Problems and Solutions Section 1.10 (1.102 through 1.114)

1.102 A 2-kg mass connected to a spring of stiffness 10^3 N/m has a dry sliding friction force (F_c) of 3 N. As the mass oscillates, its amplitude decreases 20 cm. How long does this take?

Solution: With m = 2kg, and k = 1000 N/m the natural frequency is just

$$\omega_n = \sqrt{\frac{1000}{2}} = 22.36 \text{ rad/s}$$

From equation (1.101): slope = $\dfrac{-2\mu mg\omega_n}{\pi k} = \dfrac{-2F_c\omega_n}{\pi k} = \dfrac{\Delta x}{\Delta t}$

Solving the last equality for Δt yields:

$$\Delta t = \frac{-\Delta x\pi k}{2f_c\omega_n} = \frac{-(0.20)(\pi)(10^3)}{2(3)(22.36)} = \underline{4.68 \text{ s}}$$

1.103 Consider the system of Figure 1.41 with m = 5 kg and $k = 9 \times 10^3$ N/m with a friction force of magnitude 6 N. If the initial amplitude is 4 cm, determine the amplitude one cycle later as well as the damped frequency.

Solution: Given m = 5 kg, $k = 9\times10^3$ N/m, f_c = 6 N, $x_0 = 0.04$ m, the amplitude after one cycle is $x_1 = x_0 - \dfrac{4f_c}{k} = 0.04 - \dfrac{(4)(6)}{9\times10^3} = \underline{0.0373 \text{ m}}$

Note that the damped natural frequency is the same as the natural frequency in the case of Coulomb damping, hence $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{9\times10^3}{5}} = \underline{42.43 \text{ rad/s}}$

1.104* Compute and plot the response of the system of Figure P1.104 for the case where x_0 = 0.1 m, v_0 = 0.1 m/s, μ_κ = 0.05, m = 250 kg, θ = 20° and k =3000 N/m. How long does it take for the vibration to die out?

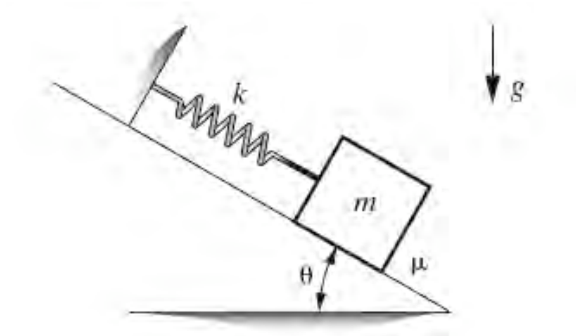
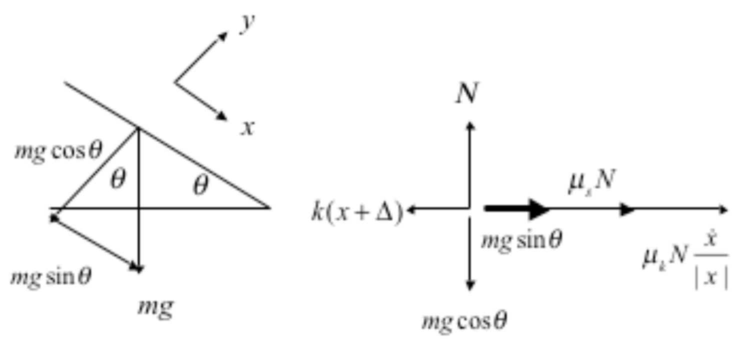


Figure P1.104

Solution: Choose the x y coordinate system to be along the incline and perpendicular to it. Let μ_s denote the static friction coefficient, μ_k the coefficient of kinetic friction and Δ the static deflection of the spring. A drawing indicating the angles and a free-body diagram is given in the figure:



For the static case

$$\sum F_x = 0 \Rightarrow k\Delta = \mu_s N + mg\sin\theta, \text{ and } \sum F_y = 0 \Rightarrow N = mg\cos\theta$$

For the dynamic case

$$\sum F_x = m\ddot{x} = -k(x+\Delta) + \mu_s N + mg\sin\theta - \mu_k N \frac{\dot{x}}{|\dot{x}|}$$

Combining these three equations yields

$$m\ddot{x} + \mu_k mg\cos\theta \frac{\dot{x}}{|\dot{x}|} + kx = 0$$

Note that as the angle θ goes to zero the equation of motion becomes that of a spring mass system with Coulomb friction on a flat surface as it should.

