Problems and Solutions for Section 3.2 (3.15 through 3.25)

3.15 Calculate the response of an overdamped single-degree-of-freedom system to an arbitrary non-periodic excitation.

Solution: From Equation (3.12): $x(t) = \int_0^t F(\tau) h(t-\tau) d\tau$

For an overdamped SDOF system (see Problem 3.4)

$$h(t-\tau) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}} e^{-\zeta\omega_n(t-\tau)}\left(e^{\omega_n\sqrt{\zeta^2-1}(t-\tau)} - e^{-\omega_n\sqrt{\zeta^2-1}(t-\tau)}\right) d\tau$$

$$x(t) = \int_0^t F(\tau)\frac{1}{2m\omega_n\sqrt{\zeta^2-1}} e^{-\zeta\omega_n(t-\tau)}\left(e^{\omega_n\sqrt{\zeta^2-1}(t-\tau)} - e^{-\omega_n\sqrt{\zeta^2-1}(t-\tau)}\right) d\tau$$

$$\Rightarrow x(t) = \frac{e^{-\zeta\omega_n}}{2m\omega_n\sqrt{\zeta^2-1}} \int_0^t F(\tau) e^{\zeta\omega_n\tau}\left(e^{\omega_n\sqrt{\zeta^2-1}(t-\tau)} - e^{-\omega_n\sqrt{\zeta^2-1}(t-\tau)}\right) d\tau$$

3.16 Calculate the response of an underdamped system to the excitation given in Figure P3.16.
Plot of a pulse input of the form $f(t) = F_0 \sin t$.

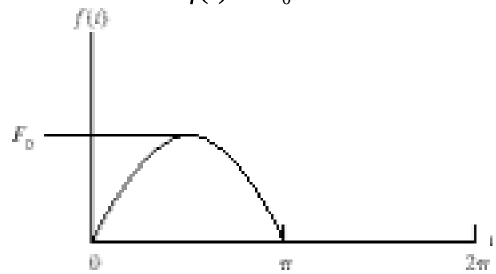


Figure P3.16

Solution:

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right] d\tau$$

$$F(t) = F_0 \sin(t) \qquad t < \pi \quad (\text{From Figure P3.16})$$

For $t \leq \pi$, $\quad x(t) = \dfrac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \displaystyle\int_0^t \left(\sin\tau e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right) d\tau$

$$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n{}^2\right]} \left\{ e^{\zeta\omega_n t} \left[(\omega_d - 1)\sin t - \zeta\omega_n \cos t \right] - (\omega_d - 1)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right.$$

$$\left. + \frac{1}{2\left[1+2\omega_d+\omega_n{}^2\right]} \left\{ e^{\zeta\omega_n t} \left[(\omega_d - 1)\sin t - \zeta\omega_n \cos t \right] + (\omega_d - 1)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right]$$

For $\tau > \pi,: \int_0^t f(\tau)h(t-\tau)d\tau = \int_0^\pi f(\tau)h(t-\tau)d\tau + \int_\pi^t (0)h(t-\tau)d\tau$

$$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\int_0^{\pi}\left(\sin\tau e^{\zeta\omega_n\tau}\sin\omega_d\left(t-\tau\right)\right)d\tau$$

$$=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n{}^2\right]}\left\{\begin{matrix} e^{\zeta\omega_n t}\left[\left(\omega_d-1\right)\sin\left[\omega_d\left(t-\pi\right)\right]-\zeta\omega_n\cos\left[\omega_d\left(t-\pi\right)\right]\right] \\ -\left(\omega_d-1\right)\sin\omega_d t-\zeta\omega_n\cos\omega_d t \end{matrix}\right\}\right.$$
$$\left.+\frac{1}{2\left[1+2\omega_d+\omega_n{}^2\right]}\left\{\begin{matrix} e^{\zeta\omega_n t}\left[\left(\omega_d+1\right)\sin\left[\omega_d\left(t-\tau\right)\right]+\zeta\omega\cos\left[\omega_d\left(t-\pi\right)\right]\right] \\ +\left(\omega_d-1\right)\sin\omega_d t-\zeta\omega_n\cos\omega_d t \end{matrix}\right\}\right]$$

Alternately, one could take a Laplace Transform approach and assume the under-damped system is a mass-spring-damper system of the form

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

The forcing function given can be written as

$$F(t)=F_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

Normalizing the equation of motion yields

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=f_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

where $f_0=\dfrac{F_0}{m}$ and m, c and k are such that $0<\zeta<1$.

