Chapter Three Solutions

Problem and Solutions for Section 3.1 (3.1 through 3.14)

3.1 Calculate the solution to

$$\ddot{x} + 2\dot{x} + 2x = \delta(t-\pi)$$
$$x(0) = 1 \quad \dot{x}(0) = 0$$

and plot the response.

Solution: Given: $\ddot{x} + 2\dot{x} + 2x = \delta(t-\pi) \quad x(0) = 1, \quad \dot{x}(0) = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.414 \text{ rad/s}, \quad \zeta = \frac{c}{2\sqrt{km}} = 0.7071, \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 1 \text{ rad/s}$$

Total Solution: $x(t) = x_h(t) + x_p(t)$

Homogeneous: From Equation (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \qquad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] = .785 \text{ rad}$$

$$\Rightarrow x_h(t) = 1.414e^{-t}\sin(t + .785)$$

Particular: From Equation. (3.9)

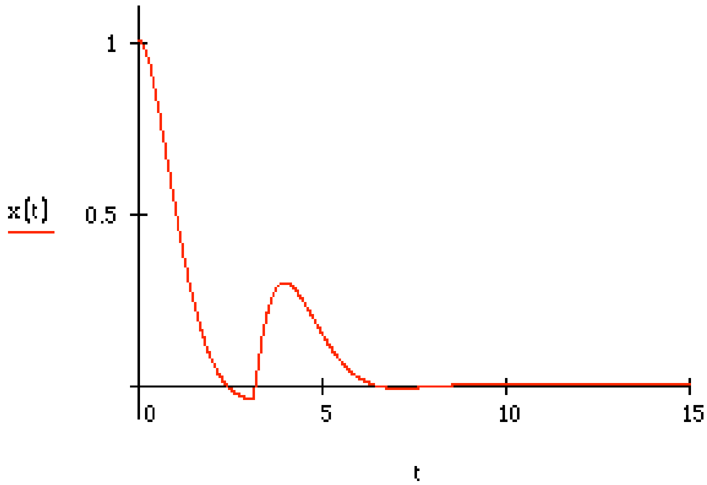
$$x_p(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau) = \frac{1}{(1)(1)}e^{-(t-\pi)}\sin(t-\pi)$$

But, $\sin(-t) = -\sin t$ So, $x_p(t) = -e^{-(t-\pi)}\sin t \quad \Rightarrow$

$$x(t) = 1.414e^{-t}\sin(t + 0.785) \qquad 0 < t < \pi$$
$$x(t) = 1.414e^{-t}\sin(t + 0.785) - e^{-(t-\pi)}\sin t \qquad t > \pi$$

This is plotted below using the Heaviside function.

$$x(t) := 1.414 \cdot e^{-t} \cdot \sin(t + 0.785) - e^{-(t-\pi)} \cdot \sin(t) \cdot \Phi(t - \pi)$$



3.2 Calculate the solution to

$$\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t-\pi)$$
$$x(0) = 0 \quad \dot{x}(0) = 1$$

and plot the response.

Solution: Given: $\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t-\pi), \quad x(0) = 0, \quad \dot{x}(0) = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.732 \text{ rad/s}, \; \zeta = \frac{c}{2\sqrt{km}} = 0.5774, \; \omega_d = \omega_n\sqrt{1-\zeta^2} = 1.414 \text{ rad/s}$$

Total Solution:

$$x(t) = x_h + x_{p1} \qquad 0 < t < \pi$$
$$x(t) = x_h + x_{p1} + x_{p2} \qquad t > \pi$$

Homogeneous: Eq. (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = Ae^{-t}\sin(1.414t + \phi)$$