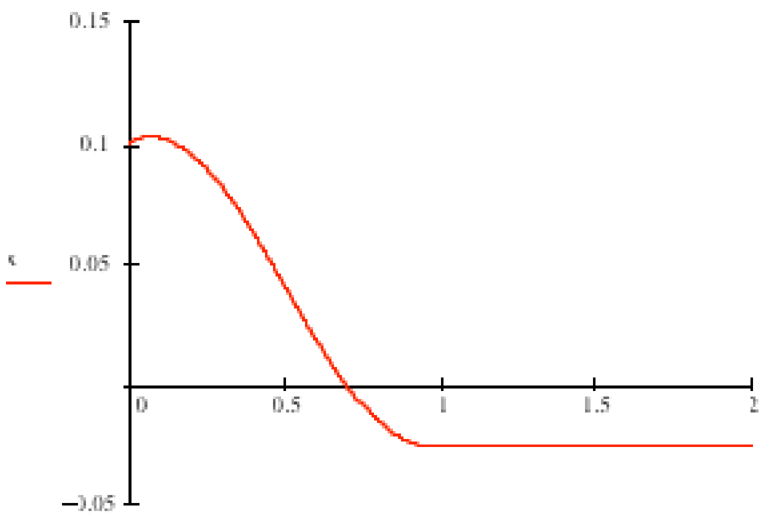
Answer: The oscillation dies out after about 0.9 s. This is illustrated in the following Mathcad code and plot.

$$X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \qquad k := 3000 \qquad m := 250 \qquad \mu := 0.05$$

$$D(t, X) := \begin{bmatrix} X_1 \\ \dfrac{-k}{m} \cdot X_0 - \cos(20\,\deg) \cdot \mu \cdot g \cdot \dfrac{X_1}{|X_1|} \end{bmatrix}$$

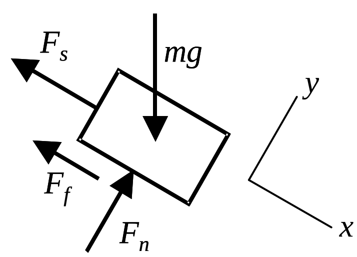
$$Z := \text{rkfixed}(X, 0, 10, 5000, D)$$

$$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle}$$



Alternate Solution (Courtesy of Prof. Chin An Tan of Wayne State University):

<u>Static Analysis</u>:



In this problem, $x(t)$ is defined as the displacement of the mass from the equilibrium position of the spring-mass system under friction. Thus, the first issue to address is how to determine this equilibrium position, or what is this equilibrium position. In reality, the mass is attached onto an initially unstretched spring on the incline. The free body diagram of the system is as shown. The governing equation of motion is:

$$m\ddot{X} = -k\overset{\text{zero initially}}{\cancel{X}} - F_f + mg\sin\theta$$

where $X(t)$ is defined as the displacement measured from the unstretched position of the spring. Note that since the spring is initially unstretched, the spring force $F_s = kX$ is zero initially. If the coefficient of static friction μ_s is sufficiently large, i.e., $\mu_s > \tan(\theta)$, then the mass remains stationary and the spring is unstretched with the mass-spring-friction in equilibrium. Also, in that case, the friction force $F_f \le \mu_s \underbrace{mg\cos\theta}_{F_N}$, not necessarily equal to the maximum static friction. In other words, these situations may hold at equilibrium: (1) the maximum static friction may not be achieved; and (2) there may be no displacement in the spring at all. In this example, $\tan(20^\circ) = 0.364$ and one would expect that μ_s (not given) should be smaller than 0.364 since $\mu_k = 0.05$ (very small). Thus, one would expect the mass to move downward initially (due to weight overcoming the maximum static friction). The mass will then likely oscillate and eventually settle into an equilibrium position with the spring stretched.

<u>Dynamic Analysis</u>:

The equation of motion for this system is:

$$m\ddot{x} = -kx - \mu mg\cos\theta \frac{\dot{x}}{|\dot{x}|}$$

where $x(t)$ is the displacement measured from the equilibrium position. Define $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$. Employing the state-space formulation, we transform the original second-order ODE into a set of two first-order ODEs. The state-space equations (for MATLAB code) are:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} x_2(t) \\ -\mu g\cos\theta \dfrac{x_2}{|x_2|} - \dfrac{kx_1}{m} \end{Bmatrix}$$

MATLAB Code:

