Problems and Solutions Section 3.3 (problems 3.26-3.32)

3.26 Derive equations (3.24). (3.25) and (3.26) and hence verify the equations for the Fourier coefficient given by equations (3.21), (3.22) and (3.23).

Solution: For $n \neq m$, integration yields:

$$\int_0^T \sin n\omega_T t \sin m\omega_T t dt = \left[\frac{\sin(n-m)\omega_T t}{\omega_T 2(n-m)} - \frac{\sin(n+m)\omega_T t}{\omega_T 2(n+m)}\right]_0^T$$

$$= \frac{\sin\left[(n-m)\left(\frac{2\pi}{T}\right)T\right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)\left(\frac{2\pi}{T}\right)T\right]}{2(n+m)\omega_T}$$

$$= \frac{\sin\left[(n-m)(2\pi)\right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)(2\pi)\right]}{2(n+m)\omega_T} = 0$$

Since m and n are integers, the sine terms are 0, so this is equal to 0.

Equation (3.24), for $m = n$:

$$\int_0^T \sin^2 n\omega_T t dt = \left[\frac{1}{2}t - \frac{1}{4n\omega_T}\sin(2n\omega_T t)\right]_0^T = \frac{T}{2} - \frac{T}{8n\pi}\sin\left[2\pi\left(\frac{2\pi}{T}\right)T\right]$$

$$= \frac{T}{2} - \frac{T}{8n\pi}\sin\left[4n\pi\right] = \frac{T}{2}$$

Since n is an integer, the sine term is 0, so this is equal to $T/2$.

So, $\displaystyle\int_0^T \sin n\omega_T t \sin m\omega_T t dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$

Equation (3.25), for $m \neq n$

$$\int_0^T \cos n\omega_T t \cos m\omega_T t dt = \left[\frac{\sin(n-m)\omega_T t}{2(n-m)\omega_T} - \frac{\sin(n+m)\omega_T t}{2(\eta+m)\omega_T} \right]\Bigg|_0^T$$

$$= \frac{\sin\left[(n-m)\left(\frac{2\pi}{T}\right)T\right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)\left(\frac{2\pi}{T}\right)T\right]}{2(n+m)\omega_T}$$

$$= \frac{\sin\left[(n-m)(2\pi)\right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)(2\pi)\right]}{2(n+m)\omega_T} = 0$$

Since *m* and *n* are integers, the sine terms are 0, so this is equal to 0.

Equation (3.25), for $m = n$ becomes:

$$\int_0^T \cos^2 n\omega_T t dt = \left[\frac{1}{2}t + \frac{1}{4n\omega_T}\sin(2n\omega_T t) \right]\Bigg|_0^T = \frac{T}{2} + \frac{T}{8n\pi}\sin\left[2n\left(\frac{2\pi}{T}\right)T\right]$$

$$= \frac{T}{2} + \frac{T}{8n\pi}\sin\left[4n\pi\right] = \frac{T}{2}$$

Since *n* is an integer, the sine term is 0, so this is equal to $T/2$.

So, $\int_0^T \cos n\omega_T t \cos m\omega_T t dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$

Equation (3.26), for $m \neq n$:

$$\int_0^T \cos n\omega_T t \sin m\omega_T t dt = \left[\frac{\cos(n-m)\omega_T t}{2\omega_T(n-m)} - \frac{\cos(n+m)\omega_T t}{2\omega_T(n+m)} \right]\Bigg|_0^T$$

$$= \frac{\cos\left[(n-m)\left(\frac{2\pi}{T}\right)T\right]}{2(n-m)\omega_T} - \frac{\cos\left[(n+m)\left(\frac{2\pi}{T}\right)T\right]}{2(n+m)\omega_T} - \frac{1}{2(m-n)\omega_T} + \frac{1}{2(m+n)\omega_T}$$

$$= \frac{\cos\left[(n-m)(2\pi)\right]}{2(n-m)\omega_T} - \frac{\cos\left[(n+m)(2\pi)\right]}{2(n+m)\omega_T} - \frac{1}{2(m-n)\omega_T} + \frac{1}{2(m+n)\omega_T} = 0$$

Since *n* is an integer, the cosine term is 1, so this is equal to 0.