Particular: #1 (Chapter 2)

$$x_{p1}(t) = X\sin(\omega t - \theta), \text{ where } \omega = 1 \text{ rad/s . Note that } f_0 = \frac{F_0}{m} = 1$$

$$\Rightarrow X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = 0.3536, \text{ and } \theta = \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right] = 0.785 \text{ rad}$$

$$\Rightarrow x_{p1}(t) = 0.3536\sin(t - 0.7854)$$

Particular: #2 Equation 3.9

$$x_{p2}(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\pi)}\sin\omega_d(t-\tau) = \frac{1}{(1)(1.414)}e^{-(t-\pi)}\sin 1.414(t-\pi)$$

$$\Rightarrow x_{p2}(t) = 0.7071e^{-(t-\pi)}\sin 1.414(t-\pi)$$

The total solution for $0 < t < \pi$ becomes:

$$x(t) = Ae^{-t}\sin(1.414t + \phi) + 0.3536\sin(t - 0.7854)$$
$$\dot{x}(t) = -Ae^{-t}\sin(1.414t + \phi) + 1.414Ae^{-t}\cos(1.414t + \phi) + 0.3536\cos(t - 0.7854)$$

$$x(0) = 0 = A\sin\phi - 0.25 \Rightarrow A = \frac{0.25}{\sin\phi}$$

$$\dot{x}(0) = 1 = -A\sin\phi + 1.414A\cos\phi + 0.25 \Rightarrow 0.75 = 0.25 - 1.414(0.25)\frac{1}{\tan\phi}$$

$$\Rightarrow \phi = 0.34 \text{ and } A = 0.75$$

Thus for the first time interval, the response is

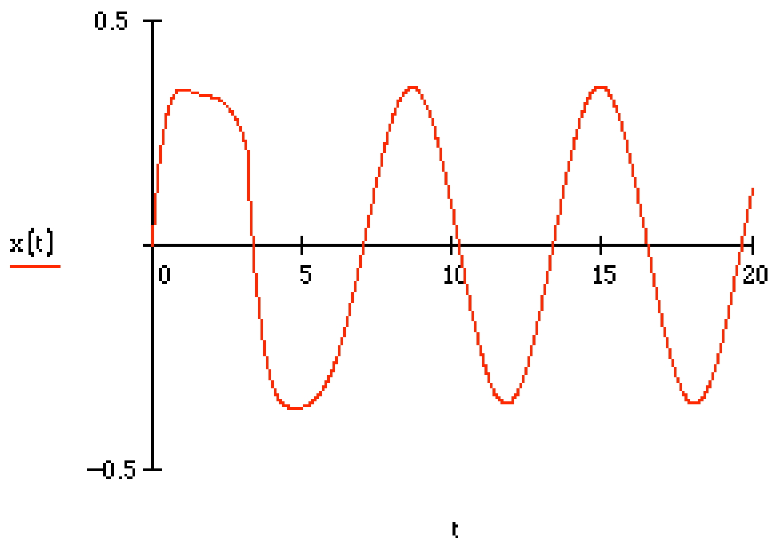
$$x(t) = 0.75e^{-t}\sin(1.414t + 0.34) + 0.3536\sin(t - 0.7854) \qquad 0 < t < \pi$$

Next consider the application of the impulse at $t = \pi$:

$$x(t) = x_h + x_{p1} + x_{p2}$$

$$x(t) = -0.433e^{-t}\sin(1.414t + 0.6155) + 0.3536\sin(t - 0.7854) - 0.7071e^{-(t-\pi)}\sin(1.414t - \pi) \quad t > \pi$$

The response is plotted in the following (from Mathcad):



3.3 Calculate the impulse response function for a critically damped system.

Solution:

The change in the velocity from an impulse is $v_0 = \frac{\hat{F}}{m}$, while $x_0 = 0$. So for a critically damped system, we have from Eqs. 1.45 and 1.46 with $x_0 = 0$:

$$x(t) = v_0 t e^{-\omega_n t}$$

$$\Rightarrow x(t) = \frac{\hat{F}}{m} t e^{-\omega_n t}$$

3.4 Calculate the impulse response of an overdamped system.

Solution:

The change in velocity for an impulse $v_0 = \frac{\hat{F}}{m}$, while $x_0 = 0$. So, for an overdamped system, we have from Eqs. 1.41, 1.42 and 1.43:

$$x(t) = e^{-\zeta\omega_n t}\left[\frac{-v_0}{2\omega_n\sqrt{\zeta^2-1}}e^{-\omega(_n\sqrt{\zeta^2-1})t} + \frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}e^{-\omega(_n\sqrt{\zeta^2-1})t}\right]$$

$$x(t) = \frac{\hat{F}}{2m\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\left[e^{-\omega(_n\sqrt{\zeta^2-1})t} - e^{-\omega(_n\sqrt{\zeta^2-1})t}\right]$$