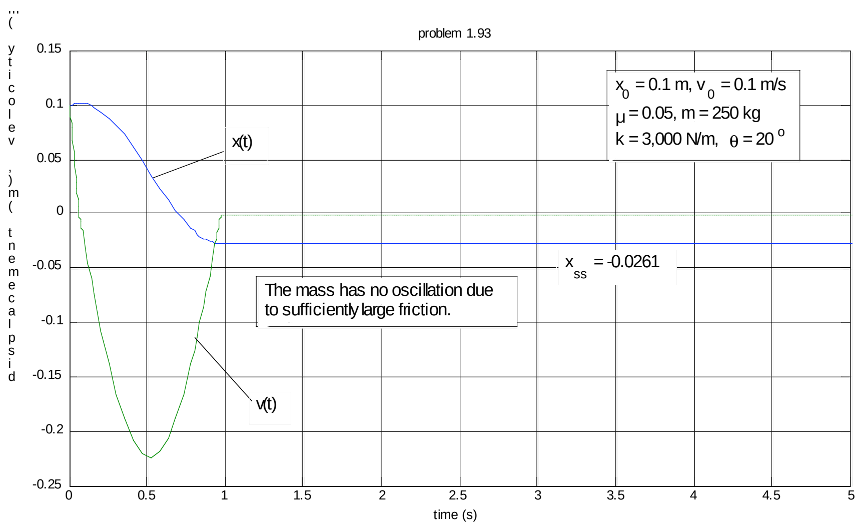
```
x0=[0.1, 0.1];
ts=[0, 5];
[t,x]=ode45('f1_93',ts,x0);
plot(t,x(:,1), t,x(:,2))
title('problem 1.93'); grid on;
xlabel('time (s)');ylabel('displacement (m), velocity (m/s)');

%-----------------------------------------------
function xdot = f1_93(t,x)
% computes derivatives for the state-space ODEs
m=250; k=3000; mu=0.05; g=9.81;
angle = 20*pi/180;
xdot(1) = x(2);
xdot(2) = -k/m*x(1) - mu*g*cos(angle)*sign(x(2));
% use the sign function to improve computation time
xdot = [xdot(1); xdot(2)];
```

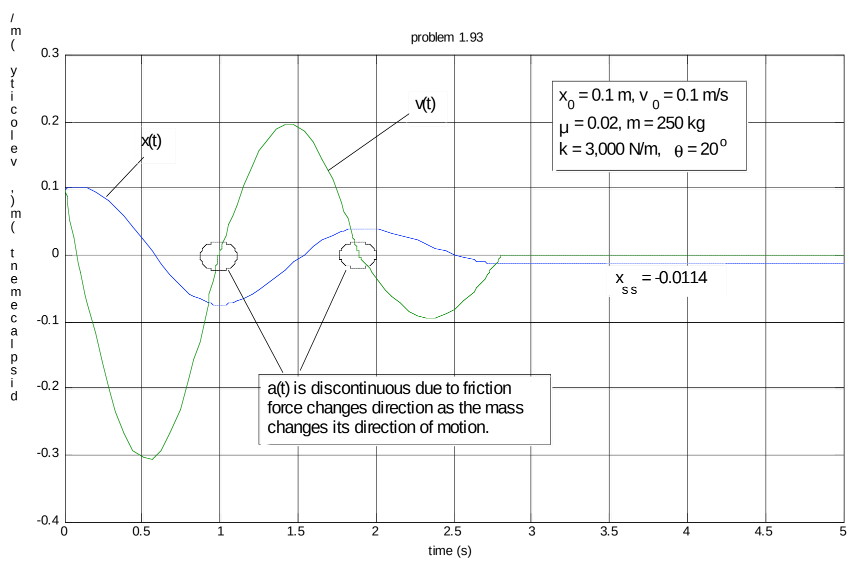
Plots for $\mu = 0.05$ and $\mu = 0.02$ cases are shown. From the $\mu = 0.05$ simulation results, the oscillation dies out after about 0.96 seconds (using `ginput(1)` command to estimate). Note that the acceleration may be discontinuous at $v = 0$ due to the nature of the friction force.

<u>Effects of μ</u>:

Comparing the figures, we see that reducing μ leads to more oscillations (takes longer time to dissipate the energy). Note that since there is a positive initial velocity, the mass is bounded to move down the incline initially. However, if μ is sufficiently large, there may be no oscillation at all and the mass will just come to a stop (as in the case of

$\mu = 0.05$). This is analogous to an overdamped mass-damper-spring system. On the other hand, when μ is very small (say, close to zero), the mass will oscillate for a long time before it comes to a stop.





<u>Discussion on the ceasing of motion</u>:

Note that when motion ceases, the mass reaches another state of equilibrium. In both simulation cases, this occurs while the mass is moving upward (negative velocity). Note that the steady-state value of $x(t)$ is very small, suggesting that this is indeed the true equilibrium position, which represents a balance of the spring force, weight component along the incline, and the static friction.

1.105* Compute and plot the response of a system with Coulomb damping of equation (1.90) for the case where $x_0 = 0.5$ m, $v_0 = 0$, $\mu = 0.1$, $m = 100$ kg and $k = 1500$ N/m. How long does it take for the vibration to die out?