Assuming initial conditions, transforming the equation of motion into the Laplace domain yields

$$X(s)=\frac{f_0\left(1+e^{-\pi s}\right)}{\left(s^2+1\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

The above expression can be converted to partial fractions

$$X(s)=f_0\left(1+e^{-\pi s}\right)\left(\frac{As+B}{s^2+1}\right)+f_0\left(1+e^{-\pi s}\right)\left(\frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

where A, B, C, and D are found to be

$$A = \frac{-2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$
$$B = \frac{\omega_n^2 - 1}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$
$$C = \frac{2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$
$$D = \frac{\left(1-\omega_n^2\right) + \left(2\zeta\omega_n\right)^2}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

Notice that $X(s)$ can be written more attractively as

$$\begin{aligned} X(s) &= f_0\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right) + f_0 e^{-\pi s}\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right) \\ &= f_0\left(G(s) + e^{-\pi s}G(s)\right) \end{aligned}$$

Performing the inverse Laplace Transform yields

$$x(t) = f_0\left(g(t) + H(t-\pi)g(t-\pi)\right)$$

where $g(t)$ is given below

$$g(t) = A\cos(t) + B\sin(t) + Ce^{-\zeta\omega_n t}\cos(\omega_d t) + \left(\frac{D - C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

ω_d is the damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Let m=1 kg, c=2 kg/sec, k=3 N/m, and F_0=2 N. The system is solved numerically. Both exact and numerical solutions are plotted below

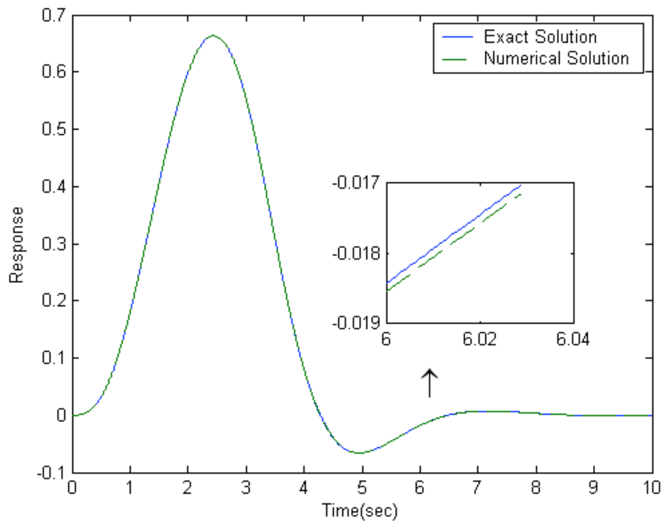


Figure 1 Analytical vs. Numerical Solutions

Below is the code used to solve this problem

```
% Establish a time vector
t=[0:0.001:10];

% Define the mass, spring stiffness and damping coefficient
m=1;
c=2;
k=3;

% Define the amplitude of the forcing function
F0=2;

% Calculate the natural frequency, damping ratio and normalized force amplitude
zeta=c/(2*sqrt(k*m));
wn=sqrt(k/m);
f0=F0/m;

% Calculate the damped natural frequency
wd=wn*sqrt(1-zeta^2);

% Below is the common denominator of A, B, C and D (partial fractions
% coefficients)
dummy=(1-wn^2)^2+(2*zeta*wn)^2;

% Hence, A, B, C, and D are given by
A=-2*zeta*wn/dummy;
B=(wn^2-1)/dummy;
C=2*zeta*wn/dummy;
```

```
D=((1-wn^2)+(2*zeta*wn)^2)/dummy;

% EXACT SOLUTION
%
***********************************************************************
*
%
***********************************************************************
*
for i=1:length(t)
   % Start by defining the function g(t)
   g(i)=A*cos(t(i))+B*sin(t(i))+C*exp(-zeta*wn*t(i))*cos(wd*t(i))+((D-
C*zeta*wn)/wd)*exp(-zeta*wn*t(i))*sin(wd*t(i));
   % Before t=pi, the response will be only g(t)
   if t(i)<pi
      xe(i)=f0*g(i);
      % d is the index of delay that will correspond to t=pi
      d=i;
   else
      % After t=pi, the response is g(t) plus a delayed g(t). The amount
      % of delay is pi seconds, and it is d increments
      xe(i)=f0*(g(i)+g(i-d));
   end;
end;

% NUMERICAL SOLUTION
%
***********************************************************************
*
%
***********************************************************************
*

% Start by defining the forcing function
for i=1:length(t)
   if t(i)<pi
      f(i)=f0*sin(t(i));
   else
      f(i)=0;
   end;
end;

% Define the transfer functions of the system
% This is given below
%        1
% ---------------------------
```

```
% s^2+2*zeta*wn+wn^2

% Define the numerator and denominator
num=[1];
den=[1 2*zeta*wn wn^2];
% Establish the transfer function
sys=tf(num,den);

% Obtain the solution using lsim
xn=lsim(sys,f,t);

% Plot the results
figure;
set(gcf,'Color','White');
plot(t,xe,t,xn,'--');
xlabel('Time(sec)');
ylabel('Response');
legend('Forcing Function','Exact Solution','Numerical Solution');
text(6,0.05,'\uparrow','FontSize',18);
axes('Position',[0.55 0.3/0.8 0.25 0.25])
plot(t(6001:6030),xe(6001:6030),t(6001:6030),xn(6001:6030),'--');
```

