Problems and Solutions Section 2.1 (2.1 through 2.15)

2.1 To familiarize yourself with the nature of the forced response, plot the solution of a forced response of equation (2.2) with ω = 2 rad/s, given by equation (2.11) for a variety of values of the initial conditions and ω_n as given in the following chart:

Case	x_0	v_0	f_0	ω_n
1	0.1	0.1	0.1	1
2	-0.1	0.1	0.1	1
3	0.1	0.1	1.0	1
4	0.1	0.1	0.1	2.1
5	1	0.1	0.1	1

Solution: Given: ω = 2 rad/sec.
From equation (2.11):

$$x(t) = \frac{v_0}{\omega_n}\sin \omega_n t + (x_0 - \frac{f_0}{\omega_n^2 - \omega^2})\cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos \omega t$$

Insert the values of x_0, v_0, f_0, and ω_n for each of the five cases.

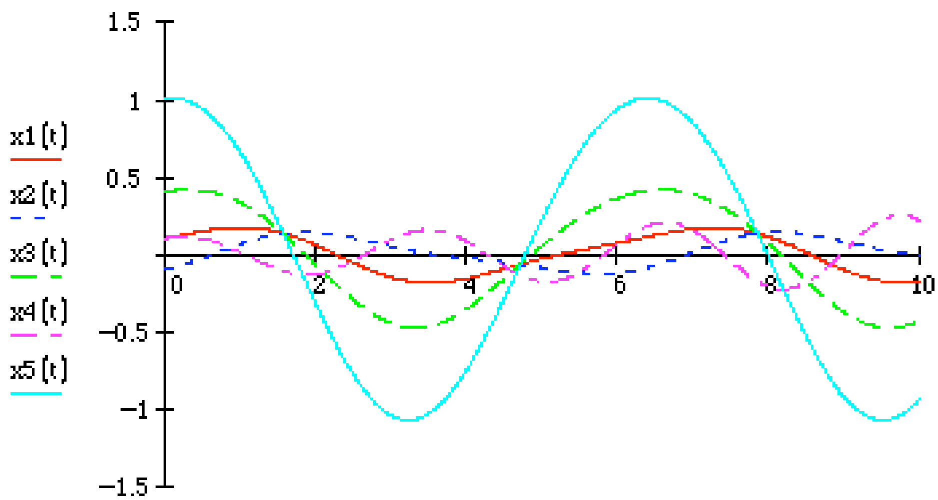
$$x1(t) := 0.1\cdot\sin(t) + 0.133\cdot\cos(t) - 0.0333\cdot\cos(2\cdot t)$$

$$x2(t) := 0.1\cdot\sin(t) - 0.0667\cdot\cos(t) - 0.0333\cdot\cos(2\cdot t)$$

$$x3(t) := 0.1\cdot\sin(t) + 0.433\cdot\cos(t) - 0.0333\cdot\cos(2\cdot t)$$

$$x4(t) := 0.0467\cdot\sin(2.1\cdot t) + 0.244\cdot\cos(2\cdot t) - 0.144\cdot\cos(2.1\cdot t)$$

$$x5(t) := 0.1\cdot\sin(t) + 1.033\cdot\cos(t) - 0.0333\cdot\cos(2\cdot t)$$



2.2 Repeat the calculation made in Example 2.1.1 for the mass of a simple spring-mass system where the mass of the spring is considered and known to be 1 kg.

Solution: Given: m_{sp} = 1 kg, Example 1.4.4 yields that the effective mass is

$$m_e = m + \frac{m_{sp}}{3} = 10 + \frac{1}{3} = 10.333 \text{ kg.}$$

Thus the natural frequency, X and the coefficients in equation (2.11) for the system now become

$$\omega_n = \sqrt{\frac{1000}{10 + 1/3}} = 9.837 \text{ rad/s}, \ \omega = 2\omega_n = 19.675 \text{ rad/s}$$

$$X = \frac{f_0}{\omega_n^2 - \omega^2} = \frac{2.338}{9.837^2 - 19.675^2} = -8.053 \times 10^{-3} \text{ m}, \ \frac{v_0}{\omega_n} = 0.02033 \text{ m}$$

Thus the response as given by equation (2.11) is

$$x(t) = 0.02033 \sin 9.837t + 8.053 \times 10^{-3}(\cos 9.837t - \cos 19.675t) \text{ m}$$

2.3 A spring-mass system is driven from rest harmonically such that the displacement response exhibits a beat of period of 0.2π s. The period of oscillation is measured to be 0.02π s. Calculate the natural frequency and the driving frequency of the system.