3.5 Derive equation (3.6) from equations (1.36) and (1.38).

Solution:

Equation 1.36: $x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$

Equation 1.38: $A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$, $\phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right]$

Since $x_0 = 0$ and $v_0 = \frac{\hat{F}}{m}$, Equation 1.38 becomes

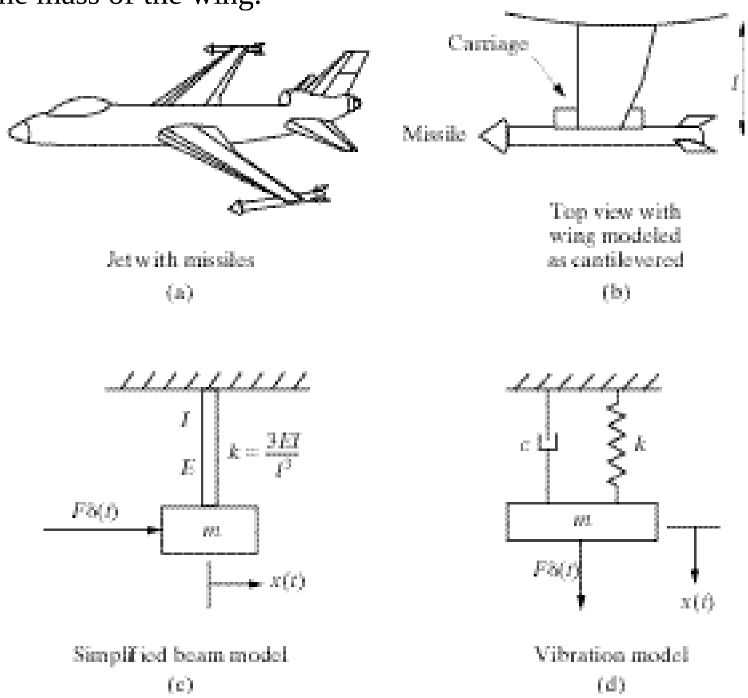
$$A = \frac{v_0}{\omega_d} = \frac{\hat{F}}{m\omega_d}$$

$$\phi = \tan^{-1}(0) = 0$$

So Equation 1.36 becomes

$$x(t) = \frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t) \text{ which is Equation 3.6}$$

3.6 Consider a simple model of an airplane wing given in Figure P3.6. The wing is approximated as vibrating back and forth in its plane, massless compared to the missile carriage system (of mass *m*). The modulus and the moment of inertia of the wing are approximated by *E* and *I*, respectively, and *l* is the length of the wing. The wing is modeled as a simple cantilever for the purpose of estimating the vibration resulting from the release of the missile, which is approximated by the impulse funciton $F\delta(t)$. Calculate the response and plot your results for the case of an aluminum wing 2 m long with $m = 1000$ kg, $\zeta = 0.01$, and $I = 0.5\ m^4$. Model *F* as 1000 N lasting over 10^{-2}s.
Modeling of wing vibration resulting from the release of a missile. (a) system of interest; (b) simplification of the detail of interest; (c) crude model of the wing: a cantilevered beam section (recall Figure 1.24); (d) vibration model used to calculate the response neglecting the mass of the wing.



Solution: Given:

$$\begin{array}{ll} m = 1000\text{ kg} & \zeta = 0.01 \\ l = 4\text{ m} & I = 0.5\text{ m}^4 \\ F = 1000\text{ N} & \Delta t = 10^{-2}\text{ s} \end{array}$$

From Table 1.2, the modulus of Aluminum is $E = 7.1\times 10^{10}\ \text{N/m}^2$

The stiffness is

$$k = \frac{3EI}{\ell^3} = \frac{3\left(7.1\times10^{10}\right)\left(0.5\right)}{4^3} = 1.664\times 10^9\ \text{N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.29\times 10^3\ \text{rad/s (205.4 Hz)}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.29\times 10^3$$

Solution (Eq. 3.6):

$$x(t) = \frac{(F\Delta t)e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = 7.753\times 10^{-6}e^{-12.9t}\sin(1290t)\text{ m}$$

