Problems and Solutions Section 2.2 (2.16 through 2.31)

2.16 Calculate the constants A and ϕ for arbitrary initial conditions, x_0 and v_0, in the case of the forced response given by Equation (2.37). Compare this solution to the transient response obtained in the case of no forcing function (i.e. $F_0 = 0$).

Solution: From equation (2.37)

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\cos(\omega t - \theta) \Rightarrow$$

$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + A\omega_d e^{-\zeta\omega_n t}\cos(\omega_d t + \phi) - X\omega\sin(\omega t - \theta)$$

Next apply the initial conditions to these general expressions for position and velocity to get:

$$x(0) = A\sin\phi + X\cos\theta$$

$$\dot{x}(0) = -\zeta\omega_n A\sin\phi + A\omega_d\cos\phi + X\omega\sin\theta$$

Solving this system of two equations in two unknowns yields:

$$\phi = \tan^{-1}\left(\frac{(x_0 - X\cos\theta)\omega_d}{v_0 + (x_0 - X\cos\theta)\zeta\omega_n - X\omega\sin\theta}\right)$$

$$A = \frac{x_0 - X\cos\theta}{\sin\phi}$$

Recall that X has the form

$$X = \frac{F_0/m}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

Now if $F_0 = 0$, then $X = 0$ and A and ϕ from above reduce to:

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + x_0\zeta\omega_n}\right)$$

$$A = \frac{x_0}{\sin\phi} = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$$

These are identical to the values given in equation (1.38).

2.17 Show that Equations (2.28) and (2.29) are equivalent by verifying Equations (2.29) and (2.30).

Solution: From equation (2.28) and expanding the trig relation yields

$$x_p = X\cos(\omega t - \theta) = X\left[\cos\omega t\cos\theta + \sin\omega t\sin\theta\right]$$
$$= \underbrace{(X\cos\theta)}_{A_s}\cos\omega t + \underbrace{(X\sin\theta)}_{B_s}\sin\omega t$$

Now with A_s and B_s defined as indicated, the magnitude is computed:

$$X = \sqrt{A_s^2 + B_s^2}$$

and

$$\frac{B_s}{A_s} = \frac{X\sin\theta}{X\cos\theta} \Rightarrow \theta = \tan^{-1}\left(\frac{B_s}{A_s}\right)$$

2.18 Plot the solution of Equation (2.27) for the case that m = 1 kg, ζ = 0.01, ω_n = 2 rad/s. F_0 = 3 N, and ω = 10 rad/s, with initial conditions x_0 = 1 m and v_0 = 1 m/s.

Solution: The particular solution is given in equations (2.36) and (2.37). Substitution of the values given yields: $x_p = 0.03125\cos(10t + 8.333\times 10^{-3})$.

Then the total solution has the form:

$$x(t) = Ae^{-0.02t}\sin(2t + \phi) + 0.03125\cos(10t + 0.008333)$$
$$= e^{-0.02t}\left(A\sin 2t + B\cos 2t\right) + 0.03125\cos(10t + 0.008333)$$

Differentiating then yields

$$\dot{x}(t) = -0.02e^{-0.02t}\left(A\sin 2t + B\cos 2t\right) + \sin(2t + \phi)$$
$$+ 2e^{-0.02t}\left(A\cos 2t - B\sin 2t\right) - 0.3125\sin(10t + 0.008333)$$

Apply the initial conditions to get:

$$x(0) = 1 = B + 0.03125\cos(0.00833) \Rightarrow \underline{B = 0.969}$$
$$\dot{x}(0) = 1 = -0.02B + 2A - 0.3125\sin(0.00833) \Rightarrow A = 0.489$$

So the solution and plot become (using Mathcad):

$F0 := 3$ $\omega := 10$ $\omega n := 2$ $\zeta := 0.02$

$$\theta := \operatorname{atan}\left(2\cdot\frac{\zeta\cdot\omega n\cdot\omega}{\omega n^2-\omega^2}\right) = -8.333\times 10^{-3}$$

$$AF := \frac{F0}{\sqrt{\left(\omega n^2-\omega^2\right)^2+(2\cdot\zeta\cdot\omega\cdot\omega n)^2}}$$