Solution: Here the solution is computed in Mathcad using the following code. Any of the codes may be used. The system dies out in about 3.2 sec.

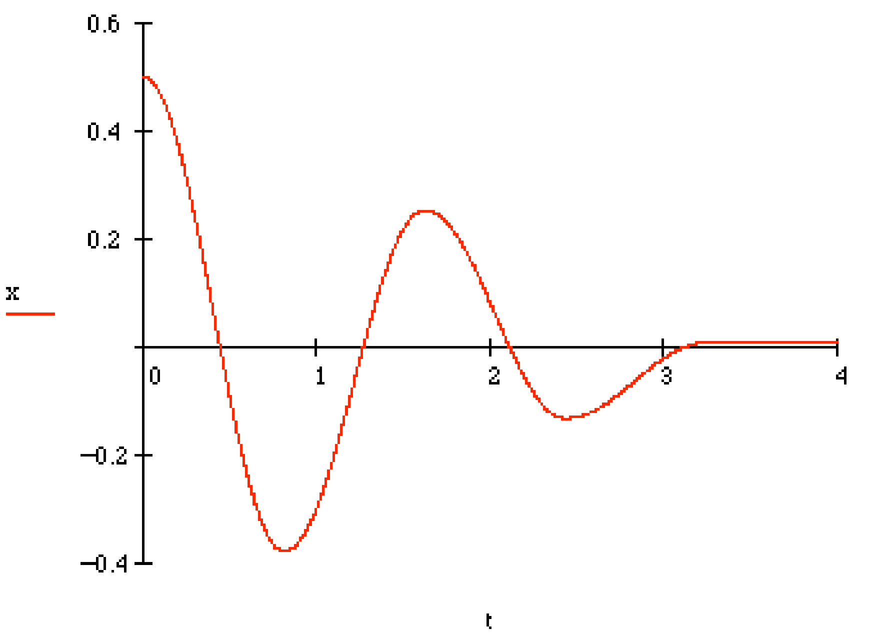
1.94

$X := \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \qquad k := 1500 \qquad m := 100 \qquad \mu := 0.1$

$$D(t, X) := \begin{bmatrix} X_1 \\ \frac{-k}{m} \cdot X_0 - \cos(20\ \mathrm{deg}) \cdot \mu \cdot g \cdot \frac{X_1}{|X_1|} \end{bmatrix}$$

$Z := \mathrm{rkfixed}(X, 0, 10, 5000, D)$

$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle}$



1.106* A mass moves in a fluid against sliding friction as illustrated in Figure P1.106. Model the damping force as a slow fluid (i.e., linear viscous damping) plus Coulomb friction because of the sliding, with the following parameters: m = 250 kg, μ =0.01, c = 25 kg/s and k =3000 N/m . a) Compute and plot the response to the initial conditions: x_0 = 0.1 m, v_0 = 0.1 m/s. b) Compute and plot the response to the initial conditions: x_0 = 0.1 m, v_0 = 1 m/s. How long does it take for the vibration to die out in each case?

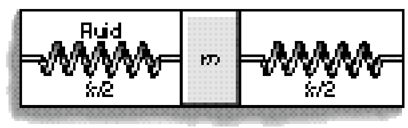
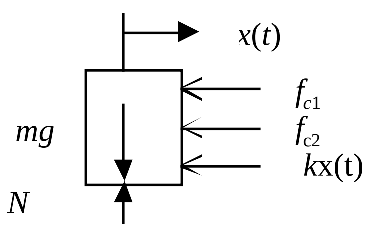


Figure P1.106

Solution: A free-body diagram yields the equation of motion.



$$m\ddot{x}(t) + \mu mg\,\mathrm{sgn}(\dot{x}) + c\dot{x}(t) + kx(t) = 0$$

where the vertical sum of forces gives the magnitude $\mu N = \mu mg$ for the Coulomb force as in figure 1.41.

The equation of motion can be solved by using any of the codes mentioned or by using the toolbox. Here a Mathcad session is presented using a fixed order Runge Kutta integration. Note that the oscillations die out after 4.8 seconds for v_0=0.1 m/s for the larger initial velocity of v_0=1 m/s the oscillations go on quite a bit longer ending only after about 13 seconds. While the next problem shows that the viscous damping can be changed to reduce the settling time, this example shows how dependent the response is on the value of the initial conditions. In a linear system the settling time, or time it takes to die out is only dependent on the system parameters, not the initial conditions. This makes design much more difficult for nonlinear systems.