So, $\int_0^T \cos n\omega_T t \sin m\omega_T t dt = 0$

Equation (3.26) for $n = m$ becomes:

$$\int_0^T \cos n\omega_T t \sin n\omega_T t dt = \left[\frac{1}{2n\omega_T} \sin^2 n\omega_T t \right]_0^T = \frac{T}{4n\pi} \sin^2 2\pi n = 0$$

$$\text{Thus } \int_0^T \cos n\omega_T t \sin n\omega_T t dt = 0$$

3.27 Calculate b_n from Example 3.3.1 and show that $b_n = 0$, $n = 1,2,\ldots,\infty$ for the triangular force of Figure 3.12. Also verify the expression a_n by completing the integration indicated. (*Hint*: Change the variable of integration from t to $x = 2\pi nt/T$.)

Solution: From Equation (3.23), $b_n = \frac{2}{T}\int_0^T F(t)\sin n\omega_T t dt$. Computing the integral yields:

$$b_n = \frac{2}{T}\left[\int_0^{T/2}\left(\frac{4}{T}t-1\right)\sin n\omega_T tdt + \int_{T/2}^{T}\left[1-\frac{4}{T}\left(t-\frac{T}{2}\right)\right]\sin n\omega_T tdt\right]$$

$$b_n = \frac{2}{T}\left[\frac{4}{T}\int_0^{T/2} t\sin n\omega_T tdt - \int_0^{T/2}\sin n\omega_T tdt + 3\int_{T/2}^{T}\sin n\omega_T tdt - \frac{4}{T}\int_{T/2}^{T} t\sin n\omega_T tdt\right]$$

Substitute $x = n\omega_T t = \frac{2\pi n}{T}t$

$$b_n = \frac{1}{\pi n}\left[\frac{2}{\pi n}\int_0^{\pi n} x\sin xdx - \int_0^{\pi n}\sin xdx + 3\int_{\pi n}^{2\pi n}\sin xdx - \frac{2}{\pi n}\int_{\pi n}^{2\pi n} x\sin xdx\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}\left(\sin x - x\cos x\right)\Big|_0^{\pi n} + \cos x\Big|_0^{\pi n} - 3\cos x\Big|_{\pi n}^{2\pi n} - \frac{2}{\pi n}\left(\sin x - x\cos x\right)\Big|_{\pi n}^{2\pi n}\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}\left(-\pi n\cos\pi n\right) + \cos\pi n - 1 - 3 + 3\cos\pi n - \frac{2}{\pi n}\left(-2\pi n + \pi n\cos\pi n\right)\right]$$

$$= \frac{1}{\pi n}\left[-2\cos\pi n + 4\cos\pi n - 4 + 4 - 2\cos\pi n\right] = \frac{1}{\pi n}\left[0\right] = 0$$

From equation (3.22), $a_n = \frac{2}{T}\int_0^T F(t)\cos n\omega_T tdt$

$$a_n = \frac{2}{T}\left[\int_0^{T/2}\left(\frac{4}{T}t-1\right)\cos n\omega_T tdt + \int_{T/2}^{T}\left[1-\frac{4}{T}\left(t-\frac{T}{2}\right)\right]\cos n\omega_T tdt\right]$$

$$a_n = \frac{2}{T}\left[\frac{4}{T}\int_0^{T/2} t\cos n\omega_T tdt - \int_0^{T/2}\cos n\omega_T tdt + 3\int_{T/2}^{T}\cos n\omega_T tdt - \frac{4}{T}\int_{T/2}^{T} t\cos n\omega_T tdt\right]$$

Substitute $x = n\omega_T t = \frac{2\pi n}{T}t$

$$a_n = \frac{1}{\pi n}\left[\frac{2}{\pi n}\int_0^{\pi n} x\cos x dx - \int_0^{\pi n} \cos x dx + 3\int_{\pi n}^{2\pi n} \cos x dx - \frac{2}{\pi n}\int_{\pi n}^{2\pi n} x\cos x dx\right]$$

