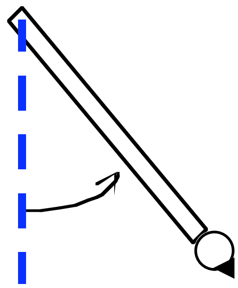
Problems and Solutions Section 1.4 (problems 1.52 through 1.65)

1.52 Calculate the frequency of the compound pendulum of Figure 1.20(b) if a mass m_T is added to the tip, by using the energy method.

Solution Using the notation and coordinates of Figure 1.20 and adding a tip mass the diagram becomes:



If the mass of the pendulum bar is m, and it is lumped at the center of mass the energies become:

Potential Energy:

$$U = \frac{1}{2}(\ell - \ell\cos\theta)mg + (\ell - \ell\cos\theta)m_t g$$
$$= \frac{\ell}{2}(1 - \cos\theta)(mg + 2m_t g)$$

Kinetic Energy:

$$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_t\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2 + \frac{1}{2}m_t\ell^2\dot{\theta}^2$$
$$= (\frac{1}{6}m + \frac{1}{2}m_t)\ell^2\dot{\theta}^2$$

Conservation of energy (Equation 1.52) requires $T + U$ = constant:

$$\frac{\ell}{2}(1 - \cos\theta)(mg + 2m_t g) + (\frac{1}{6}m + \frac{1}{2}m_t)\ell^2\dot{\theta}^2 = C$$

Differentiating with respect to time yields:

$$\frac{\ell}{2}(\sin\theta)(mg + 2m_t g)\dot{\theta} + (\frac{1}{3}m + m_t)\ell^2\dot{\theta}\ddot{\theta} = 0$$
$$\Rightarrow (\frac{1}{3}m + m_t)\ell\ddot{\theta} + \frac{1}{2}(mg + 2m_t g)\sin\theta = 0$$

Rearranging and approximating using the small angle formula $\sin\theta \sim \theta$, yields:

$$\ddot{\theta}(t)+\left(\frac{\frac{m}{2}+m_t}{\frac{1}{3}m+m_t}\frac{g}{\ell}\right)\theta(t)=0\Rightarrow \underline{\omega_n=\sqrt{\frac{3m+6m_t}{2m+6m_t}}\sqrt{\frac{g}{\ell}}\text{ rad/s}}$$

Note that this solution makes sense because if $m_t = 0$ it reduces to the frequency of the pendulum equation for a bar, and if $m = 0$ it reduces to the frequency of a massless pendulum with only a tip mass.

1.53 Calculate the total energy in a damped system with frequency 2 rad/s and damping ratio $\zeta = 0.01$ with mass 10 kg for the case $x_0 = 0.1$ and $v_0 = 0$. Plot the total energy versus time.

Solution: Given: $\omega_n = 2$ rad/s, $\zeta = 0.01$, $m = 10$ kg, $x_0 = 0.1$ mm, $v_0 = 0$.

Calculate the stiffness and damped natural frequency:

$$k = m\omega_n^2 = 10(2)^2 = 40 \text{ N/m}$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 2\sqrt{1-0.01^2} = 2 \text{ rad/s}$$

The total energy of the damped system is

$$E(t)=\frac{1}{2}m\dot{x}^2(t)+\frac{1}{2}kx(t)$$

where
$$x(t) = Ae^{-0.02t}\sin(2t+\phi)$$
$$\dot{x}(t) = -0.02Ae^{-0.02t}\sin(2t+\phi)+2Ae^{-0.02t}\cos(2t+\phi)$$

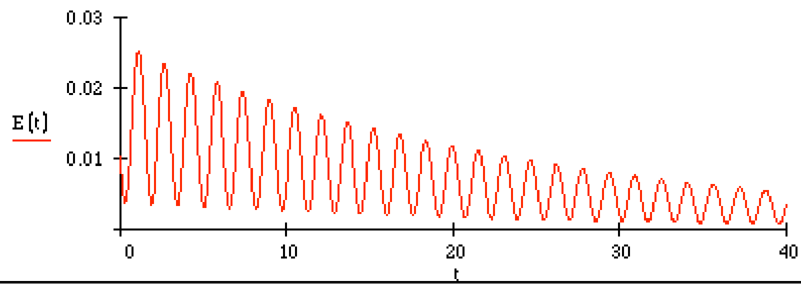
Applying the initial conditions to evaluate the constants of integration yields:

$$x(0) = 0.1 = A\sin\phi$$
$$\dot{x}(0) = 0 = -0.02A\sin\phi + 2A\cos\phi$$
$$\Rightarrow \phi = 1.56 \text{ rad/s}, \quad A = 0.1 \text{ m}$$