The following m-file

```
t=(0:0.0001:0.5);
F=1000;dt=0.01;m=1000;zeta=0.01;E=7.1*10^10;I=0.5;L=4;
wn=sqrt((3*I*E/L^3)/m);
wd=wn*sqrt(1-zeta^2);
x=(F*dt/(m*wd))*exp(-zeta*wn*t).*sin(wd*t);
plot(t,x)
```

The solution worked out in Mathcad is given in the following:

$E := 7.1\cdot 10^{10}$

$m := 1000$ $\quad L := 4$ $\quad I := 0.5$ $\quad F := 1000$ $\quad \Delta t := 10^{-2}$ sec

$\zeta := 0.01$

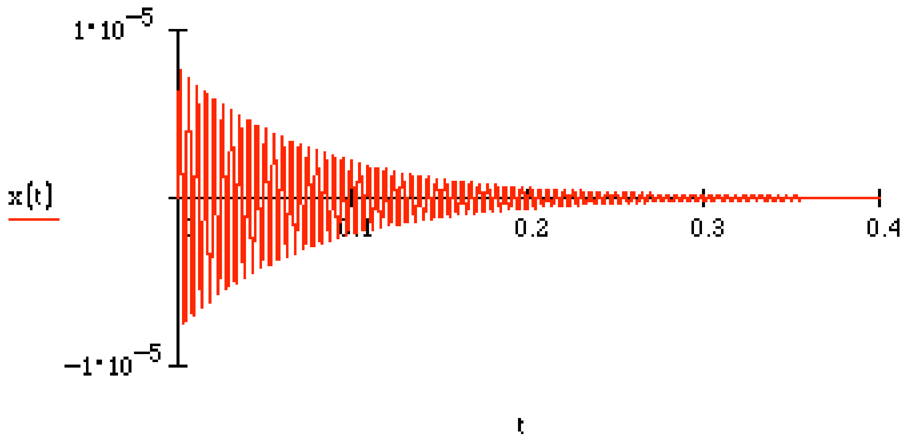
$k := \dfrac{3\cdot I\cdot E}{L^3}$ $\quad \omega n := \sqrt{\dfrac{k}{m}}$ $\quad k = 1.664\cdot 10^9$ $\quad \omega n = 1.29\cdot 10^3$

$fn := \dfrac{\omega n}{2\cdot \pi}$ $\quad fn = 205.308$ Hz $\quad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$ $\quad \omega d = 1.29\cdot 10^3$

$\zeta\cdot\omega n = 12.9$ $\quad \dfrac{F}{m\cdot\omega d}\cdot(\Delta t) = 7.752\cdot 10^{-6}$

$t := 0, 0.00001 .. 0.4$

$x(t) := \dfrac{F}{m\cdot\omega d}\cdot\left(\Delta t\cdot e^{-\zeta\cdot\omega n\cdot t}\right)\cdot\sin(\omega d\cdot t)$ meters



3.7 A cam in a large machine can be modeled as applying a 10,000 N-force over an interval of 0.005 s. This can strike a valve that is modeled as having physical parameters: m = 10 kg, c = 18 N•s/m, and stiffness k = 9000 N/m. The cam strikes the valve once every 1 s. Calculate the vibration response, $x(t)$, of the valve once it has been impacted by the cam. The valve is considered to be closed if the distance between its rest position and its actual position is less than 0.0001 m. Is the valve closed the very next time it is hit by the cam?

Solution: Given:

$$F = 10{,}000 \text{ N} \qquad \Delta t = 0.005 \text{ s}$$

$$m = 10 \text{ kg} \qquad c = 18 \text{ N}\cdot\text{s/m} \qquad k = 9000 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 30 \text{ rad/s} \quad \zeta = \frac{c}{2\sqrt{km}} = 0.03 \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 29.99 \text{ rad/s}$$

Solution Eq. (3.6):

$$x(t) = \frac{(F\Delta t)e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

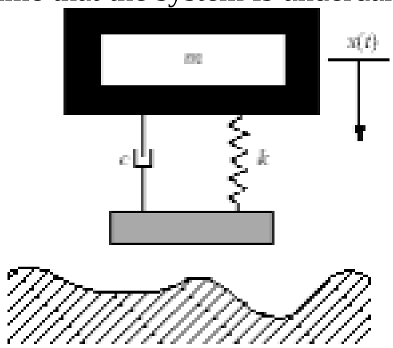
$$x(t) = \frac{(10{,}000)(0.005)e^{-(0.03)(30)t}}{(10)(29.99)}\sin(29.99t)$$

$$x(t) = 0.1667e^{-0.9t}\sin(29.99t)\text{m}$$

At t=1 s: $\qquad x(1) = 0.1667e^{-0.9}\sin(29.99) = -.06707 \text{ m}$

Since $|x(1)| = 0.06707 > 0.0001,$ the valve is not closed.