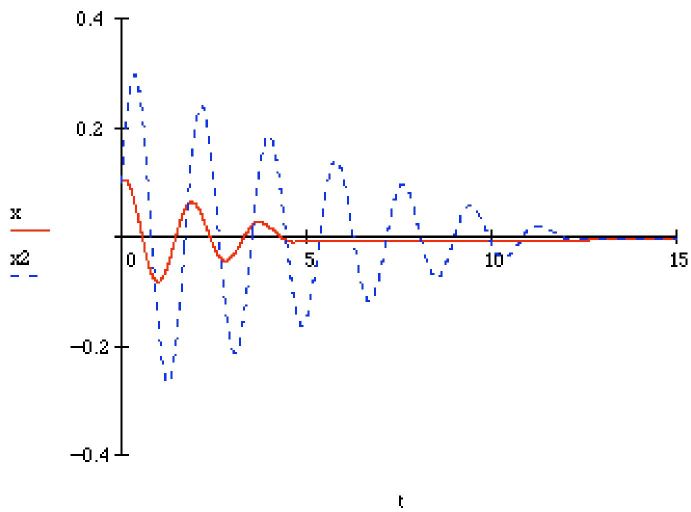
1.95

$c := 25 \quad k := 3000 \quad m := 250 \quad \mu := 0.01 \quad X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \quad X2 := \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$

$$D(t, X) := \begin{bmatrix} X_1 \\ \frac{-k}{m} \cdot X_0 - \mu \cdot g \cdot \frac{X_1}{|X_1|} - \frac{c}{m} \cdot X_1 \end{bmatrix} \qquad D2(t, X2) := \begin{bmatrix} X2_1 \\ \frac{-k}{m} \cdot X2_0 - \mu \cdot g \cdot \frac{X2_1}{|X2_1|} - \frac{c}{m} \cdot X2_1 \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 15, 3000, D) \qquad Z2 := \text{rkfixed}(X2, 0, 15, 3000, D2)$

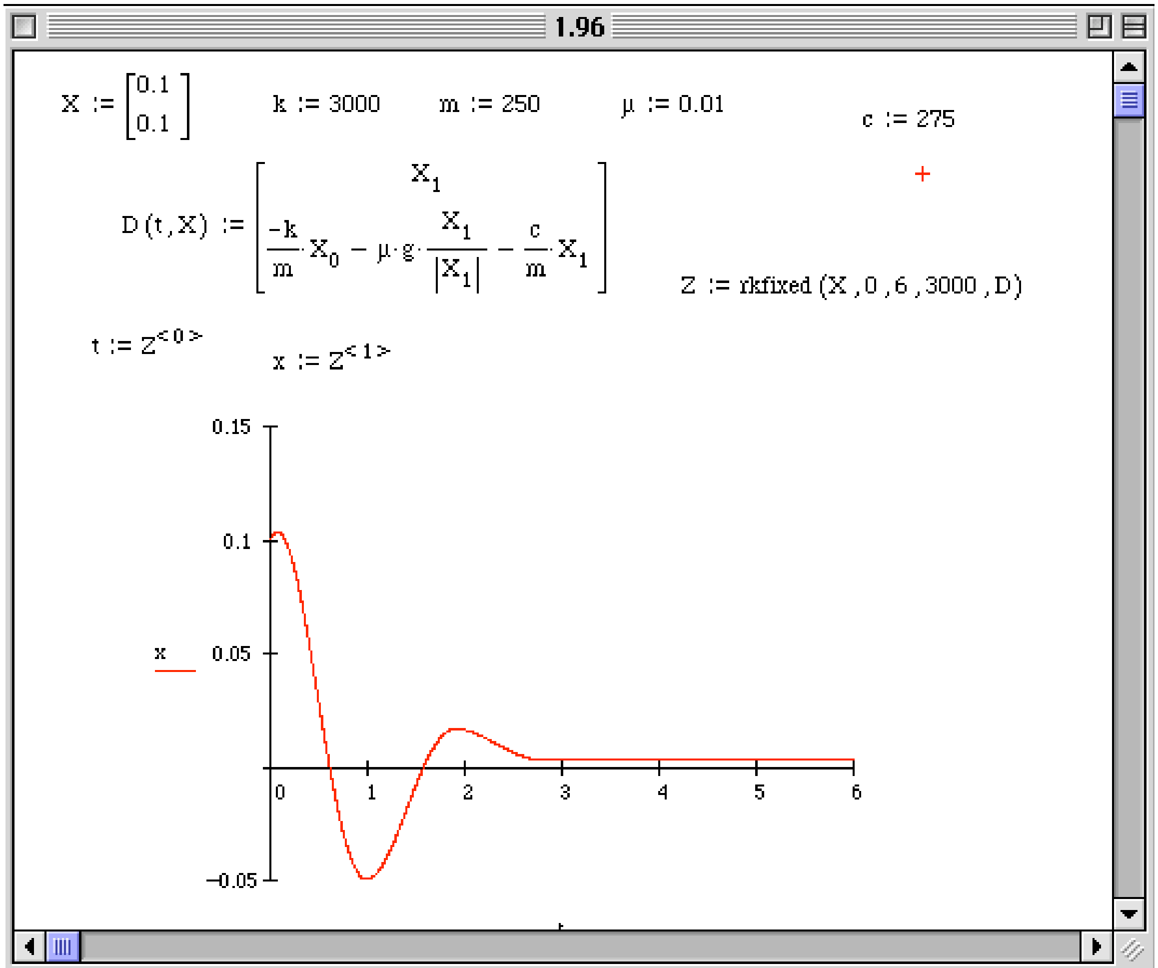
$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle} \qquad x2 := Z2^{\langle 1 \rangle}$