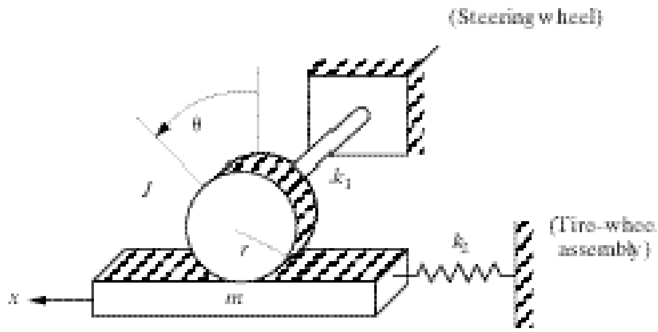
Substitution of these values into $E(t)$ yields:

1.48

$$E(t) := \left[\frac{e^{-0.04 \cdot t}}{2} \cdot \left[((0.1 \cdot \sin(2 \cdot t + 1.56))) + .2 \cdot \cos(2 \cdot t + 1.56)\right]^2\right] + 0.5 \cdot e^{-0.04 \cdot t} \cdot (0.1 \cdot \sin(2 \cdot t + 1.56))^2$$



1.54 Use the energy method to calculate the equation of motion and natural frequency of an airplane's steering mechanism for the nose wheel of its landing gear. The mechanism is modeled as the single-degree-of-freedom system illustrated in Figure P1.54.



The steering wheel and tire assembly are modeled as being fixed at ground for this calculation. The steering rod gear system is modeled as a linear spring and mass system (m, k_2) oscillating in the x direction. The shaft-gear mechanism is modeled as the disk of inertia J and torsional stiffness k_2. The gear J turns through the angle θ such that the disk does not slip on the mass. Obtain an equation in the linear motion x.

Solution: From kinematics: $x = r\theta, \Rightarrow \dot{x} = r\dot{\theta}$

Kinetic energy: $$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$$

Potential energy: $$U = \frac{1}{2}k_2 x^2 + \frac{1}{2}k_1\theta^2$$

Substitute $\theta = \dfrac{x}{r}: T + U = \dfrac{1}{2}\dfrac{J}{r^2}\dot{x}^2 + \dfrac{1}{2}m\dot{x}^2 + \dfrac{1}{2}k_2 x^2 + \dfrac{1}{2}\dfrac{k_1}{r^2}x^2$

Derivative: $$\frac{d(T+U)}{dt} = 0$$

$$\frac{J}{r^2}\ddot{x}\dot{x} + m\ddot{x}\dot{x} + k_2 x\dot{x} + \frac{k_1}{r^2}x\dot{x} = 0$$

$$\left[\left(\frac{J}{r^2} + m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x\right]\dot{x} = 0$$

Equation of motion: $$\left(\frac{J}{r^2} + m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x = 0$$

Natural frequency: $$\omega_n = \sqrt{\frac{k_2 + \dfrac{k_1}{r^2}}{\dfrac{J}{r^2} + m}} = \sqrt{\frac{k_1 + r^2 k_2}{J + mr^2}}$$

1.55 A control pedal of an aircraft can be modeled as the single-degree-of-freedom system of Figure P1.55. Consider the lever as a massless shaft and the pedal as a lumped mass at the end of the shaft. Use the energy method to determine the equation of motion in θ and calculate the natural frequency of the system. Assume the spring to be unstretched at θ = 0.