3.8 The vibration packages dropped from a height of *h* meters can be approximated by considering Figure P3.8 and modeling the point of contact as an impulse applied to the system at the time of contact. Calculate the vibration of the mass *m* after the system falls and hits the ground. Assume that the system is underdamped.



Solution: When the system hits the ground, it responds as if an impulse force acted on it.

From Equation (3.6): $x(t) = \dfrac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$ where $\dfrac{\hat{F}}{m} = v_0$

Calculate v_0:

For falling mass: $x = \dfrac{1}{2}at^2$

So, $v_0 = gt^*$, where t^* is the time of impact from height *h*

$$h = \frac{1}{2}gt^{*2} \Rightarrow t^* = \sqrt{\frac{2h}{g}}$$
$$v_0 = \sqrt{2gh}$$

Let $t = 0$ when the end of the spring hits the ground

The response is $x(t) = \dfrac{\sqrt{2gh}}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$

Where ω_n, ω_d, and ζ are calculated from *m, c, k*. Of course the problem could be solved as a free response problem with $x_0 = 0$, $v_0 = \sqrt{2gh}$ or an impulse response with impact model as the unit velocity given.

3.9 Calculate the response of

$$3\ddot{x}(t) + 12\dot{x}(t) + 12x(t) = 3\delta(t)$$

for zero initial conditions. The units are in Newtons. Plot the response.

Solution: Dividing the equation of motion by 3 reveals;

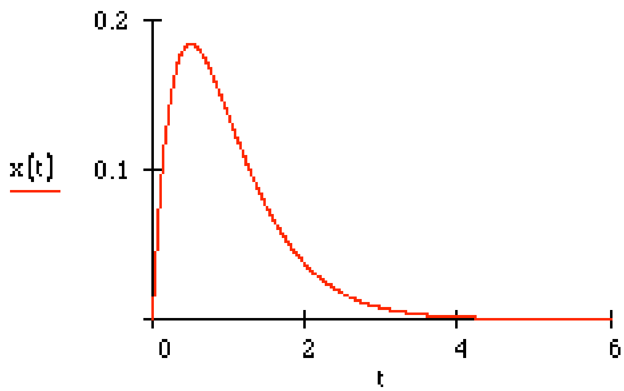
$$\omega_n = \sqrt{4} = 2 \text{ rad/s} \quad \zeta = \frac{12}{2(3)(2)} = 1 \Rightarrow \text{critically damped}$$

$$\hat{F} = 3 \qquad v_0 = \frac{F\Delta t}{m}, \qquad x_0 = 0$$

$$x = (a_1 + a_2 t)e^{-\omega_n t} \quad a_1 = 0 \quad a_2 = \frac{F\Delta t}{m}$$

$$\Rightarrow x(t) = \frac{\hat{F}}{m} t e^{-2t} = \frac{3}{3} t e^{-2t} = t e^{-2t}$$

$x(t) := t \cdot e^{-2 \cdot t}$



3.10 Compute the response of the system:

$$3\ddot{x}(t) + 12\dot{x}(t) + 12x(t) = 3\delta(t)$$

subject to the initial conditions $x(0) = 0.01$ m and $v(0) = 0$. The units are in Newtons. Plot the response.