3.17 Speed bumps are used to force drivers to slow down. Figure P3.17 is a model of a car going over a speed bump. Using the data from Example 2.4.1 and an undamped model of the suspension system ($k = 4 \times 10^5$ N/m, $m = 1007$ kg), find an expression for the maximum relative deflection of the car's mass versus the velocity of the car. Model the bump as a half sine of length 40 cm and height 20 cm. Note that this is a moving base problem.

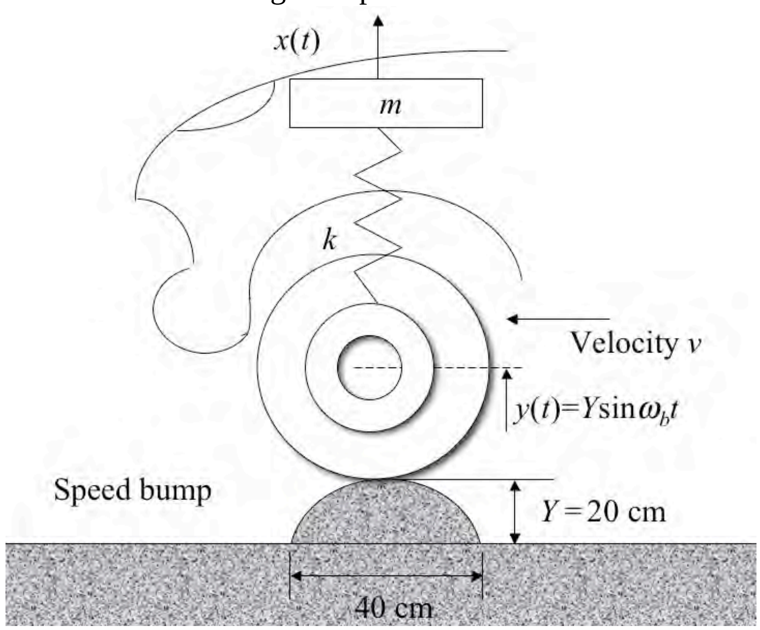


Figure P3.17 Model of a car driving over a speed bump.

Solution: This is a base motion problem, so the first step is to translate the equation of motion into a useable form. Summing forces yields in the vertical direction yields

$$m\ddot{x}(t) + k\big(x(t) - y(t)\big) = 0$$

were the displacement *y*(*t*) is prescribed. Next defined the relative displacement to be *z*(*t*) = x(*t*)-*y*(*t*), the relative motion between the car's wheel and body. The equation of motion becomes:

$$m\ddot{z}(t) + m\ddot{y}(t) + kz(t) = 0 \Rightarrow m\ddot{z}(t) + kz(t) = -m\ddot{y}(t)$$

Substitution of the form of *y*(*t*) into this last expression yields:

$$m\ddot{z}(t) + kz(t) = mY\omega_b^2 \sin\omega_b t\big(\Phi(t) - \Phi(t - t_1)\big)$$

where Φ is the Heavyside step function and ω_b is the frequency associated with the bump. The relationship between the bump frequency and the car's constant velocity is

$$\omega_b = \frac{2\pi}{2\ell}v = \frac{\pi}{\ell}v$$

where *v* is the speed of the car in m/s. For constant velocity, the time $t_1 = v\ell$, when the car finishes going over the bump.
Here, *z*(*t*) is From equation (3.13) with zero damping the solution is:

$$z(t) = \frac{1}{m\omega_n}\int_0^t f(t-\tau)\sin\omega_n\tau d\tau \qquad t < t_1$$