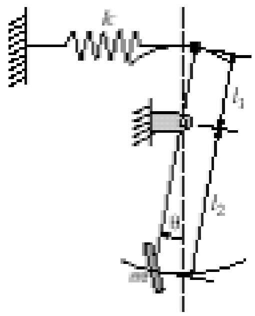


Figure P1.55

Solution: In the figure let the mass at θ = 0 be the lowest point for potential energy. Then**,** the height of the mass *m* is $(1-\cos\theta)\ell_2$.

Kinematic relation: $x = \ell_1\theta$

Kinetic Energy: $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\ell_2^2\dot{\theta}^2$

Potential Energy: $U = \frac{1}{2}k(\ell_1\theta)^2 + mg\ell_2(1-\cos\theta)$

Taking the derivative of the total energy yields:

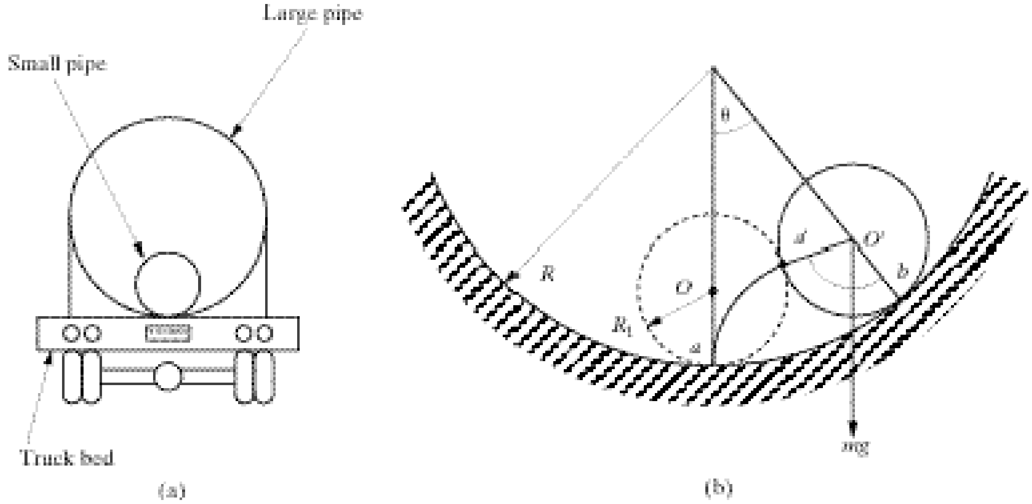
$$\frac{d}{dt}(T+U) = m\ell_2^2\dot{\theta}\ddot{\theta} + k(\ell_1^2\theta)\dot{\theta} + mg\ell_2(\sin\theta)\dot{\theta} = 0$$

Rearranging, dividing by $d\theta/dt$ and approximating sinθ with θ yields:

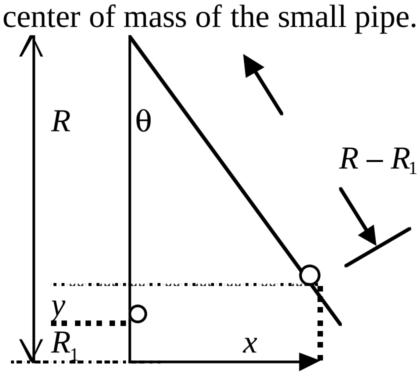
$$m\ell_2^2\ddot{\theta} + (k\ell_1^2 + mg\ell_2)\theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{k\ell_1^2 + mg\ell_2}{m\ell_2^2}}$$

1.56 To save space, two large pipes are shipped one stacked inside the other as indicated in Figure P1.56. Calculate the natural frequency of vibration of the smaller pipe (of radius R_1) rolling back and forth inside the larger pipe (of radius R). Use the energy method and assume that the inside pipe rolls without slipping and has a mass m.



Solution: Let θ be the angle that the line between the centers of the large pipe and the small pipe make with the vertical and let α be the angle that the small pipe rotates through. Let R be the radius of the large pipe and R_1 the radius of the smaller pipe. Then the kinetic energy of the system is the translational plus rotational of the small pipe. The potential energy is that of the rise in height of the center of mass of the small pipe.