Solution: From the previous problem the system is critically damped with a solution of the form

$$x(t) = (a_1 + a_2 t)e^{-2t}.$$

Applying the given initial conditions yields

$$x(0) = 0.01 = a_1 \quad \text{and} \quad \dot{x}(0) = 0 = -2(0.01 + a_2 0) + a_2$$

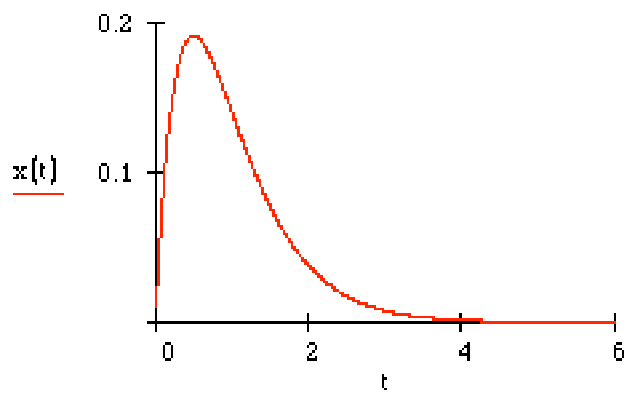
$$\Rightarrow x(t) = (0.01 + 0.02t)e^{-2t}$$

Next add to this the solution due to the unit impulse, which was calculated in Problem 3.9 to get:

$$x(t) = te^{-2t} + (0.01 + 0.02t)e^{-2t}$$

$$\Rightarrow \underline{x(t) = (0.01 + 1.02t)e^{-2t}}$$

$$x(t) := (0.01 + 1.02 \cdot t) \cdot e^{-2 \cdot t}$$



3.11 Calculate the response of the system

$$3\ddot{x}(t) + 6\dot{x}(t) + 12x(t) = 3\delta(t) - \delta(t-1)$$

subject to the initial conditions $x(0)$ =0.01 m and $v(0)$ = 1 m/s. The units are in Newtons. Plot the response.

Solution: First compute the natural frequency and damping ratio:

$$\omega_n = \sqrt{\frac{12}{3}} = 2 \text{ rad/s}, \quad \zeta = \frac{6}{2\cdot 2\cdot 3} = 0.5, \quad \omega_d = 2\sqrt{1-0.5^2} = 1.73 \text{ rad/s}$$

so that the system is underdamped. Next compute the responses to the two impulses:

$$x_1(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t = \frac{3}{3(1.73)} e^{-(t-1)} \sin 1.73(t-1) = 0.577 e^{-t} \sin 1.73 t, t > 0$$

$$x_2(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n (t-1)} \sin\omega_d (t-1) = \frac{1}{3(1.73)} e^{-t} \sin 1.73 t = 0.193 e^{-(t-1)} \sin 1.73(t-1), t > 1$$

Now compute the response to the initial conditions from Equation (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \qquad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] = 0.071 \text{ rad}$$

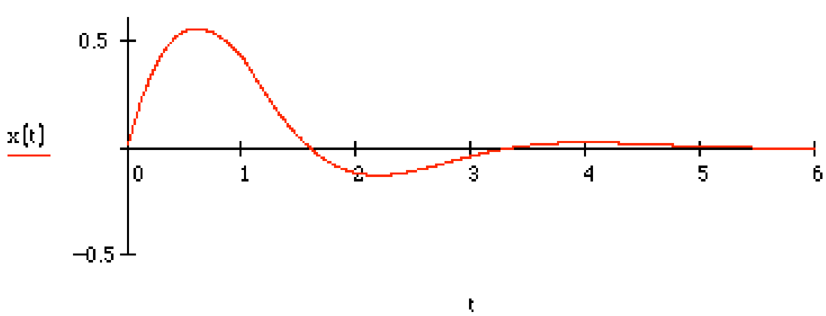
$$\Rightarrow x_h(t) = 0.5775 e^{-t} \sin(t + 0.017)$$

Using the Heaviside function the total response is

$$x(t) = 0.577 e^{-t} \sin 1.73 t + 0.583 e^{-t} \sin(t + 0.017) + 0.193 e^{-(t-1)} \sin 1.73(t-1)\Phi(t-1)$$

This is plotted below in Mathcad:

$$x(t) := \left(\frac{e^{-\zeta\cdot\omega n\cdot t}}{\omega d}\cdot\sin(\omega d\cdot t) + A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t + \phi)\right) + \left[\frac{e^{-\zeta\cdot\omega n\cdot(t-1)}}{-3\cdot\omega d}\cdot\sin[\omega d\cdot(t-1)]\right]\cdot\Phi(t-1)$$



Note the slight bump in the response at t = 1 when the second impact occurs.