Solution: Given: Beat period: $T_b = 0.2\pi$ s, Oscillation period: $T_0 = 0.02\pi$ s

Equation (2.13): $$x(t) = \frac{2 f_0}{\omega_n^2 - \omega^2} \sin\left[\frac{\omega_n - \omega}{2} t\right] \sin\left[\frac{\omega_n + \omega}{2} t\right]$$

So, $$T_b = 0.2\pi = \frac{4\pi}{\omega_n - \omega}$$

$$\omega_n - \omega = \frac{4\pi}{0.2\pi} = 20 \text{ rad/s}$$

$$T_0 = 0.02\pi = \frac{4\pi}{\omega_n + \omega}$$

$$\omega_n + \omega = \frac{4\pi}{0.02\pi} = 200 \text{ rad/s}$$

Solving for ω_n and ω gives:
Natural frequency: ω_n = 110 rad/s
Driving frequency: ω = 90 rad/s

2.4 An airplane wing modeled as a spring-mass system with natural frequency 40 Hz is driven harmonically by the rotation of its engines at 39.9 Hz. Calculate the period of the resulting beat.

Solution: Given: $\omega_n = 2\pi\,(40) = 80\pi$ rad/s, $\omega = 2\pi\,(39.9) = 79.8\pi$ rad/s

Beat period: $T_b = \dfrac{4\pi}{\omega_n - \omega} = \dfrac{4\pi}{80\pi - 79.8\pi} = 20$ s.

2.5 Derive Equation 2.13 from Equation 2.12 using standard trigonometric identities.

Solution: Equation (2.12): $x(t) = \dfrac{f_0}{\omega_n^2 - \omega^2}\,[\cos\,\omega t - \cos\,\omega_n t]$

Let $A = \dfrac{f_0}{\omega_n^2 - \omega^2}$

$$
\begin{aligned}
x(t) =\ & A\,[\cos\omega t - \cos\,\omega_n t] \\
&= A\,[1 + \cos\omega t - (1 + \cos\,\omega_n t)] \\
&= A\,[2\cos^2 \frac{\omega}{2}t\ - 2\cos^2 \frac{\omega_n}{2}t\,] \\
&= 2A\,[(\cos^2 \frac{\omega}{2}t\ - \cos^2 \frac{\omega_n}{2}\ \cos^2 \frac{\omega}{2}t\,) - (\cos^2 \frac{\omega_n}{2}t\ - \cos^2 \frac{\omega_n}{2}t\ \cos^2 \frac{\omega}{2}t\,)] \\
&= 2A\,[(1 - \cos^2 \frac{\omega_n}{2}t\,)\cos^2 \frac{\omega}{2}t\ - (1 - \cos^2 \frac{\omega}{2}t\,)\cos^2 \frac{\omega_n}{2}t\,] \\
&= 2A\,[\sin^2 \frac{\omega}{2}t\ \cos^2 \frac{\omega}{2}t\ - \cos^2 \frac{\omega}{2}t\ \sin^2 \frac{\omega}{2}t\,] \\
&= 2A\,[\sin\frac{\omega_n}{2}t\ \cos\frac{\omega}{2}t\ - \cos\frac{\omega_n}{2}t\ \sin\frac{\omega}{2}t\,]\,[\sin\frac{\omega_n}{2}t\ \cos\frac{\omega}{2}t\ - \cos\frac{\omega_n}{2}t\ \sin\frac{\omega}{2}t\,] \\
&= 2A\,\sin\left[\frac{\omega_n - \omega}{2}t\right]\,\sin\left[\frac{\omega_n + \omega}{2}t\right]
\end{aligned}
$$

$x(t) = \dfrac{2f_0}{\omega_n^2 - \omega^2}\ \sin\left[\dfrac{\omega_n - \omega}{2}t\right]\,\sin\left[\dfrac{\omega_n + \omega}{2}t\right]$ which is Equation (2.13).

2.6 Compute the total response of a spring-mass system with the following values: k = 1000 N/m, m = 10 kg, subject to a harmonic force of magnitude F_0 = 100 N and frequency of 8.162 rad/s, and initial conditions given by x_0 = 0.01 m and v_0 = 0.01 m/s. Plot the response.

