Problems and Solutions Section 3.5 (3.39 through 3.42)

3.39 Calculate the mean-square response of a system to an input force of constant PSD, S_0, and frequency response function $H(\omega) = \dfrac{10}{(3+2j\omega)}$

Solution:

Given: $S_{ff} = S_0$ and $H(\omega) = \dfrac{10}{3+2j\omega}$

The mean square of the response can be found from Eqs (3.66) and (3.68):

$$\bar{x}^2 = E[x^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{ff}(\omega) d\omega$$

$$\bar{x}^2 = S_0 \int_{-\infty}^{\infty} \left|\frac{10}{3+2j\omega}\right|^2 d\omega$$

Using Eq. (3.67) yields

$$\bar{x}^2 = \frac{50\pi S_0}{3}$$

3.40 Consider the base excitation problem of Section 2.4 as applied to an automobile model of Example 2.4.1 and illustrated in Figure 2.16. In this problem let the road have a random stationary cross section producing a PSD of S_0. Calculate the PSD of the response and the mean-square value of the response.

Solution: Given: $S_{ff} = S_0$

From example 2.4.1: $m = 1007$ kg, $c = 2000$ kg/s, $k = 40{,}000$ N/m

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{2000}{2\sqrt{40000 \bullet 1007}} = 0.157 \qquad \text{(underdamped)}$$

So,

$$H(\omega) = \frac{1}{k - m\omega^2 + jc\omega} = \frac{1}{4\times 10^4 - 1007\omega^2 + 2000\, j\omega}$$

$$\left|H(\omega)\right|^2 = \frac{1}{\left(4\times 10^4 - 1007\omega^2\right)^2 + \left(2000\right)^2 j\omega^2}$$

$$\left|H(\omega)\right|^2 = \frac{1}{1.01\times 10^6 \omega^4 - 4.06\times 10^7 \omega^2 + 1.6\times 10^9}$$

The PSD is found from equation (3.62):

$$S_{xx}(\omega) = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

$$S_{xx}(\omega) = \frac{1}{1.01\times 10^6 \omega^4 - 8.46\times 10^7 \omega^2 + 1.6\times 10^9}$$

The mean square value is found from equation (3.68):

$$\bar{x}^2 = E\left[x^2\right] = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{ff}(\omega)\, d\omega$$

$$\bar{x}^2 = S_0 \int_{-\infty}^{\infty} \left|\frac{1}{4\times 10^4 - 1007\omega^2 + 2000\, j\omega}\right|^2 d\omega$$

Using equation (3.70) yields

$$\bar{x}^2 = \frac{\pi S_0}{8\times 10^{10}}$$

3.41 To obtain a feel for the correlation functions, compute autocorrelation $R_{xx}(\tau)$ for the deterministic signal $A\sin\omega_n t$.

Solution: The autocorrelation is found from

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} A\sin(\omega_n t)A\sin(\omega_n(t+\tau))dt$$

$$= \lim_{T\to\infty}\frac{A^2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} \sin(\omega_n t)\sin(\omega_n t)\cos(\omega_n\tau)dt$$

$$+\underbrace{\lim_{T\to\infty}\frac{A^2}{T}\int_0^T \sin(\omega_n t)\cos(\omega_n t)\sin(\omega_n\tau)dt}_{\to 0}$$

Simplifying yields:

$$R_{xx}(\tau) = \frac{A^2\cos(\omega_n\tau)}{2}$$

3.42 Verify that the average $x-\bar{x}$ is zero by using the definition given in equation (3.47).

Solution:

The definition is $\bar{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)dt$. Let

$$f(t) = x(t) - \bar{x},$$

so that $$\bar{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(x(t)-\bar{x}\right)dt$$

$$\bar{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)dt - \lim_{T\to\infty}\frac{1}{T}\int_0^T \bar{x}dt$$

$$= \bar{x} - \bar{x}\lim_{T\to\infty}\frac{1}{T}\int_0^T dt = \bar{x}-\bar{x} = 0$$