From the drawing:

$$y + (R - R_1)\cos\theta + R_1 = R$$

$$\Rightarrow y = (R - R_1)(1 - \cos\theta)$$

$$\Rightarrow \dot{y} = (R - R_1)\sin(\theta)\dot{\theta}$$

Likewise examination of the value of x yields:

$$x = (R - R_1)\sin\theta$$

$$\Rightarrow \dot{x} = (R - R_1)\cos\theta\dot{\theta}$$

Let β denote the angle of rotation that the small pipe experiences as viewed in the inertial frame of reference (taken to be the truck bed in this case). Then the total kinetic energy can be written as:

$$T = T_{trans} + T_{rot} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

$$= \frac{1}{2}m(R - R_1)^2(\sin^2\theta + \cos^2\theta)\dot{\theta}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

$$\Rightarrow T = \frac{1}{2}m(R - R_1)^2\dot{\theta}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

The total potential energy becomes just:

$V = mgy = mg(R - R_1)(1 - \cos\theta)$

Now it remains to evaluate the angel β. Let α be the angle that the small pipe rotates in the frame of the big pipe as it rolls (say) up the inside of the larger pipe. Then

$\beta = \theta - \alpha$

were α is the angle "rolled" out as the small pipe rolls from *a* to *b* in figure P1.56. The rolling with out slipping condition implies that arc length *a'b* must equal arc length *ab*. Using the arc length relation this yields that $R\theta = R_1\alpha$. Substitution of the expression $\beta = \theta - \alpha$ yields:

$$R\theta = R_1(\theta - \beta) = R_1\theta - R_1\beta \Rightarrow (R - R_1)\theta = -R_1\beta$$

$$\Rightarrow \beta = \frac{1}{R_1}(R_1 - R)\theta \text{ and } \dot{\beta} = \frac{1}{R_1}(R_1 - R)\dot{\theta}$$

which is the relationship between angular motion of the small pipe relative to the ground (β) and the position of the pipe (θ). Substitution of this last expression into the kinetic energy term yields:

$$T = \frac{1}{2}m(R - R_1)^2\dot{\theta}^2 + \frac{1}{2}I_0\left(\frac{1}{R_1}(R_1 - R)\dot{\theta}\right)^2$$

$$\Rightarrow T = m(R - R_1)^2\dot{\theta}^2$$

Taking the derivative of $T + V$ yields

$$\frac{d}{d\theta}(T + V) = 2m(R - R_1)^2\dot{\theta}\ddot{\theta} + mg(R - R_1)\sin\theta\dot{\theta} = 0$$

$$\Rightarrow 2m(R - R_1)^2\ddot{\theta} + mg(R - R_1)\sin\theta = 0$$

Using the small angle approximation for sine this becomes

$$2m(R - R_1)^2\ddot{\theta} + mg(R - R_1)\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{g}{2(R - R_1)}\theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{g}{2(R - R_1)}}$$

1.57 Consider the example of a simple pendulum given in Example 1.4.2. The pendulum motion is observed to decay with a damping ratio of ζ = 0.001. Determine a damping coefficient and add a viscous damping term to the pendulum equation.

Solution: From example 1.4.2, the equation of motion for a simple pendulum is

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

So $\omega_n = \sqrt{\frac{g}{\ell}}$. With viscous damping the equation of motion in normalized form becomes:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \text{ or with } \zeta \text{ as given :}$$
$$\Rightarrow \ddot{\theta} + 2(.001)\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

The coefficient of the velocity term is

$$\frac{c}{J} = \frac{c}{m\ell^2} = (.002)\sqrt{\frac{g}{\ell}}$$
$$c = (0.002)m\sqrt{g\ell^3}$$

1.58 Determine a damping coefficient for the disk-rod system of Example 1.4.3. Assuming that the damping is due to the material properties of the rod, determine c for the rod if it is observed to have a damping ratio of $\zeta = 0.01$.