1.107* Consider the system of Problem 1.106 part (a), and compute a new damping coefficient, *c*, that will cause the vibration to die out after one oscillation.

Solution: Working in any of the codes, use the simulation from the last problem and change the damping coefficient *c* until the desired response is obtained. A Mathcad solution is given which requires an order of magnitude higher damping coefficient,

<u>c = 275 kg/s</u>



1.108 Compute the equilibrium positions of $\ddot{x} + \omega_n^2 x + \beta x^2 = 0$. How many are there?

Solution: The equation of motion in state space form is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_n^2 x_1 - \beta x_1^2$$

The equilibrium points are computed from:

$$x_2 = 0$$
$$-\omega_n^2 x_1 - \beta x_1^2 = 0$$

Solving yields the two equilibrium points:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\omega_n^2 / \beta \\ 0 \end{bmatrix}$$

1.109 Compute the equilibrium positions of $\ddot{x} + \omega_n^2 x - \beta^2 x^3 + \gamma x^5 = 0$. How many are there?

Solution: The equation of motion in state space form is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_n^2 x_1 + \beta^2 x_1^3 - \gamma x_1^5$$

The equilibrium points are computed from:

$$x_2 = 0$$
$$-\omega_n^2 x_1 + \beta^2 x_1^2 - \gamma x_1^5 = 0$$

Solving yields the five equilibrium points (one for each root of the previous equation). The first equilibrium (the linear case) is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Next divide $-\omega_n^2 x_1 + \beta^2 x_1^2 - \gamma x_1^5 = 0$ by x_1 to obtain:

$$-\omega_n^2 + \beta^2 x_1^2 - \gamma x_1^4 = 0$$

which is quadratic in $x_1{}^2$ and has the following roots which define the remaining four equilibrium points: $x_2 = 0$ and

$$\underline{\underline{x_1 = \pm\sqrt{\frac{-\beta^2 + \sqrt{\beta^4 - 4\gamma\omega_n^2}}{-2\gamma}}}}$$

$$\underline{\underline{x_1 = \pm\sqrt{\frac{-\beta^2 - \sqrt{\beta^4 - 4\gamma\omega_n^2}}{-2\gamma}}}}$$

1.110* Consider the pendulum example 1.10.3 with length of 1 m an initial conditions of $\theta_0 = \pi/10$ rad and $\dot{\theta}_0 = 0$. Compare the difference between the response of the linear version of the pendulum equation (i.e. with $\sin(\theta) = \theta$) and the response of the nonlinear version of the pendulum equation by plotting the response of both for four periods.

Solution: First consider the linear solution. Using the formula's given in the text the solution of the linear system is just: $\theta(t) = 0.314\sin(3.132t + \pi/2)$. The following Mathcad code, plots the linear solution on the same plot as a numerical solution of the nonlinear system.