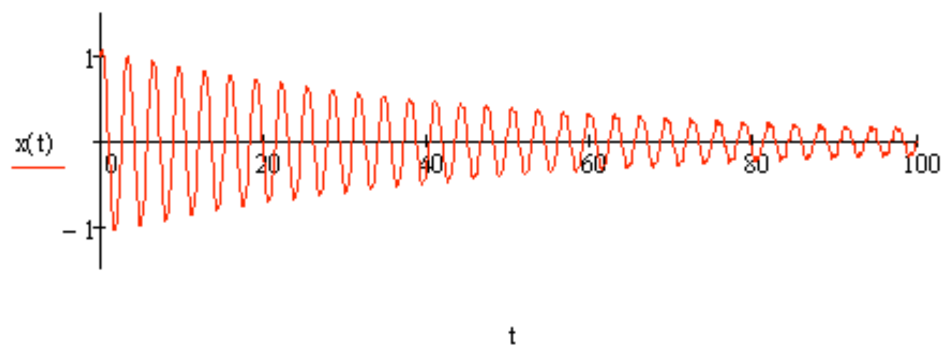
$AF = 0.03125$

$B := 1 - AF\cdot\cos(\theta)$ $B = 0.969$

$$A1 := \frac{1 + 10\cdot AF\cdot\sin(\theta) - 0.02\cdot B}{2}$$

$A1 = 0.489$

$$x(t) := e^{-0.02\cdot t}\cdot(A1\cdot\sin(2\cdot t) + B\cdot\cos(2\cdot t)) + AF\cdot\cos(10\cdot t - \theta)$$



2.19 A 100 kg mass is suspended by a spring of stiffness 30×10^3 N/m with a viscous damping constant of 1000 Ns/m. The mass is initially at rest and in equilibrium. Calculate the steady-state displacement amplitude and phase if the mass is excited by a harmonic force of 80 N at 3 Hz.

Solution: Given *m* = 100kg, *k* =30,000 N/m, *c* = 1000 Ns/m, F_0 = 80 N and ω = 6π rad/s:

$$f_0 = \frac{F_0}{m} = \frac{80}{100} = 0.8 \text{ m/s}^2, \quad \omega_n = \sqrt{\frac{k}{m}} = 17.32 \text{ rad/s}$$

$$\zeta = \frac{c}{2\sqrt{km}} = 0.289$$

$$X = \frac{0.8}{\sqrt{\left(17.32^2 + 36\pi^2\right)^2 + \left(2(0.289)(17.32)(6\pi)\right)^2}} = 0.0041 \text{ m}$$

Next compute the angle from

$$\theta = \tan^{-1}\left(\frac{188.702}{-55.323}\right)$$

Since the denominator is negative the angle must be found in the 4th quadrant. To find this use Window 2.3 and then in Matlab type atan2(188.702,-55.323) or use the principle value and add π to it. Either way the phase is <u>θ =1.856 rad.</u>

2.20 Plot the total solution of the system of Problem 2.19 including the transient.

Solution: The total response is given in the solution to Problem 2.16. For the values given in the previous problem, and with zero initial conditions the response is determined by the formulas:

$$X = 0.0041, \quad \theta = 1.856$$

$\omega n := 17.32 \quad \zeta := 0.289$

$$\theta := \operatorname{atan}\left[\frac{2 \cdot 0.289 \cdot 17.32 \cdot 6 \cdot \pi}{17.32^2 - (6 \cdot \pi)^2}\right] + \pi$$

