Problems and Solutions Section 1.6 (1.75 through 1.81)

1.75 Show that the logarithmic decrement is equal to

$$\delta = \frac{1}{n}\ln\frac{x_0}{x_n}$$

where x_n is the amplitude of vibration after n cycles have elapsed.

Solution:

$$\ln\left[\frac{x(t)}{x(t+nT)}\right] = \ln\left[\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}{Ae^{-\zeta\omega_n(t+nt)}\sin(\omega_d t+\omega_d nT+\phi)}\right] \qquad (1)$$

Since $n\omega_d T = n(2\pi)$, $\sin(\omega_d t + n\omega_d T + \phi) = \sin(\omega_d t + \phi)$

Hence, Eq. (1) becomes

$$\ln\left[\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}{Ae^{-\zeta\omega_n(t+nT)}e^{-\zeta\omega_n nt}\sin(\omega_d t+\omega_d nt+\phi)}\right] = \ln\left(e^{\zeta\omega_n nT}\right) = n\zeta\omega_n T$$

Since $$\ln\left[\frac{x(t)}{x(t+T)}\right] = \zeta\omega_n T = \delta,$$

Then $$\ln\left[\frac{x(t)}{x(t+nT)}\right] = n\delta$$

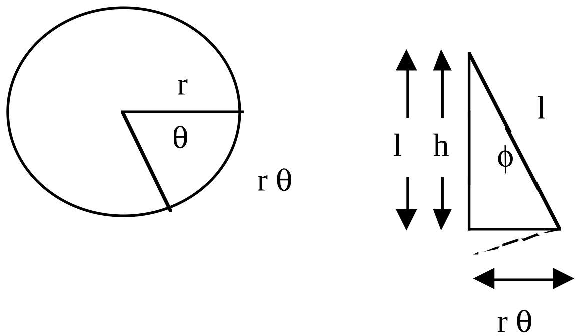
Therefore,

$$\delta = \frac{1}{n}\ln\frac{x_o}{x_n} \quad \begin{array}{l}\leftarrow \text{original amplitude}\\ \leftarrow \text{amplitude } n \text{ cycles later}\end{array}$$

Here $x_0 = x(0)$.

1.76 Derive the equation (1.70) for the trifalar suspension system.

Solution: Using the notation given for Figure 1.29, and the following geometry:



Write the kinetic and potential energy to obtain the frequency:

Kinetic energy: $T_{\max} = \frac{1}{2} I_o \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2$

From geometry, $x = r\theta$ and $\dot{x} = r\dot{\theta}$

$$T_{\max} = \frac{1}{2}\left(I_o + I\right)\frac{\dot{x}^2}{r^2}$$

Potential Energy:

$$U_{\max} = \left(m_o + m\right)g\left(l - l\cos\phi\right)$$

Two term Taylor Series Expansion of cos ϕ $\approx 1 - \frac{\phi^2}{2}$:

$$U_{\max} = \left(m_o + m\right)gl\left(\frac{\phi^2}{2}\right)$$

For geometry, sin $\phi = \frac{r\theta}{l}$, and for small ϕ, sin ϕ = ϕ so that ϕ $= \frac{r\theta}{l}$

$$U_{\max} = \left(m_o + m\right)gl\left(\frac{r^2\theta^2}{2l^2}\right)$$

$$U_{\max} = \left(m_o + m\right)g\left(\frac{r^2\theta^2}{2l}\right) \text{ where } r\theta = x$$

$$U_{\max} = \frac{\left(m_o + m\right)g}{2l}x^2$$

Conservation of energy requires that:

$$T_{\max} = U_{\max} \quad \Rightarrow$$

$$\frac{1}{2}\frac{\left(I_o + I\right)}{r^2}\dot{x}^2 = \frac{\left(m_o + m\right)g}{2l}x^2$$

At maximum energy, x = A and $\dot{x} = \omega_n A$

$$\frac{1}{2}\frac{\left(I_o + I\right)}{r^2}\omega_n^2 A^2 = \frac{\left(m_o + m\right)g}{2l}A^2$$

$$\Rightarrow \left(I_o + I\right) = \frac{gr^2\left(m_o + m\right)}{\omega_n^2 l}$$

Substitute $\omega_n = 2\pi f_n = \dfrac{2\pi}{T}$

$$\left(I_o + I\right) = \frac{gr^2\left(m_o + m\right)}{\left(2\pi / T\right)^2 l}$$

$$I = \frac{gT^2 r^2\left(m_o + m\right)}{4\pi^2 l} - I_o$$

were T is the period of oscillation of the suspension.

1.77 A prototype composite material is formed and hence has unknown modulus. An experiment is performed consisting of forming it into a cantilevered beam of length 1 m and $I = 10^{-9}$ m^4 with a 6-kg mass attached at its end. The system is given an initial displacement and found to oscillate with a period of 0.5 s. Calculate the modulus E.