Substitution of *f*(*t*) =*y*(*t*) yields:

$$z(t) = \frac{Y\omega_b^2}{\omega_n}\int_0^t \sin(\omega_b t - \omega_b\tau)\sin\omega_n\tau d\tau =$$

$$= \frac{Y\omega_b^2}{\omega_n}\frac{1}{2}\left[\frac{\sin\big(\omega_b t - (\omega_n + \omega_b)\tau\big)}{-(\omega_n + \omega_b)} - \frac{\sin\big(\omega_b t + (\omega_n - \omega_b)\tau\big)}{\omega_n - \omega_b}\right]_0^t$$

$$= \frac{Y\omega_b^2}{\omega_n}\frac{1}{\omega_n^2 - \omega_b^2}\big(\omega_n\sin\omega_b t - \omega_b\sin\omega_n t\big) \qquad t < t_1$$

where the integral has been evaluated symbolically. Clearly a resonance situation prevails. Consider two cases, high speed $(\omega_b >> \omega_n)$ and low speed $((\omega_b << \omega_n))$ as when the two frequencies are near each other and obvious maximum occurs.
For high speed, the amplitude can be approximated as

$$\frac{Y\omega_b^2}{\omega_n}\frac{\omega_b}{\omega_n^2 - \omega_b^2}\Big((\omega_n/\omega_b)\sin\omega_b t - \sin\omega_n t\Big) \approx \frac{Y\omega_b^2}{\omega_n}\frac{\omega_b}{\omega_n^2 - \omega_b^2}\sin\omega_n t$$

For the values given, this has magnitude:

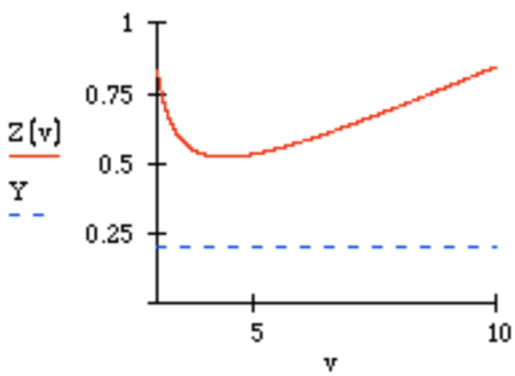
$$|Z(v)| \approx \left| \frac{Y\left(\frac{\pi}{\ell}\right)^3 v^3}{\omega_n\left(\omega_n^2 - \omega_b^2\right)} \right|$$

This increases with the cube of the velocity. Thus the faster the car is going the more sever the bump is (larger relative amplitude of vibration), hence serving to slow motorist down. A plot of magnitude versus speed shows bump size is amplified by the suspension system.

$k := 4 \cdot 10^5 \quad m := 1007 \quad \omega n := \sqrt{\frac{k}{m}} \quad \omega n = 19.93$

$L := 0.4 \quad Y := 0.2$

$$Z(v) := \frac{Y \cdot \left(\frac{\pi}{L}\right)^2 \cdot v^2}{\omega n} \cdot \left[\frac{\left(\frac{\pi}{L}\right) \cdot v}{\left| \omega n^2 - \left(\frac{\pi}{L}\right)^2 \cdot v^2 \right|} \right]$$

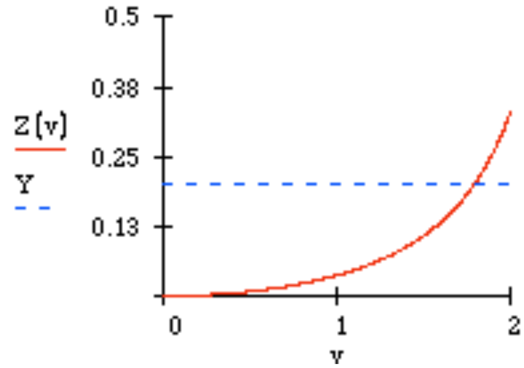


For slow speed, magnitude becomes

$$|Z(v)| \approx \left| \frac{Y\left(\frac{\pi}{\ell}\right)^2 v^2 \omega_n}{\omega_n\left(\omega_n^2 - \omega_b^2\right)} \right|$$

A plot of the approximate magnitude versus speed is given below

$$Z(v) := \frac{Y\cdot\left(\frac{\pi}{L}\right)^2\cdot v^2}{\omega n}\cdot\frac{\omega n}{\left|\omega n^2 - \left(\frac{\pi}{L}\right)^2\cdot v^2\right|}$$



Clearly at speeds above the designed velocity there is strong amplification of the bump's magnitude, causing discomfort to the driver and passengers, encouraging a slow speed when passing over the bumb.