$X := 0.0041$

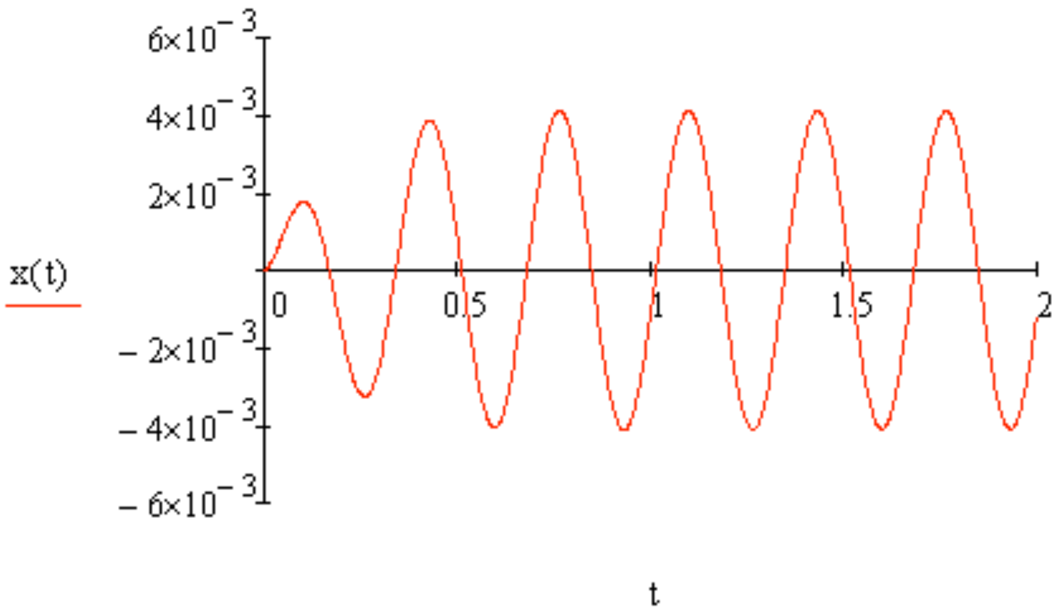
$\omega d := \omega n \cdot \sqrt{1 - \zeta^2}$ $\qquad \theta = 1.856$

$$\phi := \operatorname{atan}\left(\frac{-X \cdot \cos(\theta) \cdot \omega d}{-X \cdot \cos(\theta) \cdot \zeta \cdot \omega n - X \cdot \omega n \cdot \sin(\theta)}\right) + \pi \qquad \phi = 2.844$$

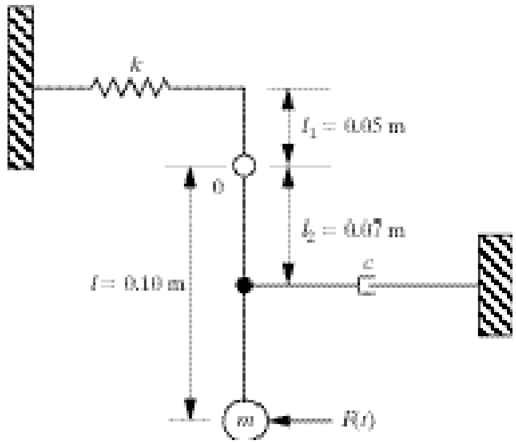
$$A := \frac{-X \cdot \cos(\theta)}{\sin(\phi)} \qquad A = 3.934 \times 10^{-3}$$

$$x(t) := \left(A \cdot e^{-\zeta \cdot \omega n \cdot t} \cdot \sin(\omega d \cdot t + \phi)\right) + X \cdot \cos(6 \cdot \pi \cdot t - \theta)$$

Plotting the result in Mathcad yields



2.21 Consider the pendulum mechanism of Figure P2.21 which is pivoted at point O. Calculate both the damped and undamped natural frequency of the system for small angles. Assume that the mass of the rod, spring, and damper are negligible. What driving frequency will cause resonance?



Solution: Assume the driving frequency to be harmonic of the standard form. To get the equation of motion take the moments about point O to get:

$$\sum M_0 = J\ddot{\theta}(t) = m\ell^2\ddot{\theta}(t)$$

$$= -k\ell_1 \sin\theta(\ell_1\cos\theta) - c\ell_2\dot{\theta}(\ell_2\cos\theta)$$

$$- mg(\ell\sin\theta) + F_0\cos\omega t(\ell\cos\theta)$$

Rearranging and approximating sinθ ~ θ and cosθ~1 yields:

$$m\ell^2\ddot{\theta}(t) + c\ell_2^2\dot{\theta}(t) + (k\ell_1^2 + mg\ell)\theta(t) = F_0\ell\cos\omega t$$

Dividing through by the coefficient of the inertia term and using the standard definitions for ζ and ω yields:

$$\omega_n = \sqrt{\frac{k\ell_1^2 + mg\ell}{m\ell^2}} \text{ which is the resnonant frequency}$$

$$\zeta = \frac{c\ell_2^2}{2\sqrt{(k\ell_1^2 + mg\ell)mg\ell}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k\ell_1^2 + mg\ell}{m\ell^2}\left(1 - \frac{c^2\ell_2^4}{4(k\ell_1^2 + mg\ell)mg\ell}\right)}$$