3.12 A chassis dynamometer is used to study the unsprung mass of an automobile as illustrated in Figure P3.12 and discussed in Example 1.4.1 and again in Problem 1.64. Compute the maximum magnitude of the center of the wheel due to an impulse of 5000 N applied over 0.01 seconds. Assume the wheel mass is m = 15 kg, the spring stiffness is k = 500,000 N/m, the shock absorber provides a damping ratio of ζ = 0.3, and the rotational inertia is J = 2.323 kg m^2. Compute and plot the response of the wheel system to an impulse of 5000 N over 0.01 s. Compare the undamped maximum amplitude to that of the maximum amplitude of the damped system (use r = 0.457 m).

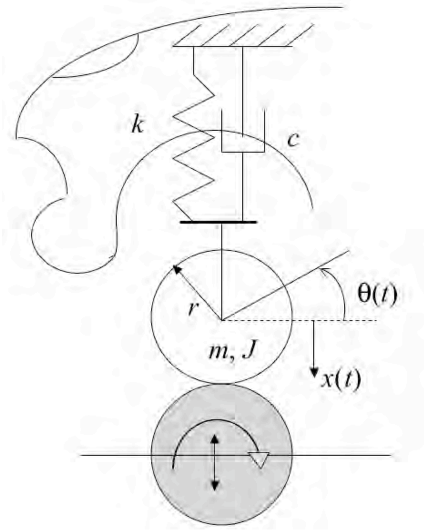


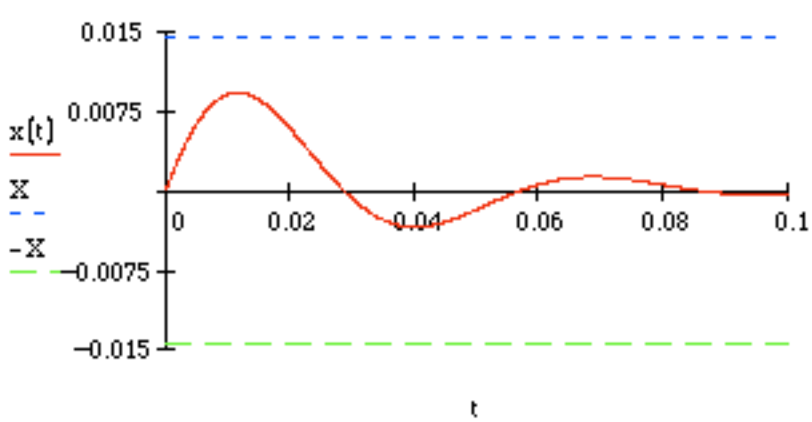
Figure P3.12 Simple model of an automobile suspension system mounted on a chassis dynamometer. The rotation of the car's wheel/tire assembly (of radius r) is given by $\theta(t)$ and is vertical deflection by $x(t)$.

Solution: With the values given the natural frequency, damped natural frequency, and impulse are calculated to be:

$$\omega_n = \sqrt{\frac{k}{m + J/r^2}} = 117.67 \text{ rad/s } = 18.73 \text{ Hz}, \;\; \omega_d = 112.25 \text{ rad/s}, \;\; X = \frac{F\Delta t}{(m + J/r^2)\omega_n} = 0.014 \text{ m}$$

The response is then plotted as

$$x(t) := \frac{F \cdot \Delta t}{\left(m + \frac{J}{r}\right) \cdot \omega n} \cdot e^{-\zeta \cdot \omega n \cdot t} \cdot \sin(\omega d \cdot t)$$



Note that the maximum amplitude of the undamped system, X, is not achieved.

3.13 Consider the effect of damping on the bird strike problem of Example 3.1.1. Recall from the example that the bird strike causes the camera to vibrate out of limits. Adding damping will cause the magnitude of the response to decrease but may not be able to keep the camera from vibrating past the 0.01 m limit. If the damping in the aluminum is modeled as $\zeta = 0.05$, approximately how long before the camera vibration reduces to the required limit? (Hint: plot the time response and note the value for time after which the oscillations remain below 0.01 m).