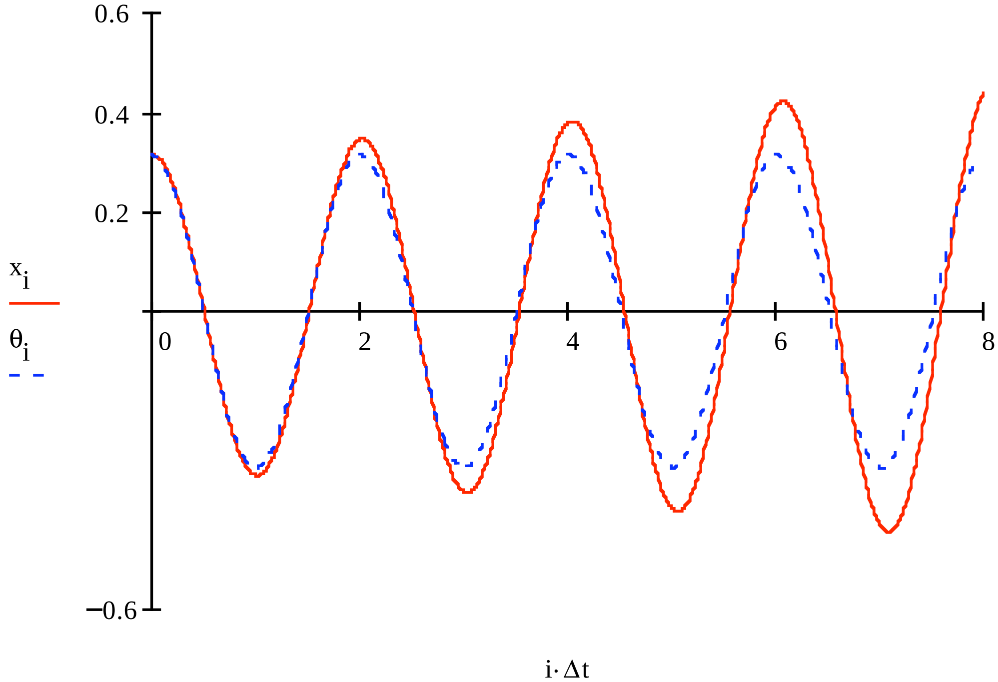
$i := 0..\ 800$

$\Delta t := 0.01$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} \dfrac{\pi}{10} \\ 0 \end{bmatrix}$$

$$\theta_i := 0.314 \cdot \sin\left(3.132 \cdot \Delta t \cdot i + \frac{\pi}{2}\right)$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + v_i \cdot \Delta t \\ v_i - \Delta t \cdot \left(\sin\left(x_i\right)\right)\ 9.81 \end{bmatrix}$$



Note how the amplitude of the nonlinear system is growing. The difference between the linear and the nonlinear plots are a function of the ration of the linear spring stiffness and the nonlinear coefficient, and of course the size of the initial condition. It is work it to investigate the various possibilities, to learn just when the linear approximation completely fails.

1.111* Repeat Problem 1.110 if the initial displacement is $\theta_0 = \pi/2$ rad.

Solution: The solution in Mathcad is:

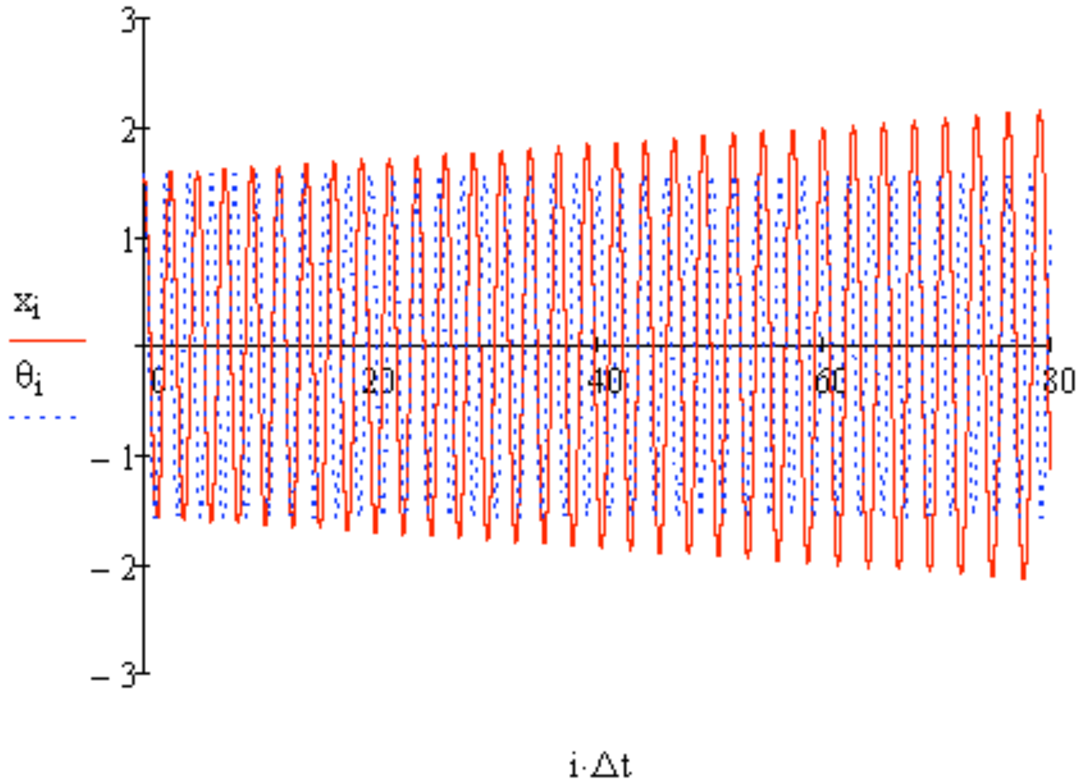
$$i := 0..80000 \qquad \omega := \sqrt{9.81}$$