Solution: Using equation (1.66) for a cantilevered beam,

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{ml^3}{3EI}}$$

Solving for E and substituting the given values yields

$$E = \frac{4\pi^2 ml^3}{3T^2 I} = \frac{4\pi^2(6)(1)^3}{3(.5)^2(10^{-9})}$$

$$\Rightarrow \underline{E = 3.16\times10^{11}\ \text{N/m}^2}$$

1.78 The free response of a 1000-kg automobile with stiffness of k = 400,000 N/m is observed to be of the form given in Figure 1.32. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, determine the damping coefficient if the displacement at t_1 is measured to be 2 cm and 0.22 cm at t_2.

Solution: Given: x_1 = 2 cm and x_2 = 0.22 cm where $t_2 = T + t_1$

Logarithmic Decrement: $\delta = \ln \frac{x_1}{x_2} = \ln \frac{2}{0.22} = 2.207$

Damping Ratio: $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{2.207}{\sqrt{4\pi^2 + (2.207)^2}} = 0.331$

Damping Coefficient: $c = 2\zeta\sqrt{km} = 2(0.331)\sqrt{(400{,}000)(1000)} = 13{,}256 \text{ kg/s}$

1.79 A pendulum decays from 10 cm to 1 cm over one period. Determine its damping ratio.

Solution: Using Figure 1.31: $x_1 = 10$ cm and $x_2 = 1$ cm

Logarithmic Decrement: $\delta = \ln \frac{x_1}{x_2} = \ln \frac{10}{1} = 2.303$

Damping Ratio: $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{2.303}{\sqrt{4\pi^2 + (2.303)^2}} = 0.344$

1.80 The relationship between the log decrement δ and the damping ratio ζ is often approximated as $\delta = 2\pi\zeta$. For what values of ζ would you consider this a good approximation to equation (1.74)?

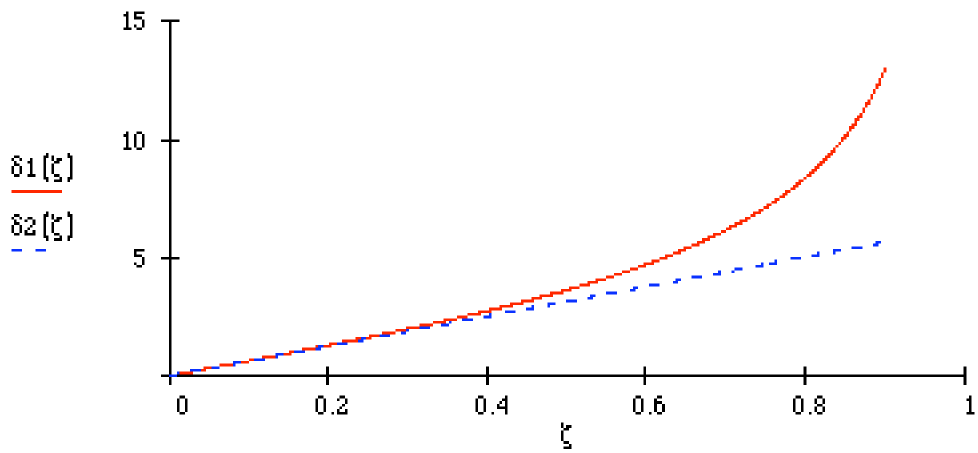
Solution: From equation (1.74), $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

For small ζ, $\delta = 2\pi\zeta$

A plot of these two equations is shown:

$\zeta := 0, 0.001 .. 0.9$

$$\delta 1(\zeta) := \frac{2\cdot\pi\cdot\zeta}{\sqrt{1-\zeta^2}} \qquad \delta 2(\zeta) := 2\cdot\pi\cdot\zeta$$



The lower curve represents the approximation for small ζ, while the upper curve is equation (1.74). The approximation appears to be valid to about $\zeta = 0.3$.

1.81 A damped system is modeled as illustrated in Figure 1.10. The mass of the system is measured to be 5 kg and its spring constant is measured to be 5000 N/m. It is observed that during free vibration the amplitude decays to 0.25 of its initial value after five cycles. Calculate the viscous damping coefficient, *c*.

Solution:

Note that for any two consecutive peak amplitudes,

$$\frac{x_o}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = e^{\delta} \text{ by definition}$$

$$\therefore \frac{x_o}{x_5} = \frac{1}{0.25} = \frac{x_0}{x_1}\cdot\frac{x_1}{x_2}\cdot\frac{x_2}{x_3}\cdot\frac{x_3}{x_4}\cdot\frac{x_4}{x_5} = e^{5\delta}$$

So,

$$\delta = \frac{1}{5}\ln(4) = 0.277$$

and

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.044$$

Solving for c,

$$c = 2\zeta\sqrt{km} = 2(0.044)\sqrt{5000(5)}$$
$$c = 13.94 \text{ N-s/m}$$