Solution: Given: k = 1000 N/m, m = 10 kg, F_0=100 N, ω = 8.162 rad/s
x_0=0.01m, v_0=0.01 m/s
From Eq. (2.11):

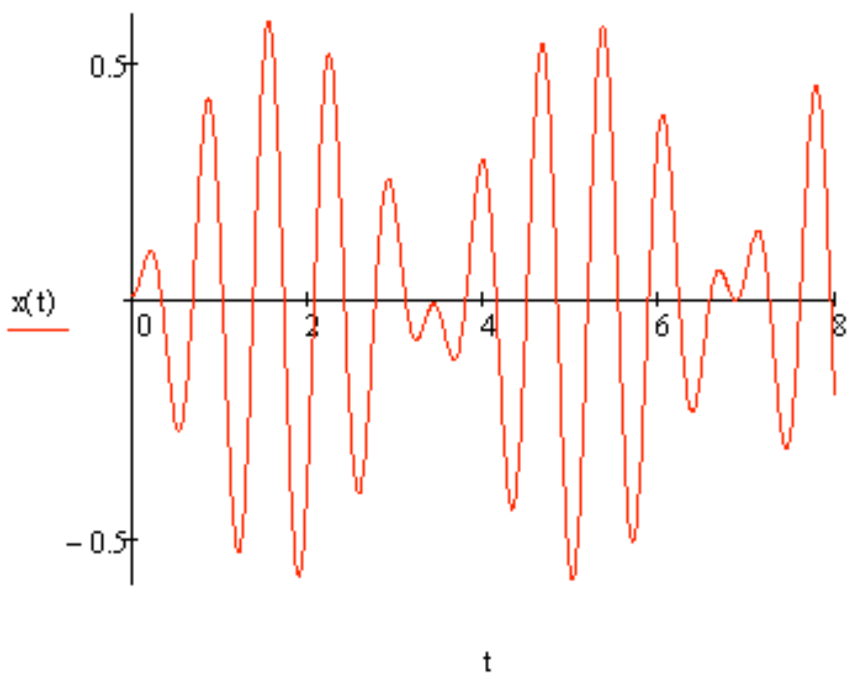
$$x(t) = \frac{v_0}{\omega_n}\sin\,\omega_n t + (x_0 - \frac{f_0}{\omega_n{}^2 - \omega^2})\cos\,\omega_n t + \frac{f_0}{\omega_n{}^2 - \omega^2}\cos\,\omega t$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10\;rad/s \qquad \mathrm{f}_0 = \frac{\mathrm{F}_0}{\mathrm{m}} = \frac{100}{10} = 10\;\mathrm{N/m}$$

In Mathcad the solution is

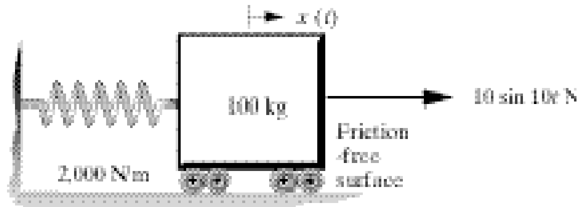
$x0 := 0.01 \qquad v0 := 0.01 \qquad \omega n := 10 \qquad \omega := 8.162 \qquad f0 := 10$

$$x(t) := \frac{v0}{\omega n}\cdot\sin(\omega n\cdot t) + \left(x0 - \frac{f0}{\omega n^2 - \omega^2}\right)\cdot\cos(\omega n\cdot t) + \frac{f0}{\left(\omega n^2 - \omega^2\right)}\cdot\cos(\omega\cdot t)$$



$$\frac{v0}{\omega n} = 1\times10^{-3} \qquad x0 - \frac{f0}{\omega n^2 - \omega^2} = -0.28956 \qquad \frac{f0}{\left(\omega n^2 - \omega^2\right)} = 0.29956$$

2.7 Consider the system in Figure P2.7, write the equation of motion and calculate the response assuming a) that the system is initially at rest, and b) that the system has an initial displacement of 0.05 m.



