**Problems and Solutions Section 1.8 (1.90 through 1.93)**

**1.90** Consider the system of Figure 1.90 and (a) write the equations of motion in terms of the angle, $\theta$, the bar makes with the vertical. Assume linear deflections of the springs and linearize the equations of motion. Then (b) discuss the stability of the linear system's solutions in terms of the physical constants, *m*, *k*, and $\ell$. Assume the mass of the rod acts at the center as indicated in the figure.

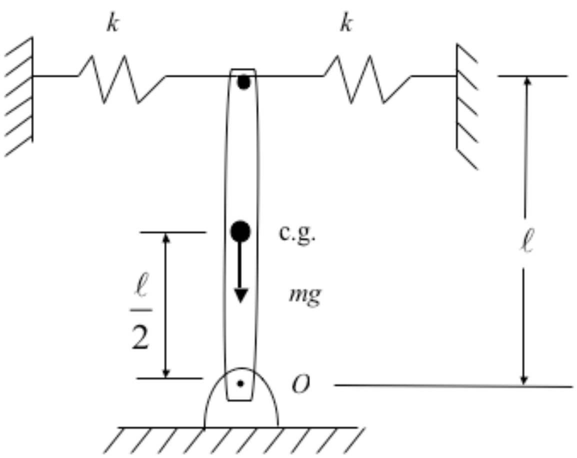



**Figure P1.90**

**Solution:** Note that from the geometry, the springs deflect a distance $kx = k(\ell \sin\theta)$ and the cg moves a distance $\ell/2 \cos\theta$. Thus the total potential energy is

$$U = 2\times\frac{1}{2}k(\ell\sin\theta)^2 - \frac{mg\ell}{2}\cos\theta$$

and the total kinetic energy is

$$T = \frac{1}{2}J_O\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2$$

The Lagrange equation (1.64) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) + \frac{\partial U}{\partial\theta} = \frac{d}{dt}\left(\frac{m\ell^2}{3}\dot{\theta}\right) + 2k\ell\sin\theta\cos\theta - \frac{1}{2}mg\ell\sin\theta = 0$$

Using the linear, small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ yields

$$\text{a)}\quad \underline{\frac{m\ell^2}{3}\ddot{\theta} + \left(2k\ell^2 - \frac{mg\ell}{2}\right)\theta = 0}$$

Since the leading coefficient is positive the sign of the coefficient of θ determines the stability.

$$\text{if}\quad 2k\ell - \frac{mg}{2} > 0 \Rightarrow 4k > \frac{mg}{\ell} \Rightarrow \text{ the system is stable}$$

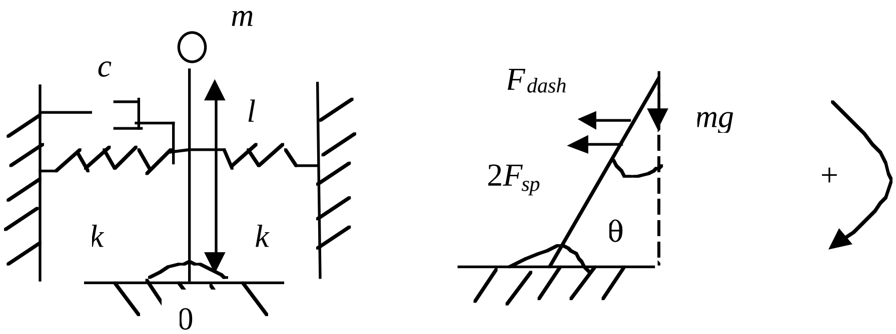
b) $$\text{if}\quad 4k = mg \Rightarrow \theta(t) = at + b \Rightarrow \text{ the system is unstable}$$

$$\text{if}\quad 2k\ell - \frac{mg}{2} < 0 \Rightarrow 4k < \frac{mg}{\ell} \Rightarrow \text{ the system is unstable}$$

Note that physically this results states that the system's response is stable as long as the spring stiffness is large enough to over come the force of gravity.

**1.91** Consider the inverted pendulum of Figure 1.37 as discussed in Example 1.8.1. Assume that a dashpot (of damping rate *c*) also acts on the pendulum parallel to the two springs. How does this affect the stability properties of the pendulum?