Solution: The equation of motion for a disc/rod in torsional vibration is

$$J\ddot{\theta} + k\theta = 0$$

or $$\ddot{\theta} + \omega_n^2\theta = 0 \quad \text{where } \omega_n = \sqrt{\frac{k}{J}}$$

Add viscous damping:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

$$\ddot{\theta} + 2(.01)\sqrt{\frac{k}{J}}\dot{\theta} + \omega_n^2\theta = 0$$

From the velocity term, the damping coefficient must be

$$\frac{c}{J} = (0.02)\sqrt{\frac{k}{J}}$$

$$\Rightarrow c = 0.02\sqrt{kJ}$$

1.59 The rod and disk of Window 1.1 are in torsional vibration. Calculate the damped natural frequency if $J = 1000$ m$^2 \cdot$ kg, $c = 20$ N$\cdot$ m$\cdot$ s/rad, and $k = 400$ N$\cdot$m/rad.

Solution: From Problem 1.57, the equation of motion is

$$J\ddot{\theta} + c\dot{\theta} + k\theta = 0$$

The damped natural frequency is

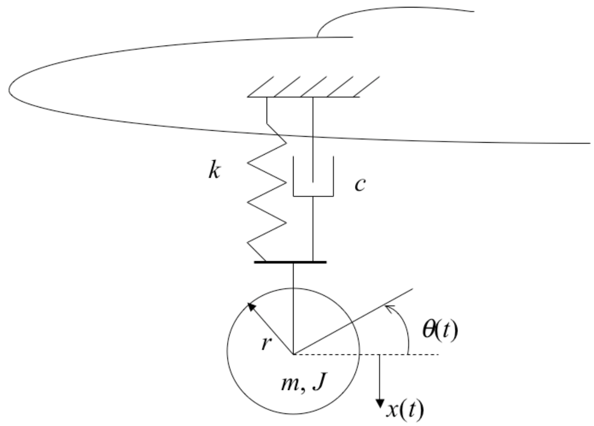
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

where $$\omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{400}{1000}} = 0.632 \text{ rad/s}$$

and $$\zeta = \frac{c}{2\sqrt{kJ}} = \frac{20}{2\sqrt{400 \times 1000}} = 0.0158$$

Thus the damped natural frequency is $\omega_d = 0.632$ rad/s

1.60 Consider the system of P1.60, which represents a simple model of an aircraft landing system. Assume, $x = r\theta$. What is the damped natural frequency?



Solution: From Example 1.4.1, the undamped equation of motion is

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + kx = 0$$

From examining the equation of motion the natural frequency is:

$$\omega_n = \sqrt{\frac{k}{m_{eq}}} = \sqrt{\frac{k}{m + \frac{J}{r^2}}}$$

An add hoc way do to this is to add the damping force to get the damped equation of motion:

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + c\dot{x} + kx = 0$$

The value of ζ is determined by examining the velocity term:

$$\frac{c}{m + \frac{J}{r^2}} = 2\zeta\omega_n \Rightarrow \zeta = \frac{c}{m + \frac{J}{r^2}} \frac{1}{2\sqrt{\frac{k}{m + \frac{J}{r^2}}}}$$

$$\Rightarrow \zeta = \frac{c}{2\sqrt{k\left(m + \frac{J}{r^2}\right)}}$$

Thus the damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m+\frac{J}{r^2}}}\sqrt{1-\left(\frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}}\right)^2}$$

$$\Rightarrow \underline{\omega_d = \sqrt{\frac{k}{m+\frac{J}{r^2}} - \frac{c^2}{4\left(m+\frac{J}{r^2}\right)^2}}} = \frac{r}{2(mr^2+J)}\sqrt{4(kmr^2+kJ)-c^2r^2}$$

1.61 Consider Problem 1.60 with k = 400,000 N·m, m = 1500 kg, J = 100 m^2·kg, r = 25 cm, and c = 8000 N·m·s. Calculate the damping ratio and the damped natural frequency. How much effect does the rotational inertia have on the undamped natural frequency?

Solution: From problem 1.60:

$$\zeta = \frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}} \text{ and } \omega_d = \sqrt{\frac{k}{m+\frac{J}{r^2}} - \frac{c^2}{4\left(m+\frac{J}{r^2}\right)^2}}$$

Given:

$$k = 4\times10^5 \text{ Nm/rad}$$
$$m = 1.5\times10^3 \text{ kg}$$
$$J = 100 \text{ m}^2\text{kg}$$
$$r = 0.25 \text{ m and}$$
$$c = 8\times10^3 \text{ N}\cdot\text{m}\cdot\text{s/rad}$$

Inserting the given values yields

$$\underline{\zeta = 0.114} \text{ and } \underline{\omega_d = 11.16 \text{ rad/s}}$$

For the undamped natural frequency, $\omega_n = \sqrt{\dfrac{k}{m+J/r^2}}$

With the rotational inertia, ω_n = 36.886 rad/s

Without rotational inertia, ω_n = 51.64 rad/s

The effect of the rotational inertia is that it lowers the natural frequency by almost 33%.

