

Problems Section 1.3 (Numbers 1.31 through 1.46)

1.31 Solve $\ddot{x} + 4\dot{x} + x = 0$ for $x_0 = 1$ mm, $v_0 = 0$ mm/s. Sketch your results and determine which root dominates.

Solution:

Given $\ddot{x} + 4\dot{x} + x = 0$ where $x_0 = 1$ mm, $v_0 = 0$

Let

Substitute these into the equation of motion to get:

$$ar^2e^{rt} + 4are^{rt} + ae^{rt} = 0$$

$$\Rightarrow r^2 + 4r + 1 = 0 \Rightarrow r_{1,2} = -2 \pm \sqrt{3}$$

So

$$x = a_1 e^{(-2+\sqrt{3})t} + a_2 e^{(-2-\sqrt{3})t}$$

$$\dot{x} = (-2 + \sqrt{3})a_1 e^{(-2+\sqrt{3})t} + (-2 - \sqrt{3})a_2 e^{(-2-\sqrt{3})t}$$

Applying initial conditions yields,

$$x_0 = a_1 + a_2 \Rightarrow x_0 - a_2 = a_1 \quad (1)$$

$$v_0 = (-2 + \sqrt{3})a_1 + (-2 - \sqrt{3})a_2 \quad (2)$$

Substitute equation (1) into (2)

$$v_0 = (-2 + \sqrt{3})(x_0 - a_2) + (-2 - \sqrt{3})a_2$$

$$v_0 = (-2 + \sqrt{3})x_0 - 2\sqrt{3}a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 + (-2 + \sqrt{3})x_0}{2\sqrt{3}}$$

Substituting the value of a_2 into equation (1), and solving for a_1 yields,

$$a_1 = \frac{v_0 + (2 + \sqrt{3})x_0}{2\sqrt{3}}$$

$$\therefore x(t) = \frac{v_0 + (2 + \sqrt{3})x_0}{2\sqrt{3}} e^{(-2+\sqrt{3})t} + \frac{-v_0 + (-2 + \sqrt{3})x_0}{2\sqrt{3}} e^{(-2-\sqrt{3})t}$$

The response is dominated by the root: $-2 + \sqrt{3}$ as the other root dies off very fast.

1.32 Solve $\ddot{x} + 2\dot{x} + 2x = 0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and sketch the response. You may wish to sketch $x(t) = e^{-t}$ and $x(t) = -e^{-t}$ first.

Solution:

Given $\ddot{x} + 2\dot{x} + x = 0$ where $x_0 = 0$, $v_0 = 1$ mm/s

Let: $x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2e^{rt}$

Substitute into the equation of motion to get

$$ar^2e^{rt} + 2are^{rt} + ae^{rt} = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow r_{1,2} = -1 \pm i$$

So

$$x = c_1e^{(-1+i)t} + c_2e^{(-1-i)t} \Rightarrow \dot{x} = (-1+i)c_1e^{(-1+i)t} + (-1-i)c_2e^{(-1-i)t}$$

Initial conditions:

$$x_0 = x(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \quad (1)$$

$$v_0 = \dot{x}(0) = (-1+i)c_1 + (-1-i)c_2 = 1 \quad (2)$$

Substituting equation (1) into (2)

$$v_0 = (-1+i)c_1 - (-1-i)c_1 = 1$$

$$c_1 = -\frac{1}{2}i, \quad c_2 = \frac{1}{2}i$$

$$x(t) = -\frac{1}{2}ie^{(-1+i)t} + \frac{1}{2}ie^{(-1-i)t} = -\frac{1}{2}ie^{-t}(e^{it} - e^{-it})$$

Applying Euler's formula

$$x(t) = -\frac{1}{2}ie^{-t}(\cos t + i\sin t - (\cos t - i\sin t))$$

$$\underline{x(t) = e^{-t} \sin t}$$

Alternately use equations (1.36) and (1.38). The plot is similar to figure 1.11.

1.33 Derive the form of λ_1 and λ_2 given by equation (1.31) from equation (1.28) and the definition of the damping ratio.

Solution:

$$\text{Equation (1.28): } \lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m} \sqrt{c^2 - 4km}$$

$$\text{Rewrite, } \lambda_{1,2} = -\left(\frac{c}{2\sqrt{m}\sqrt{m}}\right)\left(\frac{\sqrt{k}}{\sqrt{k}}\right) \pm \frac{1}{2\sqrt{m}\sqrt{m}}\left(\frac{\sqrt{k}}{\sqrt{k}}\right)\left(\frac{c}{c}\right)\sqrt{c^2 - \left(2\sqrt{km}^2\right)\left(\frac{c}{c}\right)^2}$$

$$\text{Rearrange, } \lambda_{1,2} = -\left(\frac{c}{2\sqrt{km}}\right)\left(\frac{\sqrt{k}}{\sqrt{m}}\right) \pm \frac{c}{2\sqrt{km}}\left(\frac{\sqrt{k}}{\sqrt{m}}\right)\left(\frac{1}{c}\right)\sqrt{c^2 \left[1 - \left(\frac{2\sqrt{km}}{c}\right)^2\right]}$$

Substitute:

$$\begin{aligned} \omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{c}{2\sqrt{km}} &\Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \zeta\omega_n\left(\frac{1}{c}\right)c\sqrt{1 - \left(\frac{1}{\zeta^2}\right)} \\ &\Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 \left[1 - \left(\frac{1}{\zeta^2}\right)\right]} \\ &\Rightarrow \lambda_{1,2} = \underline{-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}} \end{aligned}$$

1.34 Use the Euler formulas to derive equation (1.36) from equation (1.35) and to determine the relationships listed in Window 1.4.