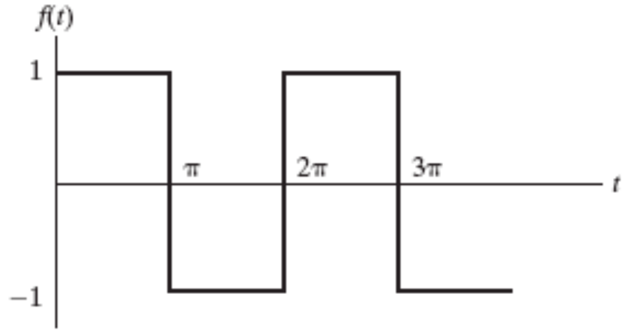
$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}\left(\cos x + x\sin x\right)\Big| - \sin x\Big|_0^{\pi n} + 3\sin x\Big|_{\pi n}^{2\pi n} - \frac{2}{\pi n}\left(\cos x - \sin x\right)\Big|_{\pi n}^{2\pi n}\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}\left(\cos\pi n - 1\right) - \frac{2}{\pi n}\left(1 - \cos\pi n\right)\right]$$

$$= \frac{2}{\pi^2 n^2}\left[\cos\pi n - 1 - 1 + \cos\pi n\right]$$

$$= \frac{4}{\pi^2 n^2}\left[\cos\pi n - 1\right] = \begin{cases} 0 & n \text{ even} \\ \dfrac{-8}{\pi^2 n^2} & n \text{ odd} \end{cases}$$

3.28 Determine the Fourier series for the rectangular wave illustrated in Figure P3.28.



Solution: The square wave of period T is described by

$$F(t)=\begin{cases}1 & 0\le t\le \pi\\ -1 & \pi\le t\le 2\pi\end{cases}$$

Determine the coefficients a_0, a_n, b_n from direct integration:

$$a_0=\frac{2}{T}\int_0^T F(t)\,dt$$
$$=\frac{2}{2\pi}\left[\int_0^{\pi}(1)\,dt+\int_{\pi}^{2\pi}(-1)\,dt\right]$$
$$=\frac{1}{\pi}\left[t\Big|_0^{\pi}-t\Big|_{\pi}^{2\pi}\,dt\right]$$
$$=\frac{1}{\pi}\left[\pi-2\pi+\pi\right]=\frac{1}{\pi}(0)\qquad\Rightarrow a_0=0$$

$$a_n=\frac{2}{T}\int_0^T F(t)\cos n\omega_T t\,dt,\text{ where }\omega_T=\frac{2\pi}{T}=\frac{2\pi}{2\pi}=1$$
$$=\frac{2}{2\pi}\left[\int_0^{\pi}\cos nt\,dt-\int_{\pi}^{2\pi}\cos nt\,dt\right]=\frac{1}{\pi}\left[\frac{1}{n}\sin nt\Big|_0^{\pi}-\frac{1}{n}\sin nt\Big|_{\pi}^{2\pi}\right]$$
$$=\frac{1}{\pi n}\left[\sin(n\pi)-\sin(n2\pi)+\sin(n\pi)\right]=0$$

$$b_n=\frac{2}{T}\int_0^T F(t)\sin\omega_T t\,dt=\frac{2}{2\pi}\left[\int_0^{\pi}\sin nt\,dt-\int_{\pi}^{2\pi}\sin nt\,dt\right]$$
$$=\frac{1}{\pi}\left[\frac{-1}{n}\cos nt\Big|_0^{\pi}-\frac{1}{n}\cos nt\Big|_{\pi}^{2\pi}\right]=\frac{1}{\pi n}\left[-\cos n\pi+1-1-\cos n\pi\right]=\frac{2}{\pi n}\left[1-\cos n\pi\right]$$

If n is even, $\cos n\pi = 1$. If n is odd, $\cos n\pi = -1$

So, $b_n=\begin{cases}0 & n\text{ even}\\ \dfrac{4}{\pi \mathrm{n}} & n\text{ odd}\end{cases}$