1.62 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.62. Model each of the brackets as a spring of stiffness *k*, and assume the inertia of the pulleys is negligible.

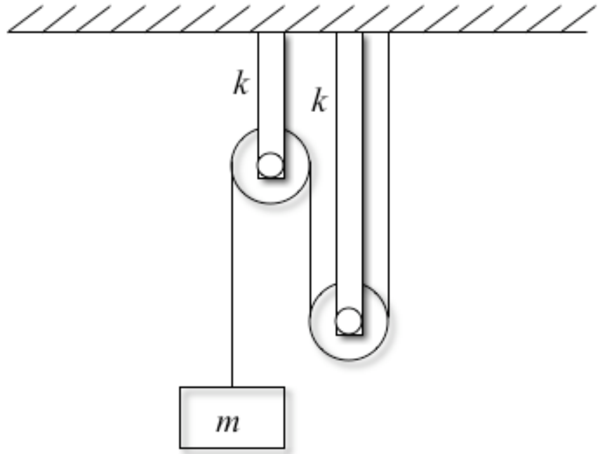


Figure P1.62

Solution: Let *x* denote the distance mass *m* moves, then each spring will deflects a distance *x*/4. Thus the potential energy of the springs is

$$U = 2\times\frac{1}{2}k\left(\frac{x}{4}\right)^2 = \frac{k}{16}x^2$$

The kinetic energy of the mass is

$$T = \frac{1}{2}m\dot{x}^2$$

Using the Lagrange formulation in the form of Equation (1.64):

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{kx^2}{16}\right) = 0 \Rightarrow \frac{d}{dt}\left(m\dot{x}\right) + \frac{k}{8}x = 0$$

$$\Rightarrow \underline{m\ddot{x} + \frac{k}{8}x = 0} \Rightarrow \underline{\omega_n = \frac{1}{2}\sqrt{\frac{k}{2m}}} \text{ rad/s}$$

1.63 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.63. This figure represents a simplified model of a jet engine mounted to a wing through a mechanism which acts as a spring of stiffness *k* and mass m_s. Assume the engine has inertial *J* and mass *m* and that the rotation of the engine is related to the vertical displacement of the engine, *x*(*t*) by the "radius" r_0 (i.e. $x = r_0\theta$).

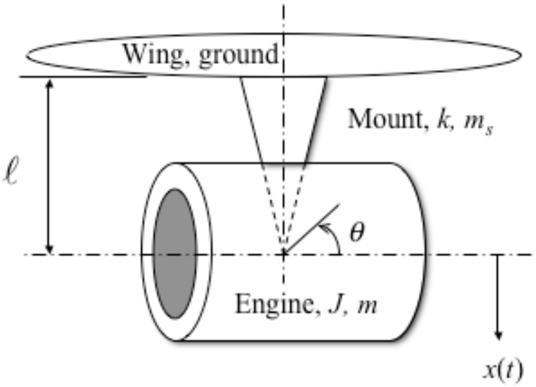


Figure P1.63

Solution: This combines Examples 1.4.1 and 1.4.4. The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + T_{\text{spring}} = \frac{1}{2}\left(m + \frac{J}{r_0^2}\right)\dot{x}^2 + T_{\text{spring}}$$

The kinetic energy in the spring (see example 1.4.4) is

$$T_{\text{spring}} = \frac{1}{2}\frac{m_s}{3}\dot{x}^2$$

Thus the total kinetic energy is

$$T = \frac{1}{2}\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\dot{x}^2$$

The potential energy is just

$$U = \frac{1}{2}kx^2$$

Using the Lagrange formulation of Equation (1.64) the equation of motion results from:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) = 0$$

$$\Rightarrow \underline{\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\ddot{x} + kx = 0}$$