3.18 Calculate and plot the response of an undamped system to a step function with a finite rise time of t_1 for the case m = 1 kg, k = 1 N/m, t_1 = 4 s and F_0 = 20 N. This function is described by

$$F(t) = \begin{cases} \dfrac{F_0 t}{t_1} & 0 \le t \le t_1 \\ F_0 & t > t_1 \end{cases}$$

Solution: Working in Mathcad to perform the integrals the solution is:

3.14

$m := 1 \quad k := 1 \quad F0 := 20 \quad \omega n := \sqrt{\frac{k}{m}}$

$$x1(t) := \frac{F0}{m \cdot \omega n \cdot 4} \cdot \left(\int_0^t \tau \cdot \sin(t-\tau)\, d\tau \right)$$

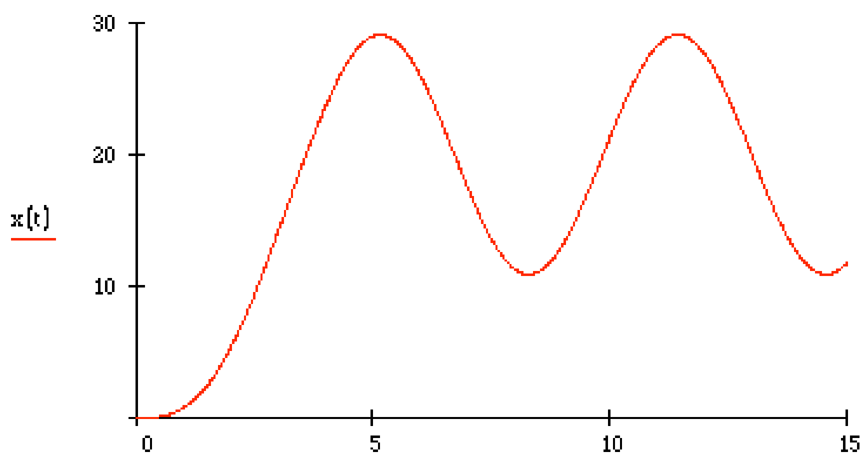
$$x1(t) := \frac{1}{4} \cdot \frac{F0}{(m \cdot \omega n)} \cdot (t - \sin(t))$$

$$x2(t) := \int_4^t \left[\frac{F0}{m \cdot \omega n} \cdot \left(1 - \frac{\tau}{4}\right) \cdot \sin(t-\tau) \right] d\tau$$

$$x2(t) := \frac{-1}{4} \cdot (t-4) \cdot \frac{F0}{(m \cdot \omega n)} + \frac{1}{4} \cdot \sin(t-4) \cdot \frac{F0}{(m \cdot \omega n)}$$

$$x(t) := x1(t) + x2(t) \cdot \Phi(t-4)$$

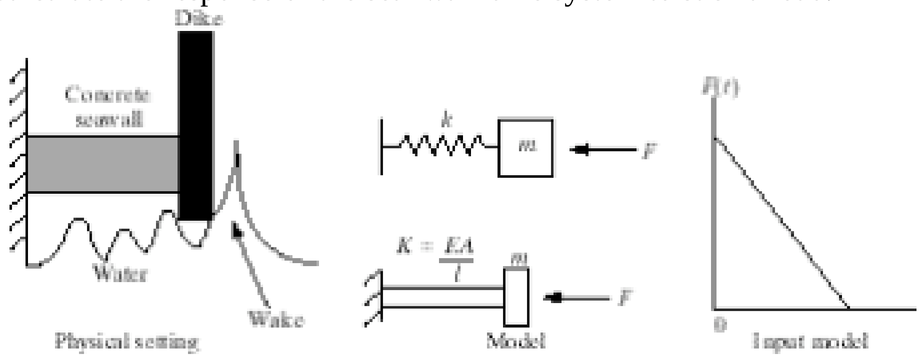
$$\frac{1}{4} \cdot \frac{F0}{(m \cdot \omega n)} = 5 \qquad \frac{F0}{(m \cdot \omega n)} = 20$$



3.19 A wave consisting of the wake from a passing boat impacts a seawall. It is desired to calculate the resulting vibration. Figure P3.19 illustrates the situation and suggests a model. This force in Figure P3.19 can be expressed as

$$F(t) = \begin{cases} F_0\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le t_0 \\ 0 & t > t_0 \end{cases}$$

Calculate the response of the seal wall-dike system to such a load.