Thus the Fourier Series collapses to a sine series of the form

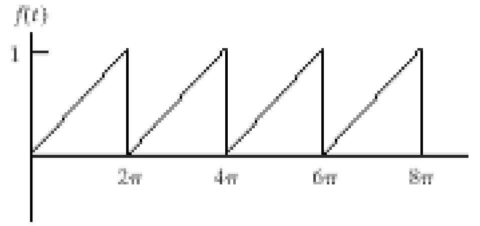
$$F(t) = \sum_{n=1}^{\infty} b_n \sin nt = \sum_{n=1,3,\cdots}^{\infty} \frac{4}{n\pi} \sin nt$$

The Vibration Toolbox can also be used:

```
t=0:pi/100:2*pi-pi/100;
f=-2*floor(t/pi)+1;
vtb3_3(f',t',100)
[a,b]=vtb3_3(f',t',100)
```

Note that vtb3_3 always gives some error on the order of delta t (.01 in this case). Using a smaller delta t reduced the error.

3.29 Determine the Fourier series representation of the sawtooth curve illustrated in Figure P3.29.



Solution: The sawtooth curve of period T is

$$F(t) = \frac{1}{2\pi}t \qquad 0 \le t \le 2\pi$$

Determine coefficients a_0, a_n, b_n :

$$a_0 = \frac{2}{T}\int_0^T F(t)\,dt = \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)dt = \left(\frac{1}{2\pi^2}\right)\frac{1}{2}t^2\Big|_0^{2\pi}$$
$$= \frac{1}{4\pi^2}\left[4\pi^2 - 0\right] = 1$$

$$a_n = \frac{2}{T}\int_0^T F(t)\cos n\omega_T t\,dt,\ \text{where } \omega_T = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$
$$= \frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\cos nt\,dt\right] = \frac{1}{2\pi^2}\left[\int_0^{2\pi} t\cos nt\,dt\right]$$
$$= \frac{1}{2\pi^2}\left[\frac{1}{n^2}\cos nt + \frac{1}{n}t\sin nt\right]\Bigg|_0^{2\pi} = \frac{1}{2\pi^2}\left[\frac{1}{n^2}(1-1) + \frac{1}{n}(0-0)\right] = 0$$

$$b_n = \frac{2}{T}\int_0^T F(t)\sin n\omega_T t\,dt = \frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\sin nt\,dt\right] = \frac{1}{2\pi^2}\left[\int_0^{2\pi} t\sin nt\,dt\right]$$
$$= \frac{1}{2\pi^2}\left[\frac{1}{n^2}\sin nt - \frac{1}{n}t\cos nt\right]\Bigg|_0^{2\pi} = \frac{1}{2\pi^2}\left[\frac{1}{n^2}(0-0) - \frac{1}{n}(2\pi - 0)\right]$$
$$= \frac{1}{2\pi^2}\left(\frac{-2\pi}{n}\right) = \frac{-1}{\pi n}$$

Fourier Series

$$F(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-1}{\pi n}\right)\sin nt$$
$$F(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin nt$$

3.30 Calculate and plot the response of the base excitation problem with base motion specified by the velocity

$$\dot{y}(t) = 3e^{-t/2}\Phi(t) \text{ m/s}$$

where $\Phi(t)$ is the unit step function and m = 10 kg, ζ = 0.01, and k = 1000 N/m. Assume that the initial conditions are both zero.