2.22 Consider the pivoted mechanism of Figure P2.21 with $k = 4 \times 10^3$ N/m. $l_1 = 0.05$ m. $l_2 = 0.07$ m. and $l = 0.10$ m. and $m = 40$ kg. The mass of the beam is 40 kg; it is pivoted at point 0 and assumed to be rigid. Design the dashpot (i.e. calculate c) so that the damping ratio of the system is 0.2. Also determine the amplitude of vibration of the steady-state response if a 10-N force is applied to the mass, as indicated in the figure, at a frequency of 10 rad/s.

Solution: This is similar to the previous problem with the mass of the beam included this time around. The equation of motion becomes:

$$m_{eq}\ddot{\theta} + c_{eq}\dot{\theta} + k_{eq}\theta = F_0\ell\cos\omega t$$

Here:

$$m_{eq} = m\ell^2 + \frac{1}{3}(\ell^3 + \ell_1^3)\frac{m_b}{\ell + \ell_1} = 0.5 \text{ kg}\cdot\text{m}^2$$

$$c_{eq} = c\ell_2^2 = 0.25c$$

$$k_{eq} = k\ell_1^2 + mg\ell + \frac{1}{2}(\ell - \ell_1)m_b g = 4.326 \times 10^3 \text{ Nm}$$

Using the formula the damping ratio and these numbers:

$$\zeta = \frac{\ell_2^2 c}{2\sqrt{m_{eq}k_{eq}}} = 0.2 \Rightarrow c = 3.797\cdot 10^3 \text{ kg/s}$$

Next compute the amplitude:

$$X = \frac{10/0.5}{\sqrt{(k_{eq}/m_{eq} - 10^2)^2 + (2\cdot 0.2\cdot 10\cdot \omega_n)^2}} = 2.336\times 10^3 \text{ rad}$$

2.23 In the design of Problem 2.22, the damping ratio was chosen to be 0.2 because it limits the amplitude of the forced response. If the driving frequency is shifted to 11 rad/s, calculate the change in damping coefficient needed to keep the amplitude less than calculated in Problem 2.22.

Solution: In this case the frequency is far away from resonance so the change in driving frequency does not matter much. This can also be seen numerically by the following Mathcad session.

$L1 := 0.05 \qquad k := 4 \cdot 10^3 \qquad L2 := 0.07 \qquad L := 0.1$

$m := 40 \qquad mb := 40$

$$meq := m \cdot L^2 + \frac{1}{3} \cdot \left(L^3 + L1^3\right) \cdot \frac{mb}{L + L1} \qquad meq = 0.5 \qquad g := 9.81$$

$$c := L2^2 \qquad c = 4.9 \cdot 10^{-3}$$

$$keq := k \cdot L1^2 + m \cdot g + \frac{1}{2 \cdot (L - L1)} \cdot (mb \cdot g) \qquad keq = 4.326 \cdot 10^3$$

$$ceq := \frac{0.2 \cdot 2 \cdot \sqrt{meq \cdot keq}}{c} \qquad ceq = 3.797 \cdot 10^3$$

$$\omega n := \sqrt{\frac{keq}{meq}} \qquad \omega n = 93.02 \qquad X := 10 \cdot \frac{L}{meq} \cdot \frac{1}{\sqrt{\left(\omega n^2 - 11^2\right)^2 + (2 \cdot .2 \cdot \omega n \cdot 11)^2}}$$

$$X = 2.341 \cdot 10^{-4}$$

The new amplitude is only slightly larger in this case. The problem would be more meaningful if the driving frequency is near resonance. Then the shift in amplitude will be more substantial and added damping may improve the response.

2.24 Compute the forced response of a spring-mass-damper system with the following values: c = 200 kg/s, k = 2000 N/m, m = 100 kg, subject to a harmonic force of magnitude F_0 = 15 N and frequency of 10 rad/s and initial conditions of x_0 = 0.01 m and v_0 = 0.1 m/s. Plot the response. How long does it take for the transient part to die off?