Solution: From Equation (3.12): $x(t) = \int_0^t F(\tau)h(t-\tau)d\tau$

From Problem 3.18, $h(t-\tau) = \dfrac{1}{m\omega_n}\sin\omega_n(t-\tau)$ for an undamped system

For $t < t_0$:

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^t F_0\left(1-\frac{\tau}{t_0}\right)\sin\omega_n(t-\tau)d\tau\right]$$

$$x(t) = \frac{F_0}{m\omega_n}\left[\int_0^t \sin\omega_n(t-\tau)d\tau - \frac{1}{t_0}\int_0^t \tau\sin\omega_n(t-\tau)d\tau\right]$$

After integrating and rearranging,

$$x(t) = \frac{F_0}{kt_0}\left[\frac{1}{\omega_n}\sin\omega_n t - t\right] + \frac{F_0}{k}\left[1-\cos\omega_n t\right] \quad t < t_0$$

For $t > t_0$: $\int_0^t f(\tau)h(t-\tau)d\tau = \int_0^{t_0} f(\tau)h(t-\tau)d\tau + \int_{t_0}^t (0)h(t-\tau)d\tau$

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{t_0} F_0\left(1-\frac{\tau}{t_0}\right)\sin\omega_n(t-\tau)d\tau\right]$$

$$x(t) = \frac{F_0}{m\omega_n}\left[\int_0^{t_0} \sin\omega_n(t-\tau)d\tau - \frac{1}{t_0}\int_0^{t_0} \tau\sin\omega_n(t-\tau)d\tau\right]$$

After integrating and rearranging,

$$x(t)=\frac{F_0}{kt_0\omega_n}\left[\sin\omega_n t-\sin\omega_n(t-t_0)\right]-\frac{F_0}{k}\left[\cos\omega_n t\right]\quad t>t_0$$

3.20 Determine the response of an undamped system to a ramp input of the form $F(t) = F_0 t$, where F_0 is a constant. Plot the response for three periods for the case $m = 1$ kg, $k = 100$ N/m and $F_0 = 50$ N.

Solution: From Eq. (3.12): $x(t)=\int_0^t F(\tau)h(t-\tau)d\tau$

From Problem 3.8, $h(t-\tau)=\frac{1}{m\omega_n}\sin\omega_n(t-\tau)$ for an undamped system.

Therefore,

$$x(t)=\frac{1}{m\omega_n}\left[\int_0^t (F_0\tau)\sin\omega_n(t-\tau)d\tau\right]=\frac{F_0}{m\omega_n}\int_0^t \tau\sin\omega_n(t-\tau)d\tau$$

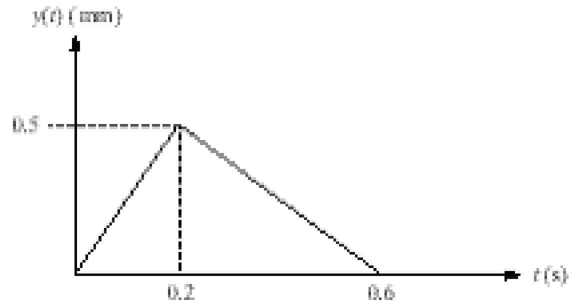
After integrating and rearranging,

$$x(t)=\frac{F_0}{m\omega_n}\left[\frac{\tau}{\omega_n}-\frac{1}{\omega_n^2}\sin\omega_n\tau\right]=\frac{F_0}{k}t-\frac{F_0}{k\omega_n}\sin\omega_n t$$

Using the values $m = 1$ kg, $k = 100$ kg, and $F_0 = 50$ N yields

$$x(t)=0.5t-.05\sin(10t)\ \text{ m}$$

3. 21 A machine resting on an elastic support can be modeled as a single-degree-of-freedom, spring-mass system arranged in the vertical direction. The ground is subject to a motion *y*(*t*) of the form illustrated in Figure P3.221. The machine has a mass of 5000 kg and the support has stiffness 1.5x10^3 N/m. Calculate the resulting vibration of the machine.