Solution: The equation of motion is

$$m\ddot{x} + kx = 10\sin 10t$$

Let us first determine the general solution for

$$\ddot{x} + \omega_n^2 x = f_0 \sin \omega t$$

Replacing the cosine function with a sine function in Eq. (2.4) and following the same argument, the general solution is:

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t$$

Using the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, a general expression for the response of a spring-mass system to a harmonic (sine) excitation is:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2}\right) \sin \omega_n t + x_0 \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t$$

Given: k=2000 N/m, m=100 kg, ω=10 rad/s,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{100}} = \sqrt{20} \text{ rad/s} = 4.472 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{10}{100} = 0.1 \text{ N/kg}$$

a) $x_0 = 0$ m, $v_0 = 0$ m/s

Using the general expression obtained above:

$$x(t) = \left(0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{\sqrt{20}^2 - 10^2}\right) \sin \sqrt{20}t + 0 + \frac{0.1}{\sqrt{20}^2 - 10^2} \sin 10t$$

$$= 2.795 \times 10^{-3} \sin 4.472t - 1.25 \times 10^{-3} \sin 10t$$

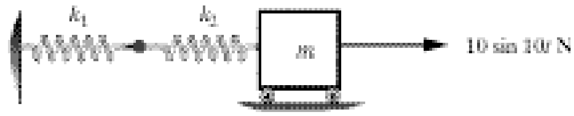
b) $x_0 = 0.05$ m, $v_0 = 0$ m/s

$$x(t) = \left(0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{\sqrt{20}^2 - 10^2}\right) \sin \sqrt{20}t + 0.05 \cos \sqrt{20}t + \frac{0.1}{\sqrt{20}^2 - 10^2} \sin 10t$$

$$= 0.002795 \sin 4.472t + 0.05 \cos 4.472t - 0.00125 \sin 10t$$

$$= 5.01 \times 10^{-2} \sin(4.472t + 1.515) - 1.25 \times 10^{-3} \sin 10t$$

2.8 Consider the system in Figure P2.8, write the equation of motion and calculate the response assuming that the system is initially at rest for the values $k_1 = 100$ N/m, $k_2 =$ 500 N/m and m = 89 kg.



Solution: The equation of motion is

$$m\ddot{x} + kx = 10\sin 10t \qquad \text{where} \quad k = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

The general expression obtained for the response of an underdamped spring-mass system to a harmonic (sine) input in Problem 2.7 was:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2}\right)\sin\omega_n t + x_0 \cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\sin\omega t$$

Substituting the following values

k = 1/(1/100+1/500)= 83.333 N/m, m = 89 kg ω = 10 rad/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{83.333}{89}} = 0.968 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{10}{89} = 0.112\ N/kg$$

and initial conditions: $x_0 = 0$, $v_0 = 0$

The response of the system is evaluated as

$$x(t) = 0.0117\sin 0.968t - 0.00113\sin 10t$$

2.9 Consider the system in Figure P2.9, write the equation of motion and calculate the response assuming that the system is initially at rest for the values θ = 30°, k = 1000 N/m and m = 50 kg.

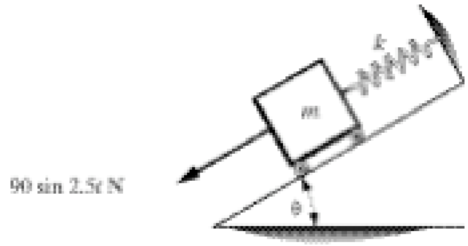
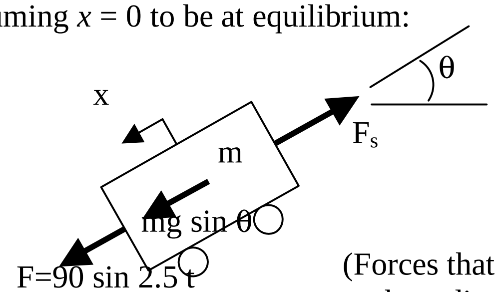


Figure P2.9

Solution: Free body diagram:
Assuming x = 0 to be at equilibrium:



(Forces that are normal to the x direction are neglected)

$$\sum F_x = m\ddot{x} = -k(x+\Delta) + mg\sin\theta + 90\sin 25t \qquad (1)$$

where Δ is the static deflection of the spring. From static equilibrium in the x direction yields

$$-k\Delta + mg\sin\theta \qquad (2)$$

Substitution of (2) onto (1), the equation of motion becomes

$$m\ddot{x} + kx = 90\sin 2.5t$$

The general expression for the response of a mass-spring system to a harmonic (sine) excitation (see Problem 2.7) is:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n}\cdot\frac{f_0}{\omega_n^{\,2}-\omega^2}\right)\sin\omega_n t + x_0\cos\omega_n t + \frac{f_0}{\omega_n^{\,2}-\omega^2}\sin\omega t$$