Solution:
Calculate the parameters

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{100}} = 4.472 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{15}{100} = 0.15 \text{ N/kg}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.472\sqrt{1-0.224^2} = 4.359 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{200}{2\cdot 100\cdot 4.472} = 0.224$$

Initial conditions: $x_0 = 0.01$ m, $v_0 = 0.1$ m/s

Using equation (2.38) and working in Mathcad yields

$$x(t) = e^{-t}(0.0104\cos 4.359t + 0.025\sin 4.359t) + 1.318\times 10^{-6}(0.335\cos 10t + 37.7\sin 10t)$$

$c := 200 \qquad k := 2000 \qquad m := 100 \qquad F0 := 15 \qquad \omega := 10$

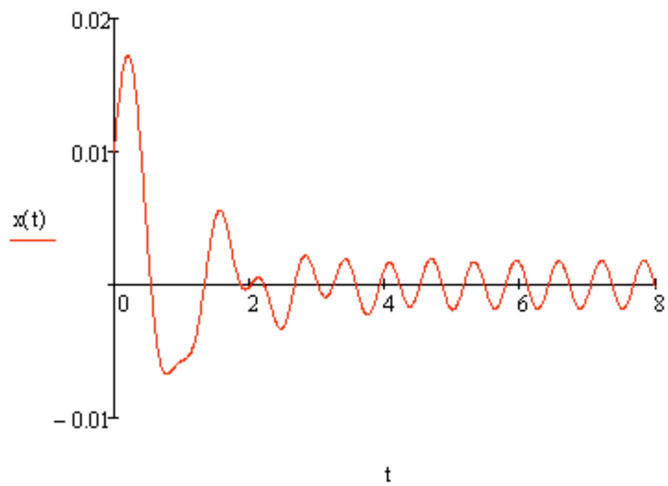
$\omega n := \sqrt{\frac{k}{m}} \qquad x0 := 0.01 \qquad v0 := 0.1 \qquad f0 := \frac{F0}{m} \qquad \zeta := \frac{c}{2\cdot m\cdot \omega n} \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$

$$A := x0 - \frac{f0\cdot\left(\omega n^2-\omega^2\right)}{\left(\omega n^2-\omega^2\right)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2}$$

$$B := \frac{\zeta\cdot\omega n}{\omega d}\cdot\left[\frac{f0\cdot\left(\omega n^2-\omega^2\right)}{\left(\omega n^2-\omega^2\right)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2}\right] - \frac{2\cdot\zeta\cdot\omega n\cdot\omega^2}{\omega d\cdot\left[\left(\omega n^2-\omega^2\right)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2\right]} + \frac{v0}{\omega d}$$

$$C := \frac{f0}{\left(\omega n^2-\omega^2\right)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2}$$

$$x(t) := e^{-\zeta\cdot\omega n\cdot t}\cdot(A\cdot\cos(\omega d\cdot t) + B\cdot\sin(\omega d\cdot t)) + C\cdot\left[\left(\omega n^2-\omega^2\right)\cdot\cos(\omega\cdot t) + 2\cdot\zeta\cdot\omega n\cdot\omega\cdot\sin(\omega\cdot t)\right]$$



a plot of m vs seconds. The time for the amplitude of the transient response to be reduced, for example, to 0.1 % of the initial (t = 0) amplitude can be determined by:

$$e^{-t} = 0.001, \text{ then } t = -\ln 0.001 = 6.908 \sec$$

2.25 Show that Equation (2.38) collapses to give Equation (2.11) in the case of zero damping.

Solution:
Eq. (2.38):

$$x(t) = e^{-\zeta\omega_n t}\left\{\begin{aligned}&\left\{(x_0 - \frac{f_0(\omega_n{}^2-\omega^2)}{(\omega_n{}^2-\omega^2)^2+(2\zeta\omega_n\omega)^2})\cos\omega_d t\right.\\ &+\left(\begin{aligned}&\frac{\zeta\omega_n}{\omega_d}(x_0-\frac{f_0(\omega_n{}^2-\omega^2)}{(\omega_n{}^2-\omega^2)^2+(2\zeta\omega_n\omega)^2})\\ &-\frac{2\zeta\omega_n\omega^2 f_0}{\omega_d\left[(\omega_n{}^2-\omega^2)^2+(2\zeta\omega_n\omega)^2\right]}+\frac{v_0}{\omega_d}\end{aligned}\right)\sin\omega_d t\end{aligned}\right\}$$

$$+\frac{f_0}{(\omega_n{}^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\left[(\omega_n{}^2-\omega^2)\cos\omega t+2\zeta\omega_n\omega\sin\omega t\right]$$