Solution: Given:

$$\dot{y}(t) = 3e^{-t/2}\mu(t) \text{ m/s}$$
$$m = 10 \text{ kg}, \ \zeta = 0.01, \ k = 1000 \text{ N/m}$$
$$x(0) = \dot{x}(0) = 0$$

From Equation (2.61):

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$
$$m\ddot{x} + c\ddot{x} + kx = c\dot{y} + ky$$

Integrate by parts to find $y(t)$:

$$y(t) = \int \ \dot{y}(t)dt = 3e^{-t/2}\mu(t)dt$$

Let

$$u = \mu(t) \qquad dv = 3e^{-t/2}dt$$
$$du = \delta(t)dt \qquad v = -6e^{-t/2}$$

When

$$t > 0, \mu(t) = 1, \ \text{so} \ \ y(t) = 6\left(1 - e^{-1/2}\right)$$

So, $m\ddot{x} + c\dot{x} + kx = c\left(3e^{-t/2}\right) + 6k\left(1 - e^{t/2}\right)$

Since $c = 2\zeta\sqrt{km} = 2$ kg/s,

$$10\ddot{x} + 2\dot{x} + 1000x = 6000 - 5994e^{-t/2}$$

The solution is given by equation (3.13):

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right] d\tau$$

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\omega_\mathrm{d} = \omega_n \sqrt{1-\zeta^2} = 10 \text{ rad/s}$$

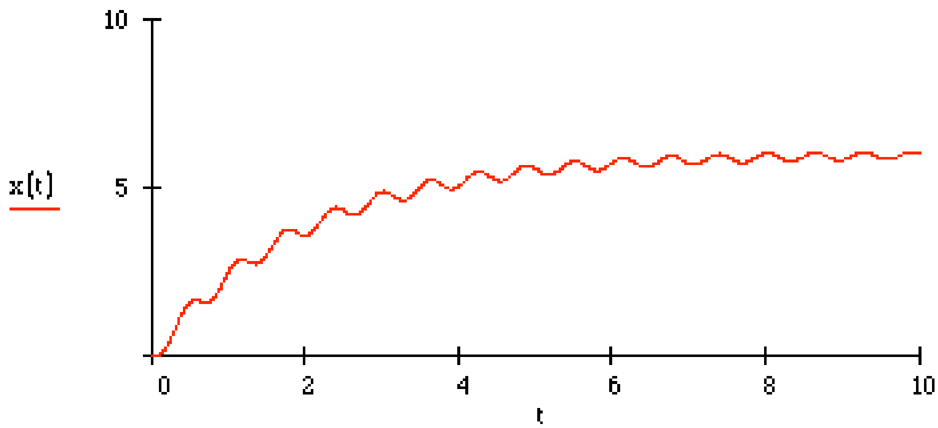
$$F(t) = 6000 - 5994 e^{-t/2}$$

$$x(t) = \frac{1}{100} e^{-0.1t} \int_0^t \left[\left(6000 - 5994 e^{-\tau/2}\right) e^{0.1\tau} \sin\left(10(t-\tau)\right) \right] d\tau$$

$$x(t) = 60 e^{-0.1t} \left\{ \int_0^t e^{0.1t} \sin\left[10(t-\tau)\right] d\tau - \int_0^t e^{-0.4t} \sin\left[10(t-\tau)\right] d\tau \right\}$$

After integrating and rearranging

$$x(t) = 6 - 5.979 e^{-t/2} - 0.0295 \cos 10t - 0.2990 \sin 10t \text{ m}$$



3.31 Calculate and plot the total response of the spring-mass-damper system of Figure 2.1 with m = 100 kg, ζ = 0.1 and k = 1000 N/m to the signal of Figure 3.12, with maximum force of 1 N. Assume that the initial conditions are zero and let $T = 2\pi$ s.

Solution: Given:

$$m = 100 \text{ kg}, \ k = 1000 \text{ N/m}, \zeta = 0.1, T = 2\pi \text{ s}, \ F_{\max} = 1N,$$

$$x(0) = \dot{x}(0) = 0, \quad \omega_n = \sqrt{\frac{k}{m}} = 3.16 \text{ rad/s}, \ \omega_d = \omega\sqrt{1-\zeta^2} = 3.15 \text{ rad/s}, \quad \omega_T = \frac{2\pi}{T} = 1 \text{ rad/s}$$

From example 3.3.1 and Figure 3.10,

$$F(t)\sum_{n=1}^{\infty} a_n \cos nt, \qquad a_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{-8}{\pi^2 n^2} & n \text{ odd} \end{cases}$$