Solution:

Equation (1.35): $x(t) = e^{-\zeta\omega_n t} (a_1 e^{j\omega_n \sqrt{1-\zeta^2} t} - a_2 e^{-j\omega_n \sqrt{1-\zeta^2} t})$

From Euler,

$$\begin{aligned} x(t) &= e^{-\zeta\omega_n t} (a_1 \cos(\omega_n \sqrt{1-\zeta^2} t) + a_1 j \sin(\omega_n \sqrt{1-\zeta^2} t) \\ &\quad + a_2 \cos(\omega_n \sqrt{1-\zeta^2} t) - a_2 j \sin(\omega_n \sqrt{1-\zeta^2} t)) \\ &= e^{-\zeta\omega_n t} (a_1 + a_2) \cos \omega_d t + j(a_1 - a_2) \sin \omega_d t \end{aligned}$$

Let: $A_1 = (a_1 + a_2)$, $A_2 = (a_1 - a_2)$, then this last expression becomes

$$x(t) = e^{-\zeta\omega_n t} A_1 \cos \omega_d t + A_2 \sin \omega_d t$$

Next use the trig identity:

$$A = \sqrt{A_1^2 + A_2^2}, \quad \phi = \tan^{-1} \frac{A_2}{A_1}$$

to get: $x(t) = e^{-\zeta\omega_n t} A \sin(\omega_d t + \phi)$

1.35 Using equation (1.35) as the form of the solution of the underdamped system, calculate the values for the constants a_1 and a_2 in terms of the initial conditions x_0 and v_0 .

Solution:

Equation (1.35):

$$x(t) = e^{-\zeta\omega_n t} \left(a_1 e^{j\omega_n \sqrt{1-\zeta^2} t} + a_2 e^{-j\omega_n \sqrt{1-\zeta^2} t} \right)$$

$$\dot{x}(t) = (-\zeta\omega_n + j\omega_n \sqrt{1-\zeta^2}) a_1 e^{(-\zeta\omega_n + j\omega_n \sqrt{1-\zeta^2}) t} + (-\zeta\omega_n - j\omega_n \sqrt{1-\zeta^2}) a_2 e^{(-\zeta\omega_n - j\omega_n \sqrt{1-\zeta^2}) t}$$

Initial conditions

$$x_0 = x(0) = a_1 + a_2 \Rightarrow a_1 = x_0 - a_2 \quad (1)$$

$$v_0 = \dot{x}(0) = (-\zeta\omega_n + j\omega_n \sqrt{1-\zeta^2}) a_1 + (-\zeta\omega_n - j\omega_n \sqrt{1-\zeta^2}) a_2 \quad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-\zeta\omega_n + j\omega_n \sqrt{1-\zeta^2} \right) (x_0 - a_2) + \left(-\zeta\omega_n - j\omega_n \sqrt{1-\zeta^2} \right) a_2$$

$$v_0 = \left(-\zeta\omega_n + j\omega_n \sqrt{1-\zeta^2} \right) x_0 - 2j\omega_n \sqrt{1-\zeta^2} a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + j\omega_n \sqrt{1-\zeta^2} x_0}{2j\omega_n \sqrt{1-\zeta^2}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + j\omega_n \sqrt{1-\zeta^2} x_0}{2j\omega_n \sqrt{1-\zeta^2}}$$

1.36 Calculate the constants A and ϕ in terms of the initial conditions and thus verify equation (1.38) for the underdamped case.

Solution:

From Equation (1.36),

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

Applying initial conditions ($t = 0$) yields,

$$x_0 = A \sin \phi \quad (1)$$

$$v_0 = \dot{x}_0 = -\zeta\omega_n A \sin \phi + \omega_d A \cos \phi \quad (2)$$

Next solve these two simultaneous equations for the two unknowns A and ϕ .

From (1),

$$A = \frac{x_0}{\sin \phi} \quad (3)$$

Substituting (3) into (2) yields

$$v_0 = -\zeta\omega_n x_0 + \frac{\omega_d x_0}{\tan \phi} \Rightarrow \tan \phi = \frac{x_0 \omega_d}{v_0 + \zeta\omega_n x_0}.$$

Hence,

$$\phi = \tan^{-1} \left[\frac{x_0 \omega_d}{v_0 + \zeta\omega_n x_0} \right] \quad (4)$$

$$\text{From (3),} \quad \sin \phi = \frac{x_0}{A} \quad (5)$$

$$\text{and From (4),} \quad \cos \phi = \frac{v_0 + \zeta\omega_n x_0}{(x_0 \omega_d)^2 + (v_0 + \zeta\omega_n x_0)^2} \quad (6)$$

Substituting (5) and (6) into (2) yields,

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0 \omega_d)^2}{\omega_d^2}}$$

which are the same as equation (1.38)

1.37 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equations (1.42) and (1.43) for the overdamped case.

