\sin(\omega_n t)$$

Now substitute the initial conditions into these expressions for the position and velocity to get:

$$x_0 = x(0) = B\sin(0) + C\cos(0) = C$$
$$v_0 = \dot{x}(0) = \omega_n B\cos(0) - \omega_n C\sin(0)$$
$$= \omega_n B(1) - \omega_n C(0) = \omega_n B$$

Solving for B and C yields:

$$B = \frac{v_0}{\omega_n}, \text{ and } C = x_0$$

Thus $$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t$$