$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)}}} \text{ rad/s}$$

1.64 Lagrange's formulation can also be used for non-conservative systems by adding the applied non-conservative term to the right side of equation (1.64) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R_i}{\partial \dot{q}_i} = 0$$

Here R_i is the *Rayleigh dissipation function* defined in the case of a viscous damper attached to ground by

$$R_i = \frac{1}{2} c\dot{q}_i^2$$

Use this extended Lagrange formulation to derive the equation of motion of the damped automobile suspension of Figure P1.64

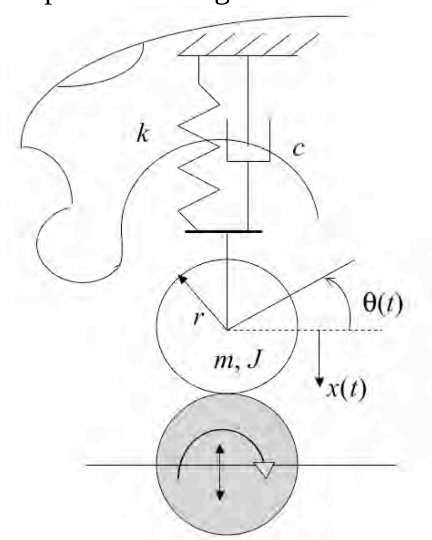


Figure P1.64

Solution: The kinetic energy is (see Example 1.4.1):

$$T = \frac{1}{2}(m + \frac{J}{r^2})\dot{x}^2$$

The potential energy is:

$$U = \frac{1}{2} kx^2$$

The Rayleigh dissipation function is

$$R = \frac{1}{2} c\dot{x}^2$$

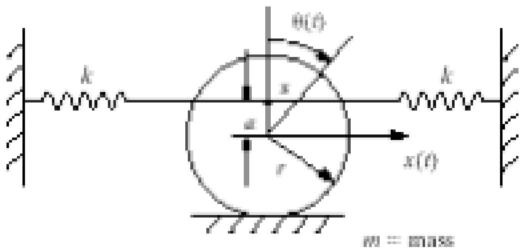
The Lagrange formulation with damping becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R_i}{\partial \dot{q}_i} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}(m + \frac{J}{r^2})\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) + \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}c\dot{x}^2\right) = 0$$

$$\Rightarrow \underline{(m + \frac{J}{r^2})\ddot{x} + c\dot{x} + kx = 0}$$

1.65 Consider the disk of Figure P1.65 connected to two springs. Use the energy method to calculate the system's natural frequency of oscillation for small angles θ(*t*).



Solution:

Known: $x = r\theta, \dot{x} = r\dot{\theta}$ and $J_o = \frac{1}{2}mr^2$

Kinetic energy:

$$T_{rot} = \frac{1}{2}J_o \quad \dot{\theta}^2 = \frac{1}{2}\left(\frac{mr^2}{2}\right) \quad \dot{\theta}^2 = \frac{1}{4}mr^2\dot{\theta}^2$$

$$T_{trans} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mr^2\dot{\theta}^2$$

$$T = T_{rot} + T_{trans} = \frac{1}{4}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2 = \frac{3}{4}mr^2\dot{\theta}^2$$

Potential energy: $U = 2\left(\frac{1}{2}k\left[(a+r)\theta\right]^2\right) = k(a+r)^2\theta^2$

Conservation of energy:

$$T + U = \text{ Constant}$$

$$\frac{d}{dt}(T+U) = 0$$

$$\frac{d}{dt}\left(\frac{3}{4}mr^2\dot{\theta}^2 + k(a+r)^2\theta^2\right) = 0$$

$$\frac{3}{4}mr^2\left(2\dot{\theta}\ddot{\theta}\right) + k(a+r)^2\left(2\dot{\theta}\theta\right) = 0$$

$$\frac{3}{2}mr^2\ddot{\theta} + 2k(a+r)^2\theta = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{2k(a+r)^2}{\frac{3}{2}mr^2}}$$

$$\omega_n = 2\frac{a+r}{r}\sqrt{\frac{k}{3m}} \text{ rad/s}$$