Solution: From Equation (1.41)

$$x(t) = e^{-\zeta\omega_n t} \left(a_1 e^{\omega_n \sqrt{\zeta^2 - 1} t} + a_2 e^{-\omega_n \sqrt{\zeta^2 - 1} t} \right)$$

taking the time derivative yields:

$$\dot{x}(t) = (-\zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1}) a_1 e^{(-\zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1})t} + (-\zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1}) a_2 e^{(-\zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1})t}$$

Applying initial conditions yields,

$$x_0 = x(0) = a_1 + a_2 \Rightarrow x_0 - a_2 = a_1 \quad (1)$$

$$v_0 = \dot{x}(0) = (-\zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1}) a_1 + (-\zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1}) a_2 \quad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$\begin{aligned} v_0 &= (-\zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1})(x_0 - a_2) + (-\zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1}) a_2 \\ v_0 &= (-\zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1}) x_0 - 2\omega_n \sqrt{\zeta^2 - 1} a_2 \end{aligned}$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + \omega_n \sqrt{\zeta^2 - 1} x_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + \omega_n \sqrt{\zeta^2 - 1} x_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

- 1.38** Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equation (1.46) for the critically damped case.

Solution:

From Equation (1.45),

$$x(t) = (a_1 + a_2 t) e^{-\omega_n t}$$

$$\Rightarrow \dot{x}_0 = -\omega_n a_1 e^{-\omega_n t} - \omega_n a_2 t e^{-\omega_n t} + a_2 e^{-\omega_n t}$$

Applying the initial conditions yields:

$$x_0 = a_1 \quad (1)$$

and

$$v_0 = \dot{x}(0) = a_2 - \omega_n a_1 \quad (2)$$

solving these two simultaneous equations for the two unknowns a_1 and a_2 .

Substituting (1) into (2) yields,

$$a_1 = x_0$$

$$a_2 = v_0 + \omega_n x_0$$

which are the same as equation (1.46).

- 1.39** Using the definition of the damping ratio and the undamped natural frequency, derive equation (1.48) from (1.47).

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{thus,} \quad \frac{k}{m} = \omega_n^2$$

$$\zeta = \frac{c}{2\sqrt{km}} \quad \text{thus,} \quad \frac{c}{m} = \frac{2\zeta\sqrt{km}}{m} = 2\zeta\omega_n$$

$$\text{Therefore, } \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

becomes,

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

- 1.40** For a damped system, m , c , and k are known to be $m = 1$ kg, $c = 2$ kg/s, $k = 10$ N/m. Calculate the value of ζ and ω_n . Is the system overdamped, underdamped, or critically damped?

Solution:

Given: $m = 1$ kg, $c = 2$ kg/s, $k = 10$ N/m

$$\text{Natural frequency: } \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16 \text{ rad/s}$$

$$\text{Damping ratio: } \zeta = \frac{c}{2\omega_n m} = \frac{2}{2(3.16)(1)} = 0.316$$

$$\text{Damped natural frequency: } \omega_d = \sqrt{10} \sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2} = 3.0 \text{ rad/s}$$

Since $0 < \zeta < 1$, the system is **underdamped**.

- 1.41** Plot $x(t)$ for a damped system of natural frequency $\omega_n = 2$ rad/s and initial conditions $x_0 = 1$ mm, $v_0 = 1$ mm, for the following values of the damping ratio: $\zeta = 0.01, \zeta = 0.2, \zeta = 0.1, \zeta = 0.4$, and $\zeta = 0.8$.

Solution:

Given: $\omega_n = 2$ rad/s, $x_0 = 1$ mm, $v_0 = 1$ mm, $\zeta_i = [0.01; 0.2; 0.1; 0.4; 0.8]$

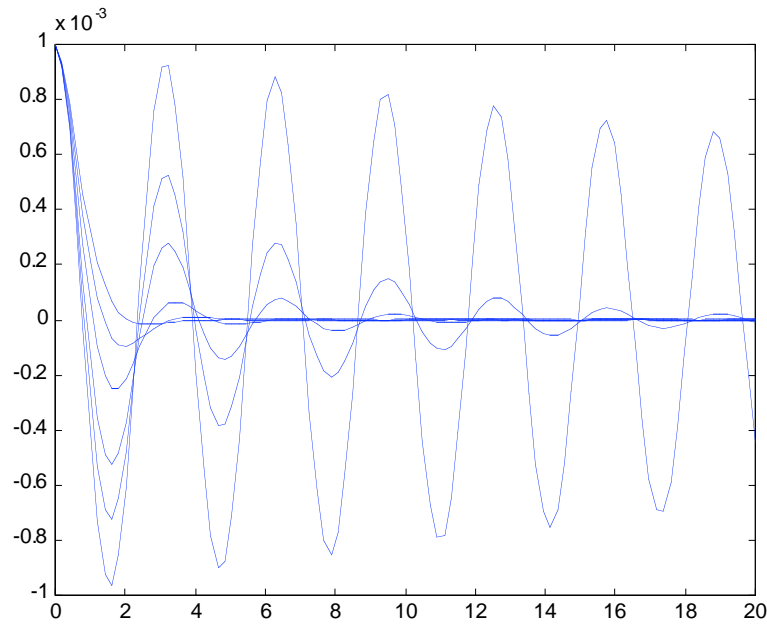
Underdamped cases:

$$\therefore \omega_{di} = \omega_n \sqrt{1 - \zeta_i^2}$$

From equation 1.38,

$$A_i = \sqrt{\frac{(v_0 + \zeta_i \omega_n x_0)^2 + (x_0 \omega_{di})^2}{\omega_{di}^2}} \quad \phi_i = \tan^{-1} \frac{x_0 \omega_{di}}{v_0 + \zeta_i \omega_n x_0}$$

The response is plotted for each value of the damping ratio in the following using Matlab:



- 1.42** Plot the response $x(t)$ of an underdamped system with $\omega_n = 2$ rad/s, $\zeta = 0.1$, and $v_0 = 0$ for the following initial displacements: $x_0 = 10$ mm and $x_0 = 100$ mm.