Given: $v_0 = 0$, $x_0 = 0$, $\omega = 2.5$ rad/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{50}} = \sqrt{20} = 4.472 \text{ rad/s} \quad , \quad f_0 = \frac{F_0}{m} = \frac{90}{50} = \frac{9}{5}\text{N/kg}$$

So the response is:

$$x(t) = -0.0732\sin 4.472t + 0.1309\sin 2.5t$$

2.10 Compute the initial conditions such that the response of :

$$m\ddot{x} + kx = F_0 \cos \omega t$$

oscillates at only one frequency (ω).

Solution: From Eq. (2.11):

$$x(t) = \frac{v_0}{\omega_n}\sin \omega_n t + (x_0 - \frac{f_0}{\omega_n^2 - \omega^2})\cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos \omega t$$

For the response of $m\ddot{x} + kx = F_0 \cos \omega t$ to have only one frequency content, namely, of the frequency of the forcing function, ω, the coefficients of the first two terms are set equal to zero. This yields that the initial conditions have to be

$$x_0 = \frac{f_0}{\omega_n^2 - \omega^2} \quad \text{and} \quad v_0 = 0$$

Then the solution becomes

$$x(t) = \frac{f_0}{\omega_n^2 - \omega^2}\cos \omega t$$

2.11 The natural frequency of a 65-kg person illustrated in Figure P.11 is measured along vertical, or longitudinal direction to be 4.5 Hz. a) What is the effective stiffness of this person in the longitudinal direction? b) If the person, 1.8 m in length and 0.58 m^2 in cross sectional area, is modeled as a thin bar, what is the modulus of elasticity for this system?

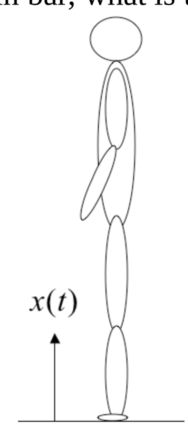


Figure P2.11 Longitudinal vibration of a person

Solution: a) First change the frequency in Hz to rad/s: $\omega_n = 4.5\frac{\text{cycles}}{\text{s}}\frac{2\pi \text{ rad}}{\text{cycles}} = 9\pi \text{ rad/s}$.

Then from the definition of natural frequency:

$$k = m\omega_n^2 = 65 \cdot (9\pi)^2 = \underline{5.196 \times 10^4 \text{ N/m}}$$

b) From section 1.4, the value of the stiffness for the longitudinal vibration of a beam is

$$k = \frac{EA}{\ell} \Rightarrow E = \frac{k\ell}{A} = \frac{\left(5.196\times 10^4\right)(1.8)}{0.58} = \underline{1.613\times 10^5 \text{ N/m}^2} = \underline{1.613\times 10^5 \text{ Pa}}$$

2.12 If the person in Problem 2.11 is standing on a floor, vibrating at 4.49 Hz with an amplitude of 1 N (very small), what longitudinal displacement would the person “feel”? Assume that the initial conditions are zero.

Solution: Using equation (2.12) for a cosine excitation and zero initial conditions yields (converting the frequency from Hertz to rad/s and using the value of *k* calculated in 2.11):

$$|X| = \frac{F_0}{m}\left|\frac{1}{\omega_n^2 - \omega^2}\right| = \frac{1}{65}\left|\frac{1}{\frac{k}{m} - (4.49\cdot 2\pi)^2}\right|$$

$$= \frac{1}{65}\left|\frac{1}{\frac{5.196\times 10^4}{65} - (4.49\cdot 2\pi)^2}\right| = 0.00443347 = \underline{0.0043\text{ m}}$$

2.13 Vibration of body parts is a significant problem in designing machines and structures. A jackhammer provides a harmonic input to the operator's arm. To model this situation, treat the forearm as a compound pendulum subject to a harmonic excitation (say of mass 6 kg and length 44.2 cm) as illustrated in Figure P2.13. Consider point *O* as a fixed pivot. Compute the maximum deflection of the hand end of the arm if the jackhammer applies a force of 10 N at 2 Hz.