In case of $\zeta = 0$, this equation becomes:

$$x(t)=1\cdot\left\{\begin{aligned}&\left\{(x_0-\frac{f_0}{(\omega_n{}^2-\omega^2)+0})\cos\omega_d t\right.\\ &+\left(0-0+\frac{v_0}{\omega_d}\right)\sin\omega_d t\end{aligned}\right\}+\frac{f_0}{(\omega_n{}^2-\omega^2)}\cos\omega t$$

$$=\frac{v_0}{\omega_n}\sin\omega_n t+(x_0-\frac{f_0}{\omega_n{}^2-\omega^2})\cos\omega_n t+\frac{f_0}{\omega_n{}^2-\omega^2}\cos\omega t$$

(Note: $\omega_d = \omega_n$ for $\zeta = 0$)

2.26 Derive Equation (2.38) for the forced response of an underdamped system.

Solution:
From Sec. 1.3, the homogeneous solution is:

$$x_h(t) = e^{-\zeta\omega_n t}(A_1\sin\omega_d t + A_2\cos\omega_d t)$$

From equations (2.29) and (2.35), the particular solution is:

$$x_p(t)=\frac{(\omega_n{}^2-\omega^2)f_0}{(\omega_n{}^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\cos\omega t+\frac{2\zeta\omega_n\omega f_0}{(\omega_n{}^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\sin\omega t$$

Then the general solution is:

$$x(t)=x_h(t)+x_p(t)=e^{-\zeta\omega_n t}(A_1\sin\omega_d t+A_2\cos\omega_d t)$$

$$+\frac{(\omega_n{}^2-\omega^2)f_0}{(\omega_n{}^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\cos\omega t+\frac{2\zeta\omega_n\omega f_0}{(\omega_n{}^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\sin\omega t$$

Using the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, the constants, A_1 and A_2, are determined:

$$A_2 = x_0 - \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$A_1 = \frac{v_0}{\omega_d} + \frac{\omega}{\omega_d} \cdot \frac{2\zeta\omega_n \omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} + \zeta \frac{\omega_n}{\omega_d}\left(x_0 - \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\right)$$

Then, Eq. (2.30) is obtained by substituting the expressions for A_1 and A_2 into the general solution and simplifying the resulting equation.

2.27 Compute a value of the damping coefficient c such that the steady state response amplitude of the system in Figure P2.27 is 0.01 m.

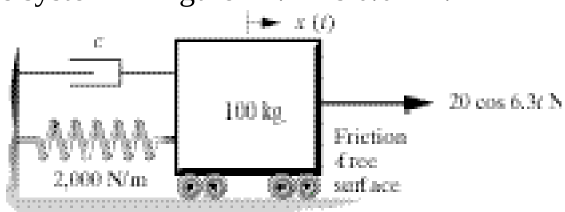


Figure P2.27

Solution:

From Eq. (2.39), the amplitude of the steady state response is given by

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

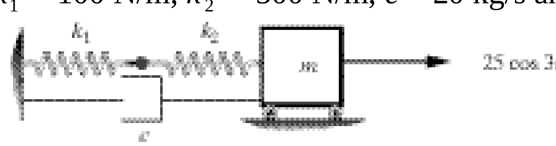
Then substitute, $2\zeta\omega_n = c/m$, $c = \sqrt{\frac{F_0^2}{\omega^2 \cdot X^2} - m^2 \frac{(\omega_n^2 - \omega^2)^2}{\omega^2}}$ into this equation and solve for c:

Given:

$X = 0.01\text{m}$ $\omega = 6.3\,\text{rad/s}$ $F_0 = 20\,\text{N}$ $m = 100\,\text{kg}$

$$\omega_n^2 = \frac{k}{m} = \frac{2000}{100} = 20\ (\text{rad/s})^2 \Rightarrow \underline{c = 55.7\ \text{kg/s}}$$

2.28 Compute the response of the system in Figure P2.28 if the system is initially at rest for the values k_1 = 100 N/m, k_2 = 500 N/m, c = 20 kg/s and m = 89 kg.