Solution:

Given: $\omega_n = 2$ rad/s, $\zeta = 0.1$, $v_0 = 0$, $x_0 = 10$ mm and $x_0 = 100$ mm.

Underdamped case:

$$\therefore \omega_d = \omega_n \sqrt{1 - \zeta^2} = 2\sqrt{1 - 0.1^2} = 1.99 \text{ rad/s}$$

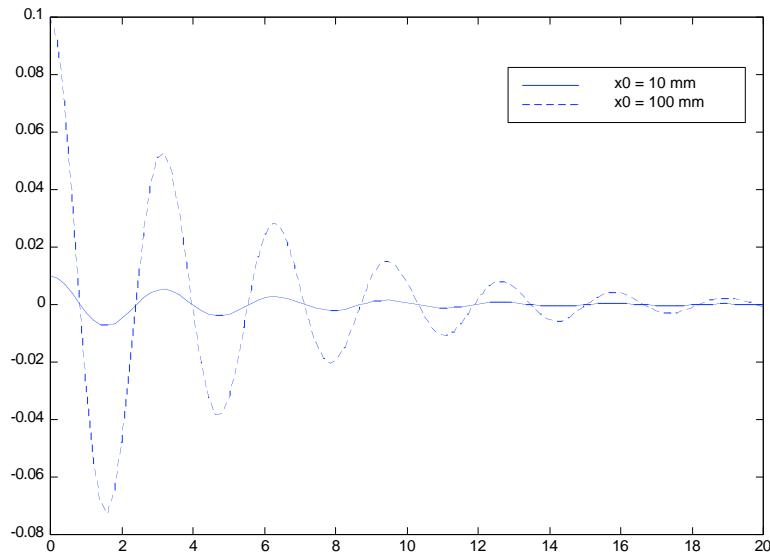
$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}} = 1.01 x_0$$

$$\phi = \tan^{-1} \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} = 1.47 \text{ rad}$$

where

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

The following is a plot from Matlab.



1.43 Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the response.

Solution: This is a problem with negative damping which can be used to tie into Section 1.8 on stability, or can be used to practice the method for deriving the solution using the method suggested following equation (1.13) and eluded to at the start of the section on damping. To this end let $x(t) = Ae^{\lambda t}$ the equation of motion to get:

$$(\lambda^2 - \lambda + 1)e^{\lambda t} = 0$$

This yields the characteristic equation:

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}j, \text{ where } j = \sqrt{-1}$$

There are thus two solutions as expected and these combine to form

$$x(t) = e^{0.5t} \left(Ae^{\frac{\sqrt{3}}{2}jt} + Be^{-\frac{\sqrt{3}}{2}jt} \right)$$

Using the Euler relationship for the term in parenthesis as given in Window 1.4, this can be written as

$$x(t) = e^{0.5t} \left(A_1 \cos \frac{\sqrt{3}}{2}t + A_2 \sin \frac{\sqrt{3}}{2}t \right)$$

Next apply the initial conditions to determine the two constants of integration:

$$x(0) = 1 = A_1(1) + A_2(0) \Rightarrow A_1 = 1$$

Differentiate the solution to get the velocity and then apply the initial velocity condition to get

$$\dot{x}(t) =$$

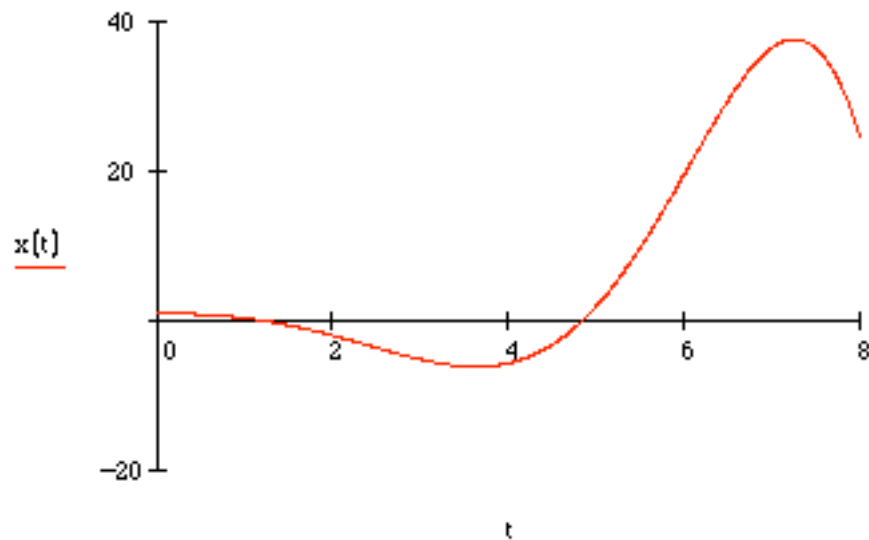
$$\frac{1}{2}e^0 \left(A_1 \cos \frac{\sqrt{3}}{2}0 + A_2 \sin \frac{\sqrt{3}}{2}0 \right) + e^0 \frac{\sqrt{3}}{2} \left(-A_1 \sin \frac{\sqrt{3}}{2}0 + A_2 \cos \frac{\sqrt{3}}{2}0 \right) = 0$$

$$\Rightarrow A_1 + \sqrt{3}(A_2) = 0 \Rightarrow A_2 = -\frac{1}{\sqrt{3}},$$

$$\Rightarrow x(t) = e^{0.5t} \left(\cos \frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}t \right)$$

This function oscillates with increasing amplitude as shown in the following plot which shows the increasing amplitude. This type of response is referred to as a flutter instability. This plot is from Mathcad.

$$x(t) := e^{0.5 \cdot t} \left[\cos\left(\frac{\sqrt{3}}{2} \cdot t\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} \cdot t\right) \right]$$



- 1.44** A spring-mass-damper system has mass of 100 kg, stiffness of 3000 N/m and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000 \text{ N/m}}{100 \text{ kg}}} = 5.477 \text{ rad/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{300}{2\sqrt{km}} = \frac{300}{2\sqrt{(3000)(100)}} = 0.274$$

Since ζ is less than 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 5.27 \text{ rad/s}$.