Solution: Given m = 5000 kg, k = 1.5x10^3 N/m, $\omega_n = \sqrt{k/m} = 0.548$ rad/s and that the ground motion is given by:

$$y(t) = \begin{cases} 2.5t & 0 \le t \le 0.2 \\ 0.75 - 1.25t & 0.2 \le t \le 0.6 \\ 0 & t \ge 0.6 \end{cases}$$

The equation of motion is $m\ddot{x} + k(x-y) = 0$ or $m\ddot{x} + kx = ky = F(t)$ The impulse response function computed from equation (3.12) for an undamped system is

$$h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n(t-\tau)$$

This gives the solution by integrating a *yh* across each time step:

$$x(t) = \frac{1}{m\omega_n}\int_0^t ky(\tau)\sin\omega_n(t-\tau)d\tau = \omega_n\int_0^t y(\tau)\sin\omega_n(t-\tau)d\tau$$

For the interval 0≤ t ≤ 0.2:

$$x(t) = \omega_n\int_0^t 2.5\tau\sin\omega_n(t-\tau)d\tau$$

$$\Rightarrow \underline{x(t) = 2.5t - 4.56\sin 0.548t \text{ mm } 0 \le t \le 0.2}$$

For the interval 0.2≤ t ≤ 0.6:

$$x(t) = \omega_n\int_0^{0.2} 2.5\tau\sin\omega_n(t-\tau)d\tau + \omega_n\int_{0.2}^t (0.75 - 1.25\tau)\sin\omega_n(t-\tau)d\tau$$

$$= 0.75 - 0.5\cos 0.548(t-0.2) - 1.25t + 2.28\sin 0.548(t-0.2)$$

Combining this with the solution from the first interval yields:

$$\underline{x(t) = 0.75 + 1.25t - 0.5\cos 0.548(t-0.2)}$$

$$\underline{+6.48\sin 0.548(t-0.2) - 4.56\sin 0.548(t-0.2) \text{ mm } 0.2 \le t \le 0.6}$$

Finally for the interval t ≥0.6:

$$x(t) = \omega_n \int_0^{0.2} 2.5t \sin \omega_n (t-\tau) d\tau + \omega_n \int_{0.2}^{0.6} (0.75 - 1.25t) \sin \omega_n (t-\tau) d\tau + \omega_n \int_0^t (0) \sin \omega_n (t-\tau) d\tau$$
$$= -0.5 \cos 0.548(t-0.2) - 2.28 \sin 0.548(t-0.6) + 2.28 \sin 0.548(t-0.2)$$

Combining this with the total solution from the previous time interval yields:

$$x(t) = -0.5 \cos 0.548(t-0.2) + 6.84 \sin 0.548(t-0.2) - 2.28 \sin 0.548(t-0.6)$$
$$- 4.56 \sin 0.548t \quad \text{mm } t \geq 0.6$$

3.22 Consider the step response described in Figure 3.7. Calculate t_p by noting that it occurs at the first peak, or critical point, of the curve.

Solution: Assume $t_0 = 0$. The response is given by Eq. (3.17):

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \phi)$$

To find t_p, compute the derivative and let $\dot{x}(t) = 0$

$$\dot{x}(t) = \frac{-F_0}{k\sqrt{1-\zeta^2}} \left[-\zeta\omega_n e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) + e^{-\zeta\omega_n t}(-\omega_d)\sin(\omega_d t - \phi) \right] = 0$$

$$\Rightarrow -\zeta\omega_n \cos(\omega_d t - \phi) - \omega_d \sin(\omega_d t - \phi) = 0$$

$$\Rightarrow \tan(\omega_d t - \phi) = \frac{-\zeta\omega_n}{\omega_d}$$

$\omega_d t - \phi - \pi = \tan^{-1}\left(\frac{-\zeta\omega_n}{\omega_d}\right)$ (π can be added or subtracted without changing the tangent of an angle)

$$t = \frac{1}{\omega_d}\left[\pi + \phi + \tan^{-1}\left(\frac{-\zeta\omega_n}{\omega_d}\right) \right]$$