Solution:

The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = 25\cos 3t \qquad \text{where} \quad k = \frac{1}{1/k_1 + 1/k_2}$$

Using Eq. (2.37) in an alternative form, the general solution is:

$$x(t) = e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t) + X\cos(\omega t - \theta)$$

where

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{25/89}{\sqrt{(0.966^2 - 3^2)^2 + (2\cdot 0.116\cdot 0.966\cdot 3)^2}} = 0.0347\,m$$

$$\theta = \tan^{-1}\cdot\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} = \tan^{-1}\cdot\frac{2\cdot 0.116\cdot 0.966\cdot 3}{0.966^2 - 3^2} = 3.058\,rad \qquad \text{(see Window 2.3)}$$

Using the initial conditions, $x(0) = 0$ and $\dot{x}(0) = 0$, the constants, A_1 and A_2, are determined:

$A_2 = 0.0345$ $\qquad$ $A_1 = -0.005$

Given: $c = 20$ kg/sec, $m = 89$ kg

$$k = \frac{1}{1/k_1 + 1/k_2} = \frac{1}{1/100 + 1/500} = 83\,\text{N/m}$$

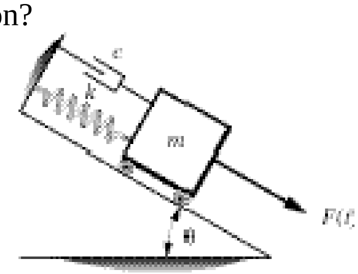
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{83}{89}} = 0.966\,\text{rad/s} \qquad \zeta = \frac{c}{2m\omega_n} = \frac{20}{2\cdot 89\cdot 0.966} = 0.116$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 0.966\sqrt{1-0.116^2} = 0.9595\,\text{rad/s}$$

Substituting the values into the general solution:

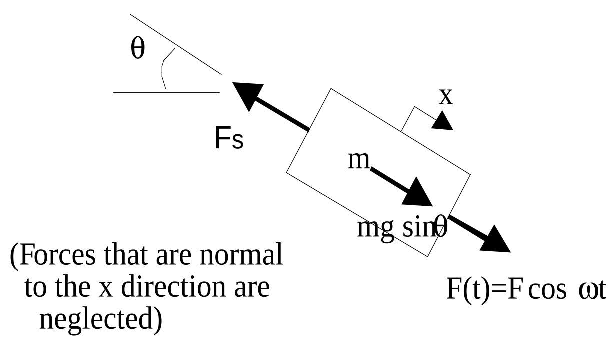
$$x(t) = e^{-0.112t}(-0.005\sin 0.9595t + 0.0345\cos 0.9595t) + 0.0347\cos(3t - 3.058)$$

2.29 Write the equation of motion for the system given in Figure P2.29 for the case that $F(t) = F\cos\omega t$ and the surface is friction free. Does the angle θ effect the magnitude of oscillation?



Solution:

Free body diagram:



Assuming x = 0 to be at the equilibrium:

$$\sum F_x = F + mg\sin\theta - F_s = m\ddot{x}$$

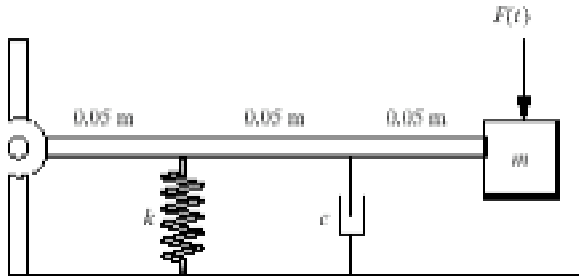
where $F_s = k(x + \frac{mg\sin\theta}{k})$ and $F(t) = F\cos\omega t$

Then the equation of motion is:

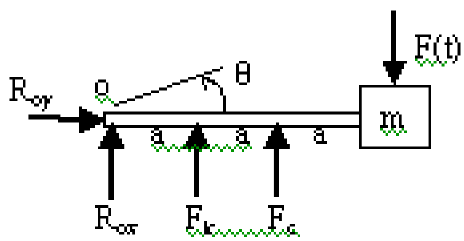
$m\ddot{x} + kx = F\cos\omega t$

Note that the equation of motion does not contain θ which means that the magnitude of the response is not affected by the angle of the incline.