- 1.45** A sketch of a valve and rocker arm system for an internal combustion engine is give in Figure P1.45. Model the system as a pendulum attached to a spring and a mass and assume the oil provides viscous damping in the range of $\zeta = 0.01$. Determine the equations of motion and calculate an expression for the natural frequency and the damped natural frequency. Here J is the rotational inertia of the rocker arm about its pivot point, k is the stiffness of the valve spring and m is the mass of the valve and stem. Ignore the mass of the spring.

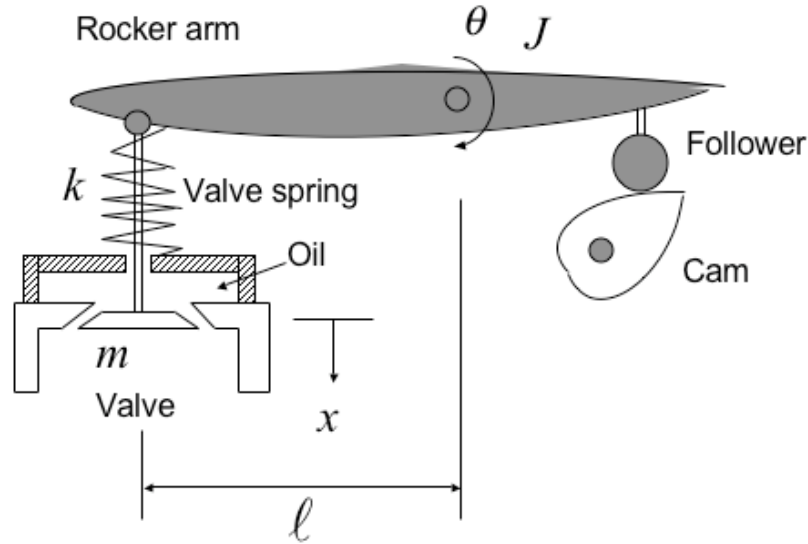
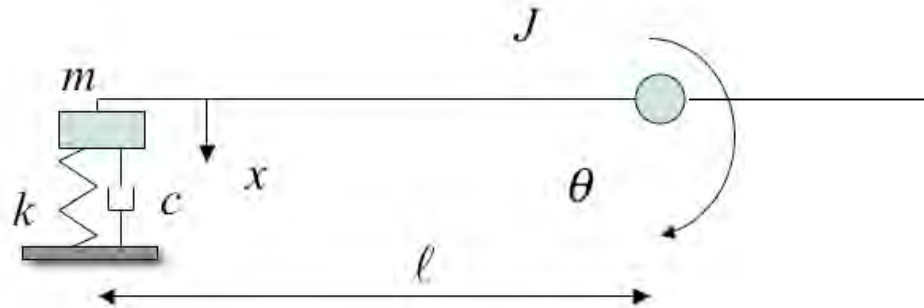


Figure P1.45

Solution: The model is of the form given in the figure. You may wish to give this figure as a hint as it may not be obvious to all students.



Taking moments about the pivot point yields:

$$\begin{aligned}(J + m\ell^2)\ddot{\theta}(t) &= -kx\ell - c\dot{x}\ell = -k\ell^2\theta - c\ell^2\dot{\theta} \\ \Rightarrow (J + m\ell^2)\ddot{\theta}(t) + c\ell^2\dot{\theta} + k\ell^2\theta &= 0\end{aligned}$$

Next divide by the leading coefficient to get;

$$\ddot{\theta}(t) + \left(\frac{c\ell^2}{J + m\ell^2} \right) \dot{\theta}(t) + \frac{k\ell^2}{J + m\ell^2} \theta(t) = 0$$

From the coefficient of q , the undamped natural frequency is

$$\omega_n = \sqrt{\frac{k\ell^2}{J + m\ell^2}} \text{ rad/s}$$

From equation (1.37), the damped natural frequency becomes

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 0.99995 \sqrt{\frac{k\ell^2}{J + m\ell^2}} \sim \sqrt{\frac{k\ell^2}{J + m\ell^2}}$$

This is effectively the same as the undamped frequency for any reasonable accuracy. However, it is important to point out that the resulting response will still decay, even though the frequency of oscillation is unchanged. So even though the numerical value seems to have a negligible effect on the frequency of oscillation, the small value of damping still makes a substantial difference in the response.

- 1.46** A spring-mass-damper system has mass of 150 kg, stiffness of 1500 N/m and damping coefficient of 200 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Is the system overdamped, underdamped or critically damped? Does the solution oscillate?

