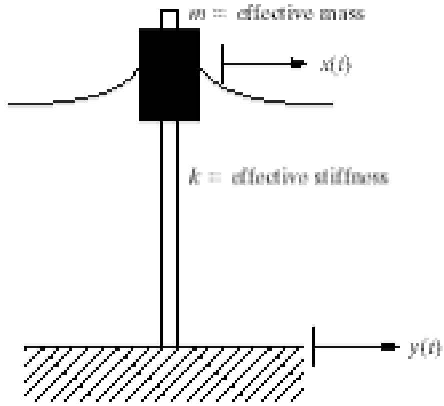
Problems and Solutions Section 3.6 (3.43 through 3.44)

3.43 A power line pole with a transformer is modeled by

$$m\ddot{x} + kx = -\ddot{y}$$

where x and y are as indicated in Figure 3.23. Calculate the response of the relative displacement $(x - y)$ if the pole is subject to an earthquake base excitation of (assume the initial conditions are zero)

$$\ddot{y}(t) = \begin{cases} A\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le 2t_0 \\ 0 & t > 2t_0 \end{cases}$$



Solution: Given: $m\ddot{x} + kx = -\ddot{y}$

$$\ddot{y} = \begin{cases} A\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le 2t_0 \\ 0 & t > 2t_0 \end{cases}$$

$$x(0) = \dot{x}(0) = 0$$

The response $x(t)$ is given by Eq. (3.12) as

$$x(t) = \int_0^t F(\tau)h(t-\tau)d\tau$$

where $h(t-\tau) = \dfrac{1}{m\omega_n}\sin\omega_n(t-\tau)$ for an undamped system

For $0 \le t \le 2t_0$,

$$x(t)=\int_0^t A\left(1-\frac{\tau}{t_0}\right)\left(\frac{1}{m\omega_n}\right)\sin\omega_n(t-\tau)\,d\tau$$

$$x(t)=\frac{A}{m\omega_n^2}\left[1-\frac{t}{t_0}+\frac{1}{t_0\omega_n}\sin\omega_n t-\cos\omega_n t\right]$$

For $t>2t_0$,

$$x(t)=\int_0^{2t_0} A\left(1-\frac{\tau}{t_0}\right)\left(\frac{1}{m\omega_n}\right)\sin\omega_n(t-\tau)\,d\tau$$

$$x(t)=\frac{A}{m\omega_n^2}\left[\frac{1}{t_0\omega_n}\left(\sin\omega_n t-\sin\omega_n(t-2t_0)\right)-\cos\omega_n t-\cos\omega_n(t-2t_0)\right]$$

Find $y(t)$ when $0\le t\le 2t_0$,

$$\ddot{y}(t)=A\left(1-\frac{t}{t_0}\right)$$

$$\dot{y}(t)=At-\frac{A}{2t_0}t^2+C_1$$

$$y(t)=\frac{A}{2}t^2-\frac{A}{6t_0}t^3+C_1t+C_2$$

Using IC's yields $C_1 = C_2 = 0$. Find $y(t)$ when $t > \mathrm{w}t_0$:

$$\ddot{y}(t)=0$$

$$\dot{y}(t)=C_3$$

$$y(t)=C_3t+C_4$$

Using IC's yields $C_3 = C_4 = 0$. The relative displacement $x(t) - y(t)$ is therefore:

For $0\le t\le 2t_0$

$$x(t)-y(t)=\frac{A}{m\omega_n^2}\left[1-\frac{t}{t_0}+\frac{1}{t_0\omega_n}\sin\omega_n t-\cos\omega_n t\right]-\frac{A}{2}t^2+\frac{A}{6t_0}t^3$$

For $t > 2t_0$,

$$x(t)-y(t)=\frac{A}{m\omega_n^2}\left[\frac{1}{t_0\omega_n}\left(\sin\omega_n t-\sin\omega_n(t-2t_0)\right)-\cos\omega_n t-\cos\omega_n(t-2t_0)\right]$$

3.44 Calculate the response spectrum of an undamped system to the forcing function

$$F(t)=\begin{cases} F_0\sin\dfrac{\pi t}{t_1} & 0\le t\le t_1 \\ 0 & t>t_1 \end{cases}$$

assuming the initial conditions are zero.

Solution: Let $\omega=\pi/t_1$. The solution is the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$ or $x(t)=x_h(t)+x_p(t)$. Thus

$$x(t)=A\cos\omega_n t+B\sin\omega_n t+\left(\frac{F_0}{k-m\omega^2}\right)\sin\omega t$$

where A and B are constants and ω_n is the natural frequency of the system:
Using the initial conditions $x(0)=\dot{x}(0)=0$ the constants A and B are

$$A=0,\quad B=\frac{-F_0\omega}{\omega_n\left(k-m\omega^2\right)}$$

so that $x(t)=\dfrac{F_0/k}{1-(\omega/\omega_n)^2}\left\{\sin\omega t-\dfrac{\omega}{\omega_n}\sin\omega_n t\right\},\ 0\le t\le t_1$

Which can be written as (where $\delta=F_0/k$ the static deflection)

$$\frac{x(t)}{\delta}=\frac{1}{1-\left(\dfrac{\tau}{2t_1}\right)^2}\left\{\sin\frac{\pi t}{t_1}-\frac{\tau}{2t_1}\sin\frac{2\pi t}{\tau}\right\},\ 0\le t\le t_1$$

and where $\tau=2\pi/\omega_n$. After t_1 the solution is a free response

$$x(t)=A'\cos\omega_n t+B'\sin\omega_n t,\ t>t_1$$

where the constants A' and B' can be found by using the values of $x(t=t_1)$ and $\dot{x}(t=t_1),\ t>t_0$. This gives

$$x(t=t_1)=a\left[-\frac{\tau}{2t_1}\sin\frac{2\pi t_1}{\tau}\right]=A'\cos\omega_n t_1+B'\sin\omega_n t_1$$

$$\dot{x}(t=t_1)=a\left\{-\frac{\pi}{t_1}-\frac{\pi}{t_1}\cos\frac{2\pi t_1}{\tau}\right\}=-\omega_n A'\sin\omega_n t_1+\omega_n B'\cos\omega_n t$$

where

$$a=\frac{\delta}{1-\left(\dfrac{\tau}{2t_1}\right)^2}$$

These are solved to yield

$$A' = \frac{a\pi}{\omega_n t_1}\sin\omega_n t_1, \quad B' = -\frac{a\pi}{\omega_n t_1}\left[1+\cos\omega_n t_1\right]$$

So that after t_1 the solution is

$$\frac{x(t)}{\delta} = \frac{(\tau/t_1)}{2\left\{1-(\tau/2t_1)^2\right\}}\left[\sin 2\pi\left(\frac{t_1}{\tau}-\frac{t}{\tau}\right)-\sin 2\pi\frac{t}{\tau}\right],\ t \ge t_1$$