Solution: Using the values given in Example 3.1.1 and equations (3.7) and (3.8), the response has the form

$$x(t) = \frac{m_b v}{m\omega_n} e^{-\zeta\omega_n t} \sin \omega_d t = 0.026 e^{-13.07t} \sin 260.976t$$

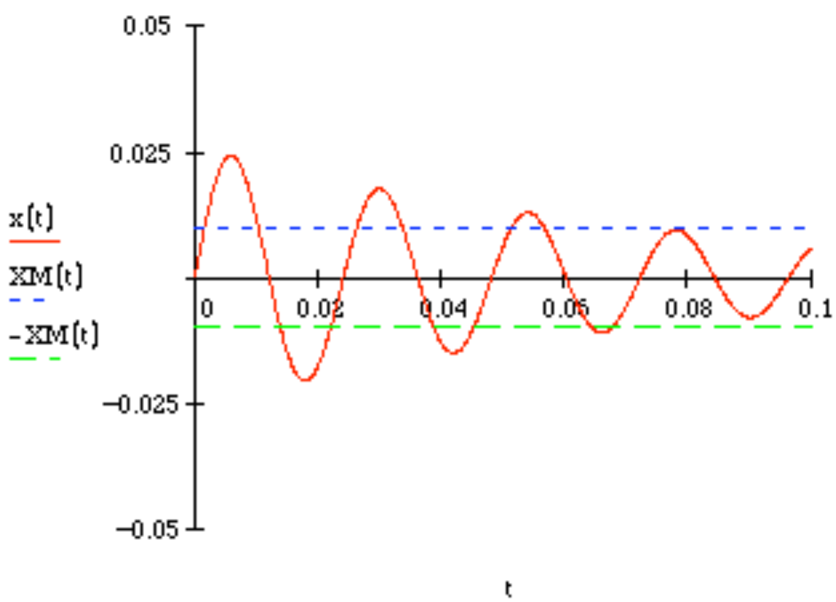
Here m_b is the mass of the bird and m is the mass of the camera. This is plotted in Mathcad below

$$X := \frac{mb \cdot v}{mc \cdot \omega n} \qquad X = 0.026 \cdot m$$

$$\zeta := 0.05 \qquad \zeta \cdot \omega n = 13.065 \cdot sec^{-1}$$

$$Y := 0.026 \qquad wn := 261.303 \qquad wd := wn \cdot \sqrt{1 - \zeta^2} \qquad wd = 260.976$$

$$x(t) := Y \cdot e^{-\zeta \cdot wn \cdot t} \cdot \sin(wd \cdot t) \qquad XM(t) := 0.01$$



From the plot, the amplitude remains below 0.01 m after about <u>0.057 s.</u>

3.14 Consider the jet engine and mount indicated in Figure P3.14 and model it as a mass on the end of a beam as done in Figure 1.24. The mass of the engine is usually fixed. Find a expression for the value of the transverse mount stiffness, *k*, as a function of the relative speed of the bird, *v*, the bird mass, the mass of the engine and the maximum displacement that the engine is allowed to vibrate.

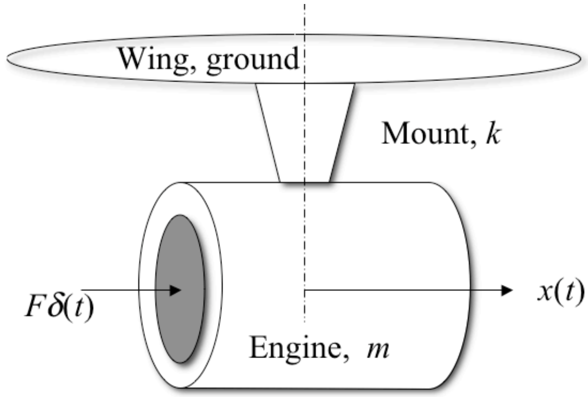


Figure P3.14 Model of a jet engine in transverse vibration due to a bird strike.

Solution: The equation of motion is

$$m\ddot{x}(t) + kx(t) = \hat{F}\delta(t)$$

From equations (3.7) and (3.8) the magnitude of the response is

$$|X| = \frac{\hat{F}}{m\omega_n}$$

for an undamped system. If the bird is moving with momentum $m_b v$ then:

$$|X| = \frac{m_b v}{m\omega_n} \Rightarrow |X| = \frac{m_b v}{\sqrt{mk}} \Rightarrow \underline{k = \frac{1}{m}\left(\frac{m_b v}{|X|}\right)^2}$$

This can be used to provide some guidance in designing the engine mount.