Solution: Working straight from the definitions:

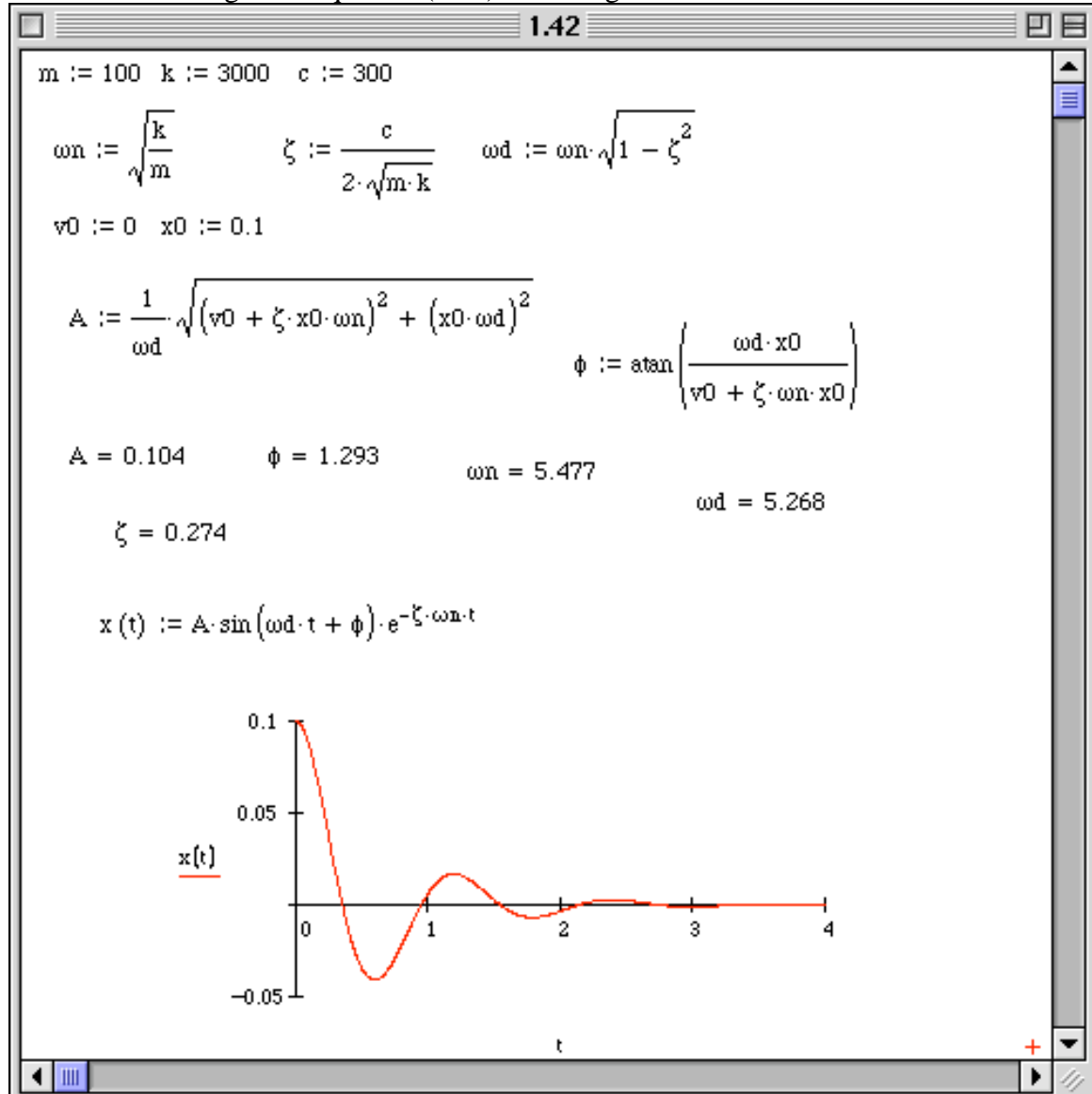
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500 \text{ N/m}}{150 \text{ kg}}} = 3.162 \text{ rad/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{200}{2\sqrt{km}} = \frac{200}{2\sqrt{(1500)(150)}} = 0.211$$

This last expression follows from the equation following equation (1.29). Since ζ is less than 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 3.091 \text{ rad/s}$, which follows from equation (1.37).