$$\Delta t := 0.001$$

$$\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} := \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix} \qquad \theta_i := \frac{\pi}{2} \cdot \sin\left(\omega \cdot \Delta t \cdot i + \frac{\pi}{2}\right)$$

$$\begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} := \begin{bmatrix} x_i + v_i \cdot \Delta t \\ v_i - \Delta t \cdot \left(\sin\left(x_i\right)\right) 9.81 \end{bmatrix}$$

Here both solutions oscillate around the "stable" equilibrium, but the nonlinear solution is not oscillating at the natural frequency and is increasing in amplitude.



1.112 If the pendulum of Example 1.10.3 is given an initial condition near the equilibrium position of $\theta_0 = \pi$ rad and $\dot{\theta}_0 = 0$, does it oscillate around this equilibrium?

Solution The pendulum will not oscillate around this equilibrium as it is unstable. Rather it will "wind" around the equilibrium as indicated in the solution to Example 1.10.4.

1.113* Calculate the response of the system of Problem 1.109 for the initial conditions of x_0 = 0.01 m, v_0 = 0, and a natural frequency of 3 rad/s and for β = 100, γ = 0.

Solution: In Mathcad the solution is given using a simple Euler integration as follows:

$\Delta t := 0.01$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \qquad \omega := 3 \qquad A := \frac{1}{\omega}\cdot\sqrt{\omega^2\cdot(x_0)^2}$$

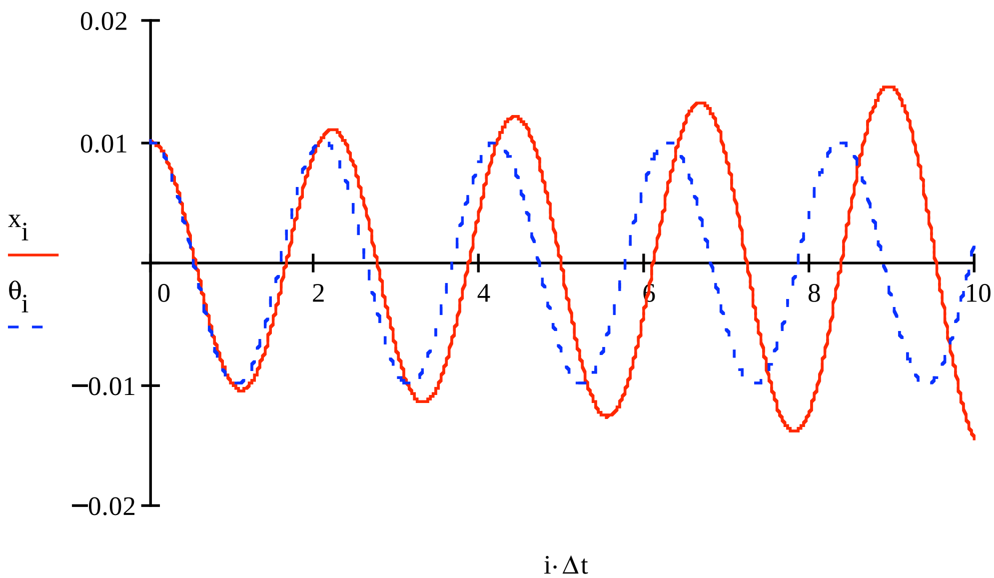
$\beta:=100$

$i := 0 .. 1000$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + v_i\cdot\Delta t \\ v_i - \Delta t\cdot\left[\omega^2\cdot x_i - \beta^2\cdot(x_i)^3\right] \end{bmatrix}$$

$$\theta_i := A\cdot\sin\left(3\cdot\Delta t\cdot i + \frac{\pi}{2}\right)$$

This is the linear solution $\theta(t)$



The other codes may be used to compute this solution as well.

1.114* Repeat problem 1.113 and plot the response of the linear version of the system (β =0) on the same plot to compare the difference between the linear and nonlinear versions of this equation of motion.

Solution: The solution is computed and plotted in the solution of Problem 1.113. Note that the linear solution starts out very close to the nonlinear solution. The two solutions however diverge. They look similar, but the nonlinear solution is growing in amplitude and period.