Problems and Solutions for Section 3.4 (3.35 through 3.38)

3.35 Calculate the response of

$$m\ddot{x} + c\dot{x} + kx = F_0\Phi(t)$$

where $\Phi(t)$ is the unit step function for the case with $x_0 = v_0 = 0$. Use the Laplace transform method and assume that the system is underdamped.

Solution:

Given:

$$m\ddot{x} + c\dot{x} + kx = F_0\mu(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\mu(t) \qquad (\zeta < 1)$$

Take Laplace Transform:

$$s^2X(s) + 2\zeta\omega_n sX(s) + \omega_n^2X(s) = \frac{F_0}{m}\left(\frac{1}{s}\right)$$

$$X(s) = \frac{F_0 / m}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)s} = \left(\frac{F_0}{m\omega_n^2}\right)\frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

Using inverse Laplace tables,

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \cos^{-1}(\zeta)\right)$$

3.36 Using the Laplace transform method, calculate the response of the system of Example 3.4.4 for the overdamped case ($\zeta > 1$). Plot the response for $m = 1$ kg, $k = 100$ N/m, and $\zeta = 1.5$.

Solution:

From example 3.4.4,

$$m\ddot{x} + c\dot{x} + kx = \delta(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}\delta(t) \qquad (\zeta > 1)$$

Take Laplace Transform:

$$s^2 X(s) + 2\zeta\omega_n sX(s) + \omega_n^2 X(s) = \frac{1}{m}$$

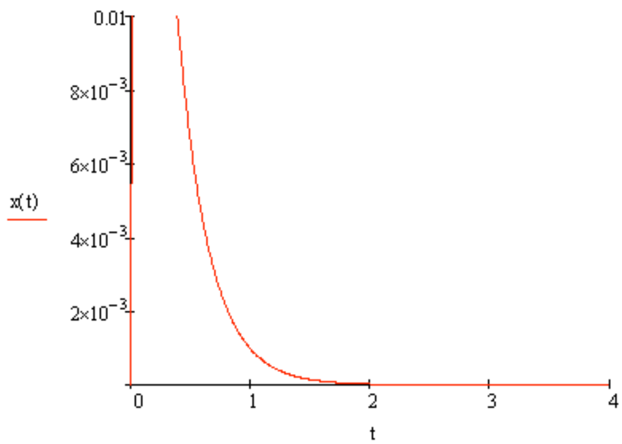
$$X(s) = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1/m}{(s+a)(s+b)}$$

Using inverse Laplace tables, $a = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$, $b = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

$$x(t) = \frac{e^{-\zeta\omega_n t}}{2m\omega_n\sqrt{\zeta^2 - 1}}\left[e^{\omega_n\sqrt{\zeta^2-1}t} - e^{-\omega_n\sqrt{\zeta^2-1}t}\right]$$

Inserting the given values yields: $x(t) = \dfrac{e^{-15t}}{22.36}\left[e^{11.18t} - e^{-11.18t}\right]$ m

$$x(t) := \frac{e^{-15\cdot t}}{22.36}\cdot\left(e^{11.18\cdot t} - e^{-11.18\cdot t}\right)$$



3.37 Calculate the response of the underdamped system given by

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{-at}$$

using the Laplace transform method. Assume $a > 0$ and that the initial conditions are all zero.

Solution:

Given:

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{-at} \qquad a > 0, \ \text{initial conditions} = 0$$

Rewrite:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m} e^{-at}$$

Take Laplace Transform:

$$s^2 X(s) + 2\zeta\omega_n s X(s) + \omega_n^2 X(s) = \frac{F_0}{m}\left(\frac{1}{s+a}\right)$$

$$X(s) = \frac{F_0 / m}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s+a)}$$

For an underdamped system, the inverse Laplace Transform is

$$x(t) = \left(\frac{F_0}{m\left(2\zeta\omega_n a - \omega_n^2 - a^2\right)}\right)\left\{e^{-\zeta\omega_n t}\left[\frac{\zeta\omega_n - a}{\omega_d}\sin(\omega_d t) + \cos(\omega_d t)\right] - e^{-at}\right\}$$

3.38 Solve the following system for the response $x(t)$ using Laplace transforms:

$$100\ddot{x}(t) + 2000x(t) = 50\delta(t)$$

where the units are in Newtons and the initial conditions are both zero.

Solution:

First divide by the mass to get

$$\ddot{x} + 20x(t) = 0.5\delta(t)$$

Take the Laplace Transform to get

$$(s^2 + 20)X(s) = 0.5$$

So

$$X(s) = \frac{0.5}{s^2 + 20}$$

Taking the inverse Laplace Transform using entry 5 of Table 3.1 yields

$$X(s) = \frac{0.5}{\sqrt{20}} \cdot \frac{\omega}{s^2 + \omega^2} \quad \text{where } \omega = \sqrt{20}$$

$$\Rightarrow x(t) = \frac{1}{4\sqrt{5}} \sin\sqrt{20}t$$