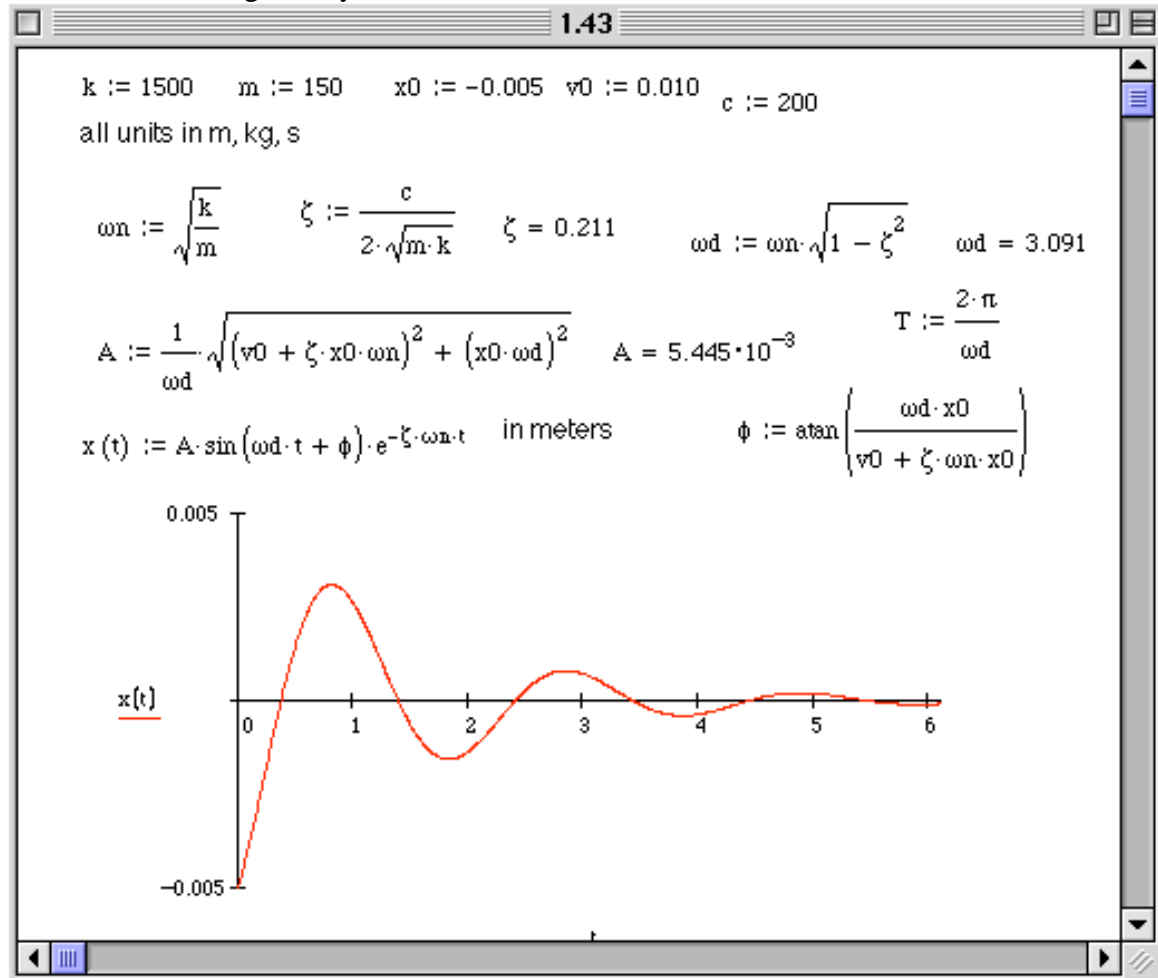
- 1.47*** The system of Problem 1.44 is given a zero initial velocity and an initial displacement of 0.1 m. Calculate the form of the response and plot it for as long as it takes to die out.

Solution: Working from equation (1.38) and using Mathcad the solution is:



1.48* The system of Problem 1.46 is given an initial velocity of 10 mm/s and an initial displacement of -5 mm. Calculate the form of the response and plot it for as long as it takes to die out. How long does it take to die out?

Solution: Working from equation (1.38), the form of the response is programmed in Mathcad and is given by:



It appears to take a little over 6 to 8 seconds to die out. This can also be plotted in Matlab, Mathematica or by using the toolbox.

1.49* Choose the damping coefficient of a spring-mass-damper system with mass of 150 kg and stiffness of 2000 N/m such that it's response will die out after about 2 s, given a zero initial position and an initial velocity of 10 mm/s.

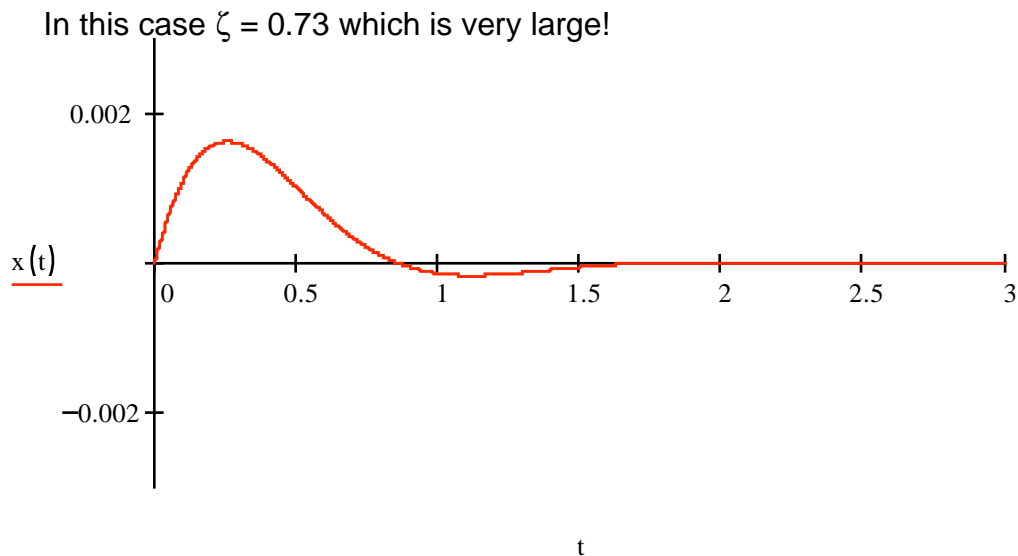
Solution: Working in Mathcad, the response is plotted and the value of c is changed until the desired decay rate is met:

$$\begin{aligned} c &:= 800 & k &:= 2000 & v_0 &:= 0.010 & x_0 &:= 0 \\ m &:= 150 \end{aligned}$$

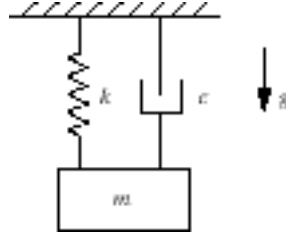
$$\zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}} \quad \omega_n := \sqrt{\frac{k}{m}} \quad \omega_d := \omega_n \cdot \sqrt{1 - \zeta^2}$$

$$x(t) := A \cdot \sin(\omega_n \cdot t + \phi) \cdot e^{-\zeta \cdot \omega_n \cdot t}$$