2.30 A foot pedal for a musical instrument is modeled by the sketch in Figure P2.30. With k = 2000 N/m, c = 25 kg/s, m = 25 kg and $F(t)$ = 50 cos $2\pi\, t$N, compute the steady state response assuming the system starts from rest. Also use the small angle approximation.



Solution: Free body diagram of pedal follows:



Summing the moments with respect to the point, O:

$$\sum M_0 = F(3\cdot a) - F_c(2\cdot a) - F_s(a) = I_o\ddot{\theta}$$

$$\text{where} \quad I_o = m(3a)^2 = 9a^2m \ , \quad F_s = ka\sin\theta$$

$$F_c = c(2\cdot a\cdot\sin\theta)' = 2ca\cos\theta\dot{\theta}$$

Substituting these equations and simplifying ($\sin\theta \approx \theta$, cosθ =1,for small θ):

$$9a^2m\ddot{\theta} + 4a^2c\dot{\theta} + a^2k\theta = 3aF(t)$$

Given: k = 2000 N/m, c = 25kg/s , m = 25 kg , $F(t) = 50\cos 2\pi t$, a = 0.05 m

The equation of motion becomes: $0.5625\ddot{\theta} + 0.25\dot{\theta} + 5\theta = 7.5\cos 2\pi t$

Observing the equation of motion, equivalent mass, damping and stiffness coefficients are:

$c_{eq} = 0.25$, $m_{eq} = 0.5625$, $k_{eq} = 5$, $f_0 = \frac{F_0}{m_{eq}} = \frac{7.5}{0.5625} = 13.33$, $\omega = 2\pi$

$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{5}{0.5625}} = 2.981$ $\qquad \zeta = \frac{c_{eq}}{2m_{eq}\omega_n} = 0.0745$

From Eq. (2.36), the steady-state response is:

$$\theta(t) = \frac{f_{0eq}}{\sqrt{(\omega_n^{\ 2} - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos(\omega t - \tan^{-1}\frac{2\zeta_{eq}\omega_n\omega}{\omega_n^{\ 2} - \omega^2})$$

$$\Rightarrow \underline{\theta(t) = 0.434\cos(2\pi t - 3.051) \text{ rad}}$$

2.31 Consider the system of Problem 2.15, repeated here as Figure P2.31 with the effects of damping indicated. The physical constants are J =25 kg m^2, k = 2000 N/m, and the applied moment is 5 Nm at 1.432 Hz acting through the distance r = 0.5 m. Compute the magnitude of the steady state response if the measured damping ratio of the spring system is ζ = 0.01. Compare this to the response for the case where the damping is not modeled (ζ = 0).

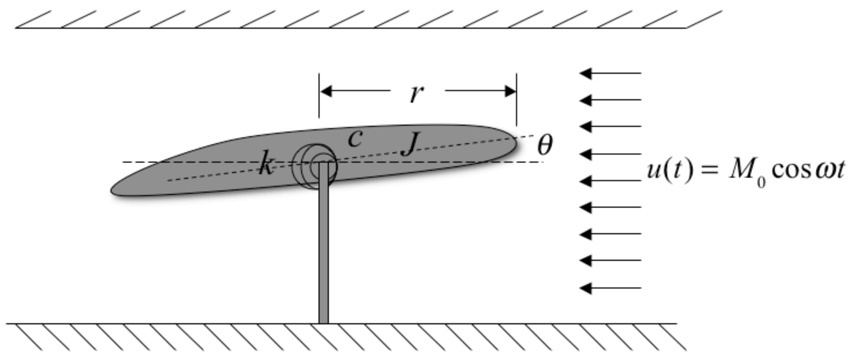


Figure P2.31 Model of an airfoil in at wind tunnel including the effects of damping.

Solution From equation (2.39) the magnitude of the steady state response for an underdamped system is

$$|\theta| = \frac{M_0 / J}{\sqrt{\left(\frac{k}{J} - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}}$$

Substitution of the given values yields (here $X = r\theta$)

$$|\theta| = 0.2 \text{ rad and } X = 0.1 \text{ m for } \zeta = 0$$

$$|\theta| = 0.106 \text{ rad and } X = 0.053 \text{ m for } \zeta = 0.01$$

where X is the vertical displacement of the wing tip. Thus a small amount of damping can greatly reduce the amplitude of vibration.