So, $m\ddot{x} + c\dot{x} + kx = \sum_{n=1}^{\infty} a_n \cos nt \qquad (n \text{ odd})$

The total solution is

$$x(t) = x_h(t) + \sum_{n=1}^{\infty} x_{cn}(t) \qquad (n \text{ odd})$$

From equation (3.33),

$$x_{cn}(t) = \frac{a_n / m}{\left[\left[\omega_n^2 - (n\omega_T)^2\right]^2 + \left[2\zeta\omega_n n\omega_T\right]^2\right]^{1/2}} \cos(n\omega_T t - \phi_n)$$

$$\phi_n = \tan^{-1}\left(\frac{2\zeta\omega_n n\omega_T}{\omega_n^2 - n^2\omega_T^2}\right) = \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)$$

$$x_{cn}(t) = \frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}} \cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]$$

So,

$$x(t) = Ae^{\zeta\omega_n t}\sin(\omega_d t - \theta) + \sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \quad (n \text{ odd})$$

$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t - \theta)$$

$$+\omega_d Ae^{-\zeta\omega_n t}\cos(\omega_d t - \theta) + \sum_{n=1}^{\infty}\left[\frac{0.00811}{n\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\sin nt - \tan^{-1}\frac{0.6325n}{10 - n^2}\right] \quad (n \text{ odd})$$

$$x(0) = 0 = -A\sin\theta + \sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \quad (n \text{ odd})$$

$$0 = -A\sin\theta - 0.00110$$

$$\dot{x}(0) = 0 = \zeta\omega_n A\sin\theta + \omega_d A\cos\theta$$

$$+\sum_{n=1}^{\infty}\left[\frac{-0.000569}{\left[n^4 - 19.6n^2 + 100\right]^{1/2}\left[0.00493n^2 + 1\right]}\right] \quad (n \text{ odd})$$

$$0 = \zeta\omega_n A\sin\theta + \omega_d A\cos\theta - 0.001186$$

So A = 0.00117 m and θ = - 1.232 rad.

The total solution is:

$$x(t) = 0.00117e^{-0.316t}\sin(3.15t + 1.23)$$

$$+\sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \text{ m} \quad (n \text{ odd})$$

3.32 Calculate the total response of the system of Example 3.3.2 for the case of a base motion driving frequency of ω_b = 3.162 rad/s.

Solution: Let ω_b = 3.162 rad/s. From Example 3.3.2,

$$F(t) = cY\omega_b \cos\omega_b t + kY\sin\omega_b t = 1.581\cos(3.162t) + 50\sin(3.162t)$$

Also,

$$\omega_n = \sqrt{\frac{k}{m}} = 31.62 \text{ rad/s and } \zeta = \frac{c}{2\sqrt{km}} = 0.158$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 31.22 \text{ rad/s}$$

The solution is

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \theta) + \omega_n Y\left[\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}\right]^{1/2}\cos(\omega_b t - \phi_1 - \phi_2)$$
$$x(t) = Ae^{-5t}\sin(31.22t + \theta) + 0.0505\cos(3.162t - \phi_1 - \phi_2)$$
$$\phi_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right) = 0.0319 \text{ rad}$$
$$\phi_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta\omega_b}\right) = 1.54 \text{ rad}$$

So,

$$x(t) = Ae^{-5t}\sin(31.22t + \theta) + 0.0505\cos(3.162t - 1.57)$$
$$\dot{x}(t) = -5Ae^{-5t}\sin(31.22t + \theta) + 31.22Ae^{-5t}\cos(31.22t + \theta) - 0.16\sin(3.162t - 1.57)$$
$$\Rightarrow x(0) = 0.01 = A\sin\theta + 0.0505(0)$$
$$\Rightarrow \dot{x}(0) = 3 - 5A\sin\theta + 31.22A\cos\theta + 0.16(1)$$

So, $A = 0.0932$ m and $\theta = 0.107$ rad

The total solution is

$$x(t) = 0.0932e^{-5t}\sin(31.22t + 0.107) + 0.0505\cos(3.162t - 1.57) \text{ m}$$