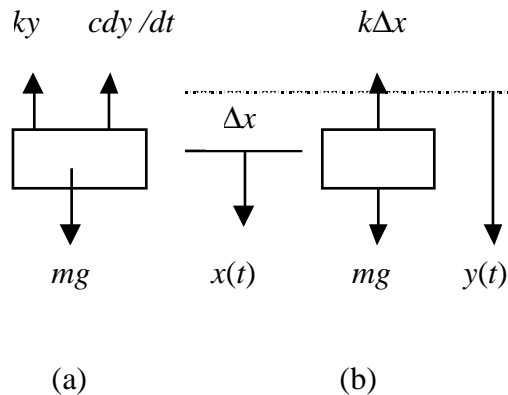
$$\phi := \operatorname{atan}\left(\frac{\omega_d \cdot x_0}{v_0 + \zeta \cdot \omega_n \cdot x_0}\right)$$



- 1.50** Derive the equation of motion of the system in Figure P1.50 and discuss the effect of gravity on the natural frequency and the damping ratio.



Solution: This requires two free body diagrams. One for the dynamic case and one to show static equilibrium.



From the free-body diagram of static equilibrium (b) we have that $mg = k\Delta x$, where Δx represents the static deflection. From the free-body diagram of the dynamic case given in (a) the equation of motion is:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) - mg = 0$$

From the diagram, $y(t) = x(t) + \Delta x$. Since Δx is a constant, differentiating and substitution into the equation of motion yields:

$$\dot{y}(t) = \dot{x}(t) \quad \text{and} \quad \ddot{y}(t) = \ddot{x}(t) \Rightarrow$$

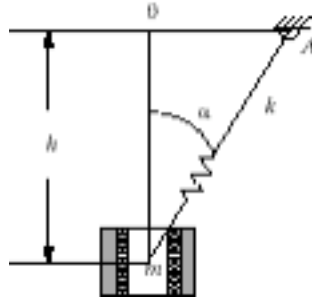
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + \underbrace{(k\Delta x - mg)}_{=0} = 0$$

where the last term is zero from the relation resulting from static equilibrium. Dividing by the mass yields the standard form

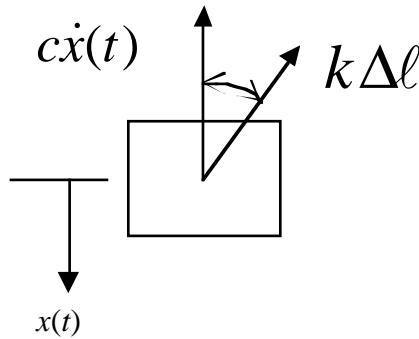
$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = 0$$

It is clear that gravity has no effect on the damping ratio ζ or the natural frequency ω_n . Note that the damping force is not present in the static case because the velocity is zero.

- 1.51** Derive the equation of motion of the system in Figure P1.46 and discuss the effect of gravity on the natural frequency and the damping ratio. You may have to make some approximations of the cosine. Assume the bearings provide a viscous damping force only in the vertical direction. (From the A. Diaz-Jimenez, *South African Mechanical Engineer*, Vol. 26, pp. 65-69, 1976)



Solution: First consider a free-body diagram of the system:



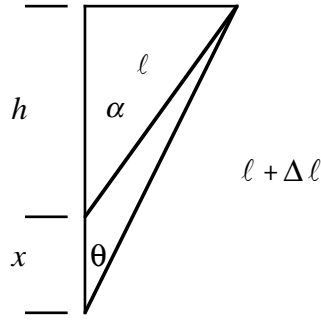
Let α be the angle between the damping and stiffness force. The equation of motion becomes

$$m\ddot{x}(t) = -c\dot{x}(t) - k(\Delta\ell + \delta_s)\cos\alpha$$

From static equilibrium, the free-body diagram (above with $c = 0$ and stiffness force $k\delta_s$) yields: $\sum F_x = 0 = mg - k\delta_s\cos\alpha$. Thus the equation of motion becomes

$$m\ddot{x} + c\dot{x} + k\Delta\ell\cos\alpha = 0 \quad (1)$$

Next consider the geometry of the dynamic state:



From the definition of cosine applied to the two different triangles:

$$\cos \alpha = \frac{h}{\ell} \quad \text{and} \quad \cos \theta = \frac{h+x}{\ell + \Delta \ell}$$

Next assume small deflections so that the angles are nearly the same $\cos \alpha = \cos \theta$, so that

$$\frac{h}{\ell} \approx \frac{h+x}{\ell + \Delta \ell} \Rightarrow \Delta \ell \approx x \frac{\ell}{h} \Rightarrow \Delta \ell \approx \frac{x}{\cos \alpha}$$

For small motion, then this last expression can be substituted into the equation of motion (1) above to yield:

$$m\ddot{x} + c\dot{x} + kx = 0, \quad \alpha \text{ and } x \text{ small}$$

Thus the frequency and damping ratio have the standard values and are not effected by gravity. If the small angle assumption is not made, the frequency can be approximated as

$$\omega_n = \sqrt{\frac{k}{m} \cos^2 \alpha + \frac{g}{h} \sin^2 \alpha}, \quad \zeta = \frac{c}{2m\omega_n}$$

as detailed in the reference above. For a small angle these reduce to the normal values of

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n}$$

as derived here.