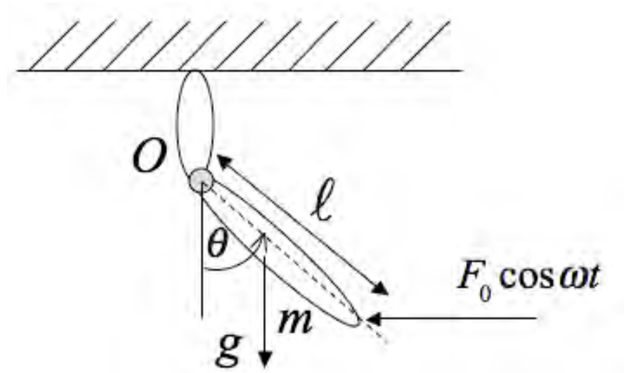


Figure P2.13 Vibration model of a forearm driven by a jackhammer

Solution: Taking moments about point *O* yields (referring to Example 1.4.6 for the inertial of a compound pendulum):

$$\frac{m\ell^2}{3}\ddot{\theta} + mg\frac{\ell}{2}\sin\theta = F_O\ell\cos\theta\cos\omega t$$

Using the linear approximation for sine and cosine and dividing through by the inertia yields:

$$\ddot{\theta} + \frac{3g}{2\ell}\theta = \frac{3F_O}{m\ell}\cos\omega t$$

Thus the natural frequency is

$$\omega_n = \sqrt{\frac{3g}{2\ell}} = \sqrt{\frac{3(9.81)}{2(0.442)}} = 5.77 \text{ rad/s} \quad (=0.92 \text{ Hz})$$

and the system is well away from resonance. Referring to equation (2.13), the amplitude for zero initial conditions is (converting the driving frequency from 2 Hertz to 2(2π) rad/s):

$$|\theta| = \left|\frac{2f_0}{\omega_n^2 - \omega^2}\right| = \left|\frac{2\left(\frac{3F_0\ell}{m\ell^2}\right)}{\frac{3g}{2\ell} - (2\cdot 2\pi)^2}\right| = \underline{0.182 \text{ rad}}$$

Note that sin(0.182) = 0.181 so the approximation made above is valid. The maximum linear displacement of the hand end of the arm is just

$$|X| = r|\theta| = 0.442\cdot 0.182 = \underline{0.08 \text{ m}}$$

2.14 Consider again the camera problem of Example 2.1.3 depicted in Figure P2.14, and determine the torsional natural frequency, the maximum torsional deflection experienced by the camera due to the wind and the linear displacement corresponding to the computed torsional deflection. Model the camera in torsional vibration as suggested in the figure where $J_P = 9.817\text{x}10^{-6}$ m^4 and $L = 0.2$ m. Use the values computed in Example 2.1.3 for the mass (m =3 kg), shaft length ($\ell = 0.55$ m), torque ($M_0 = 15 \text{ x } L$ Nm) and frequency (ω = 10 Hz). Here G is the shear modulus of aluminum and the rotational inertia of the camera is approximated by $J = mL^2$. In the example, torsion was ignored. The purpose of this problem is to determine if ignoring the torsion is a reasonable assumption or not. Please comment on this assumption based on the results of the requested calculation.

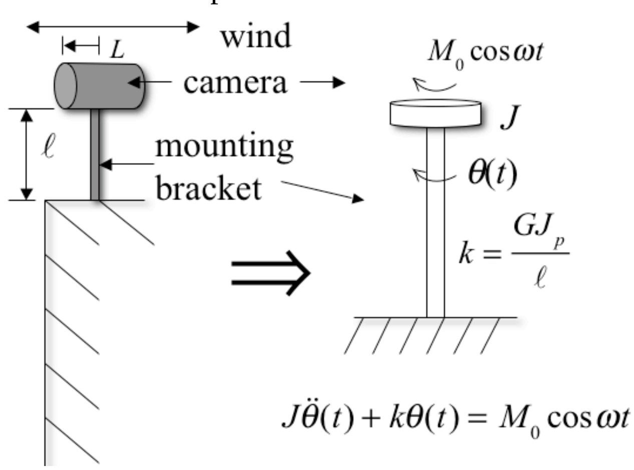


Figure P2.14 Torsional vibration of a camera

Solution: First calculate the rotational stiffness and inertia from the data given:

$$k = \frac{GJ_p}{\ell} = \frac{2.67\times 10^{10}\times 9.817\times 10^{-6}}{0.55} = 4.766\times 10^5 \text{ N}\cdot\text{m}$$

where the modulus is taken from Table 1.2 for aluminum. The inertia is approximated by

$$J = mL^2 = 3(0.2)^2 = 0.12 \text{ kg}\cdot\text{m}^2$$

The torsional natural frequency is thus

$$\omega_n = \sqrt{\frac{k}{J}} = \underline{1.993 \times 10^3 \text{ rad/s}}$$

This is well away from the driving frequency. To see the effect, recall equation magnitude of the forced response given in Example 2.1.2:

$$\left|\frac{2f_0}{\omega_n^2 - \omega^2}\right| = \left|\frac{2M_0 / J}{\omega_n^2 - \omega^2}\right| = \underline{1.26 \times 10^{-5} \text{ rad}}$$

Clearly this is very small. To change this to a linear displacement of the camera tip, use

$$X = r\theta = (0.2)(1.26 \times 10^{-5}) = \underline{2.52 \times 10^{-6} \text{ m}}$$

well within the limit imposed on the camera's vibration requirement of 0.01 m. Thus, the assumption to ignore torsional vibration in designing the length of the mounting bracket made in example 2.1.3 is justified.

2.15 An airfoil is mounted in a wind tunnel for the purpose of studying the aerodynamic properties of the airfoil's shape. A simple model of this is illustrated in Figure P2.15 as a rigid inertial body mounted on a rotational spring, fixed to the floor with a rigid support. Find a design relationship for the spring stiffness *k* in terms of the rotational inertia, *J*, the magnitude of the applied moment, M_0, and the driving frequency, ω, that will keep the magnitude of the angular deflection less then 5°. Assume that the initial conditions are zero and that the driving frequency is such that $\omega_n^2 - \omega^2 > 0$.

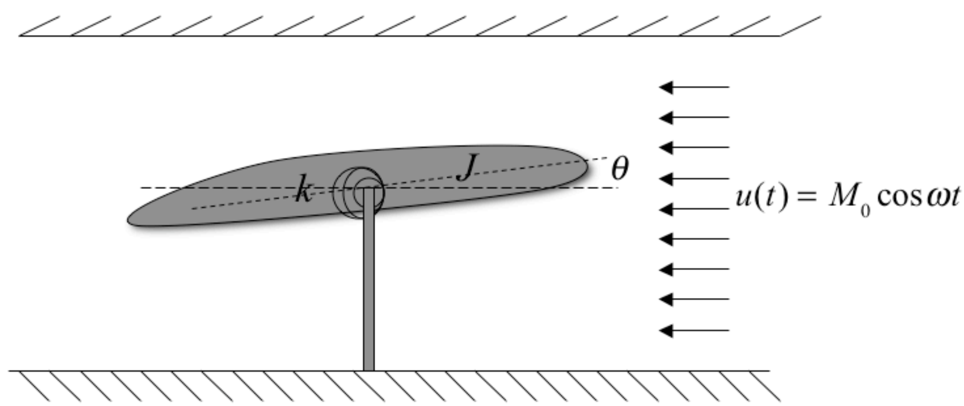


Figure P2.15 Vibration model of a wing in a wind tunnel

Solution: Assuming compatible units, the equation of motion is:

$$J\ddot{\theta}(t) + k\theta(t) = M_0 \cos\omega t \Rightarrow \ddot{\theta}(t) + \frac{k}{J}\theta(t) = \frac{M_0}{J}\cos\omega t$$

From equation (2.12) the maximum deflection for zero initial conditions is

$$\theta_{\max} = \left| \frac{2M_0 \diagup J}{\frac{k}{J} - \omega^2} \right| < 5° \frac{\pi \text{rad}}{180°} = \frac{\pi}{36} \text{rad}$$

$$\Rightarrow \frac{2M_0}{J} < (\frac{k}{J} - \omega^2) \frac{\pi}{36} \text{rad} \Rightarrow \underline{\frac{36J}{\pi}\left(\frac{2M_0}{J} + \frac{\pi\omega^2}{36} \right) < k}$$