But, $\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$

So,

$$t = \frac{1}{\omega_d}\left[\pi + \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \right]$$

$$t_p = \frac{\pi}{\omega_d}$$

3.23 Calculate the value of the overshoot (o.s.), for the system of Figure P3.7.

Solution:

The overshoot occurs at $t_p = \dfrac{\pi}{\omega_d}$

Substitute into Eq. (3.17):

$$x\left(t_p\right) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}}e^{-\zeta\omega_n\pi/\omega_d}\cos\left[\omega_d\left(\frac{\pi}{\omega_d}\right)-\theta\right]$$

The overshoot is

$$o.s. = x\left(t_p\right) - x_{ss}\left(t\right)$$

$$o.s. = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}}e^{-\zeta\omega_n\pi/\omega_d}\left(-\cos\theta\right) - \frac{F_0}{k}$$

Since $\theta = \tan^{-1}\left(\dfrac{\zeta}{\sqrt{1-\zeta^2}}\right)$, then $\cos\theta = \sqrt{1\text{-}\zeta^2}$

$$o.s. = -\frac{F_0}{k\sqrt{1-\zeta^2}}\left(e^{-\zeta\omega_n\pi/\omega_d}\right)\left(\sqrt{1-\zeta^2}\right)$$

$$o.s. = \frac{F_0}{k}e^{-\zeta\omega_n\pi/\omega_d}$$

3.24 It is desired to design a system so that its step response has a settling time of 3 s and a time to peak of 1 s. Calculate the appropriate natural frequency and damping ratio to use in the design.

Solution:

Given $t_s = 3\text{s}, t_p = 1\text{s}$

Settling time:

$$t_s = \frac{3.5}{\zeta\omega_n} = 3\text{ s} \;\Rightarrow \zeta\omega_n = \frac{3.5}{3} = 1.1667\text{ rad/s}$$

Peak time:

$$t_p = \frac{\pi}{\omega_d} = 1\text{ s} \qquad \Rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2} = \pi\text{ rad/s}$$

$$\Rightarrow \omega_n\sqrt{1-\left(\frac{1.1667}{\omega_n}\right)^2} = \pi \Rightarrow \omega_n{}^2\left[1-\left(\frac{1.1667}{\omega_n}\right)^2\right] = \pi^2$$

$$\Rightarrow \omega_n^2\left[1-\frac{1.3611}{\omega_n^2}\right] = \pi^2 \Rightarrow \omega_n^2 - 1.311 = \pi^2 \Rightarrow \underline{\omega_n = 3.35\text{ rad/s}}$$

Next use the settling time relationship to get the damping ratio:

$$\zeta = \frac{1.1667}{\omega_n} = \frac{1.1667}{3.35} \Rightarrow \underline{\zeta = \mathbf{0.3483}}$$

3.25 Plot the response of a spring-mass-damper system for this input of Figure 3.8 for the case that the pulse width is the natural period of the system (i.e., $t_1 = \pi/\omega_n$).

Solution:

The values from Figure 3.7 will be used to plot the response.

$$F_0 = 30 \text{ N}$$
$$k = 1000 \text{ N/m}$$
$$\zeta = 0.1$$
$$\omega = 3.16 \text{ rad/s}$$

From example 3.2.2 and Figure 3.7, with $t_1 = \dfrac{\pi}{\omega}$ we have for $t = 0$ to t_1,

$$x(t) = \frac{F_0}{k} - \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi) \qquad \text{where } \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$x(t) = .03 - .03015e^{-.316t}\ \cos(3.144t - .1002) \qquad 0 < t \leq t_1$

For $t > t_1$,

$$x(t) = \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\left\{e^{\zeta\omega_n t_1}\cos\left[\omega_d\left(t - \frac{\pi}{\omega_n}\right) - \phi\right] - \cos(\omega_d t - \phi)\right\}$$

$x(t) = 0.0315e^{-.316t}\ \{1.3691\cos(3.144t - 3.026) - \cos(3.144t - .1002)\}\ t > t_1$

The plot in Mathcad follows:

$\omega := 3.144$

$x(t) := 0.03 - 0.301 \cdot e^{-.316 \cdot t} \cdot \cos(\omega \cdot t - .1002) + \left[(0.0315) \cdot e^{-.316 \cdot t} \cdot \left(1.3691 \cdot \cos(\omega \cdot t - 3.226) - \cos(\omega \cdot t - .1002)\right)\right] \cdot \Phi\left(t - \frac{\pi}{\omega}\right)$