**Solution:** The equation of motion is found from the following FBD:




Moment about O: $\Sigma M_o = I\ddot{\theta}$

$$ml^2\ddot{\theta} = mgl\sin\theta - 2\frac{kl}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) - c\left(\frac{l}{2}\dot{\theta}\right)\left(\frac{l}{2}\cos\theta\right)$$

When θ is small, sinθ ≈ θ and cosθ ≈ 1

$$ml^2\ddot{\theta} + \frac{cl^2}{4}\dot{\theta} + \left(\frac{kl^2}{2} - mgl\right)\theta = 0$$

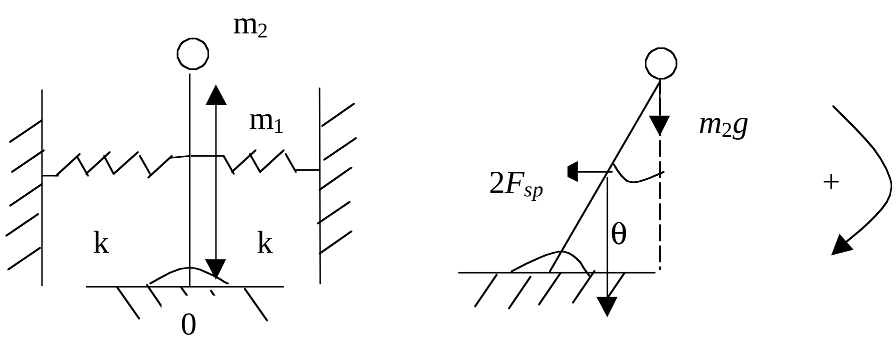
$$ml\ddot{\theta} + \frac{cl}{4}\dot{\theta} + \left(\frac{kl}{2} - mg\right)\theta = 0$$

For stability, $\frac{kl}{2} > mg$ and $c \geq 0$.

The result of ***adding a dashpot is to make the system asymptotically stable***.

**1.92** Replace the massless rod of the inverted pendulum of Figure 1.37 with a solid object compound pendulum of Figure 1.20(b). Calculate the equations of vibration and discuss values of the parameter relations for which the system is stable.

**Solution:**




Moment about O: $\Sigma M_o = I\ddot{\theta}$

$$m_1 g\frac{l}{2}\sin\theta + m_2 gl\sin\theta - 2\frac{kl}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) = \left(\frac{1}{3}m_1 l^2 + m_2 l^2\right)\ddot{\theta}$$

When θ is small, sinθ ≈ θ and cosθ ≈ 1.

$$\left(\frac{m_1}{3} + m_2\right)l^2\ddot{\theta} + \left(\frac{kl^2}{2} - \frac{m_1}{2}gl - m_2 gl\right)\theta = 0$$

$$\left(\frac{m_1}{3} + m_2\right)l\ddot{\theta} + \left[\frac{kl}{2} - \left(\frac{m_1}{2} + m_2\right)g\right]\theta = 0$$

For stability, $\frac{kl}{2} > \left(\frac{m_1}{2} + m_2\right)g.$

**1.93** A simple model of a control tab for an airplane is sketched in Figure P1.93. The equation of motion for the tab about the hinge point is written in terms of the angle $\theta$ from the centerline to be

$$J\ddot{\theta} + (c - f_d)\dot{\theta} + k\theta = 0\,.$$

Here $J$ is the moment of inertia of the tab, $k$ is the rotational stiffness of the hinge, $c$ is the rotational damping in the hinge and $f_d\dot{\theta}$ is the negative damping provided by the aerodynamic forces (indicated by arrows in the figure). Discuss the stability of the solution in terms of the parameters $c$ and $f_d$ .

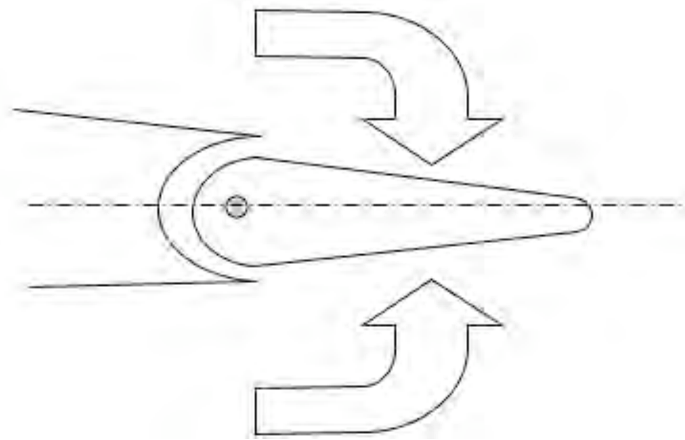


**Figure P1.93** A simple model of an airplane control tab

**Solution:** The stability of the system is determined by the coefficient of $\dot{\theta}$ since the inertia and stiffness terms are both positive. There are three cases

*Case 1* $c$ - $f_d$ > 0 and the system's solution is of the form $\theta(t) = e^{-at}\sin(\omega_n t + \phi)$ and the solution is asymptotically stable.

*Case 2* $c$ - $f_d$ < 0 and the system's solution is of the form $\theta(t) = e^{at}\sin(\omega_n t + \phi)$ and the solution is oscillates and grows without bound, and exhibits flutter instability as illustrated in Figure 1.36.

*Case 3* $c = f_d$ and the system's solution is of the form $\theta(t) = A\sin(\omega_n t + \phi)$ and the solution is stable as illustrated in Figure 1.34.