Problems and Solutions for Section 3.7 (3.45 through 3.52)

3.45 Using complex algebra, derive equation (3.89) from (3.86) with $s = j\omega$.

Solution: From equation (3.86):

$$H(s) = \frac{1}{ms^2 + cs + k}$$

Substituting $s = j\omega$ yields

$$H(j\omega) = \frac{1}{m(j\omega)^2 + c(j\omega) + k} = \frac{1}{k - m\omega^2 - cj\omega}$$

The magnitude is given by

$$\left|H(j\omega_{dr})\right| = \left[\left(\frac{1}{m(j\omega)^2 + (cj\omega) + k}\right) = \left(\frac{1}{k - m\omega^2 - cj\omega}\right)\right]^{1/2}$$

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}$$ which is Eq. (3.89)

3.46 Using the plot in Figure 3.20, estimate the system's parameters *m*, *c*, and *k*, as well as the natural frequency.

Solution: From Fig. 3.20

$$\frac{1}{k} = 2 \Rightarrow k = 0.5$$

$$\omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8$$

$$\frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

3.47 Using the values determined in Problem 3.46 plot the inertance transfer function's magnitude and phase for this system.

Solution: From Problem 3.46

$$\frac{1}{k} = 2 \Rightarrow k = 0.5, \omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8, \frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

The inertance transfer function is given by Eq. (3.88):

$$s^2 H(s) = \frac{s^2}{ms^2 + cs + k}$$

Substitute $s = j\omega$ to get the frequency response function. The magnitude is given by:

$$\left|(j\omega)^2 H(j\omega)\right| = \frac{\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}} = \frac{\omega^2}{\sqrt{\left(0.5 - 8\omega^2\right)^2 + \left(0.087\omega\right)^2}}$$

The phase is given by

$$\phi = \tan^{-1}\left(\frac{\text{Imaginary part of frequency response function}}{\text{Real part of frequency response function}}\right)$$

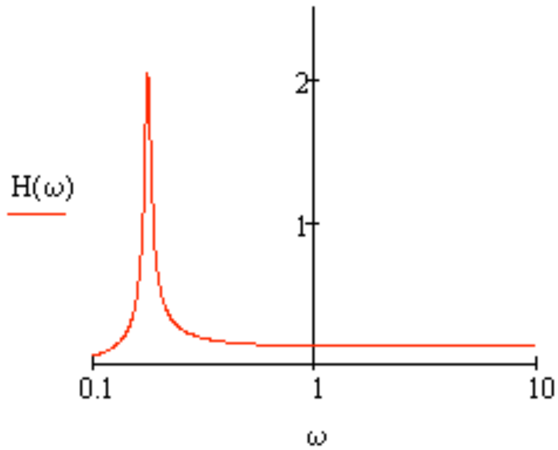
Multiply the numerator and denominator of $(j\omega)^2 H(j\omega)$ by $\left(k - m\omega^2\right) - cj\omega$ to get

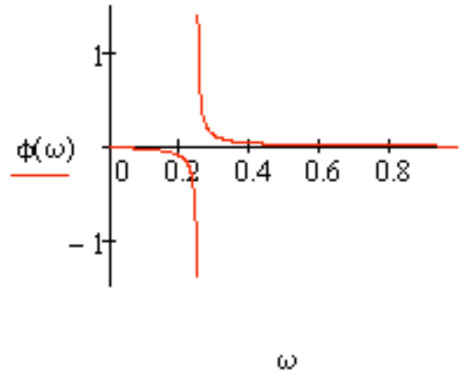
$$(j\omega)^2 H(j\omega) = \frac{-\omega^2\left(k - m\omega\right) + cj\omega^3}{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}$$

So, $\phi = \tan^{-1}\left(\dfrac{c\omega^3}{-\omega^2\left(k - m\omega^2\right)}\right) = \tan^{-1}\left(\dfrac{0.087\omega}{8\omega^2 - 0.5}\right)$

The magnitude and phase plots are shown on a semilog scale. The plots are given in the following Mathcad session

$$\phi(\omega) := \operatorname{atan}\left(\frac{0.087\cdot\omega}{8\cdot\omega^2 - 0.5}\right) \qquad H(\omega) := \frac{\omega^2}{\sqrt{\left(0.5^2 - 8\cdot\omega^2\right)^2 + (0.087\cdot\omega)^2}}$$





3.48 Using the values determined in Problem 3.46 plot the mobility transfer function's magnitude and phase for the system of Figure 3.20.

Solution: From Problem 3.46

$$\frac{1}{k} = 2 \Rightarrow k = 0.5, \omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8, \frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

The mobility transfer function is given by equation (3.87):

$$sH(s) = \frac{s}{ms^2 + cs + k}$$

Substitute $s = j\omega$ to get the frequency response function. The magnitude is given by

$$\left|(j\omega)H(j\omega)\right| = \frac{\omega}{\sqrt{\left(k - j\omega^2\right)^2 + (c\omega)^2}} = \frac{\omega}{\sqrt{\left(0.5 - 8\omega^2\right)^2 + (0.087\omega)^2}}$$

The phase is given by

$$\phi = \tan^{-1}\left(\frac{\text{Imaginary part of frequency response function}}{\text{Real part of frequency response function}}\right)$$

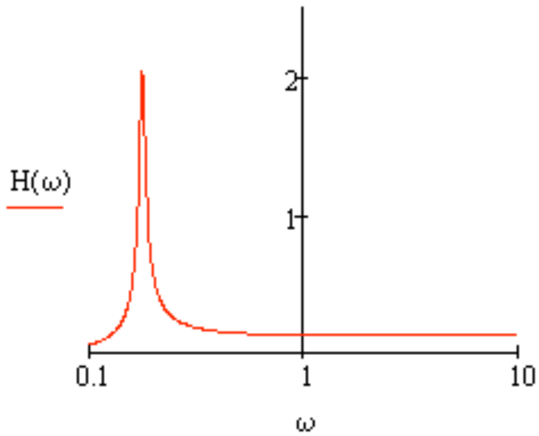
Multiply the numerator and denominator of $j\omega H(j\omega)$ by j and by $-\left(k - m\omega^2\right)j - c\omega$ to get

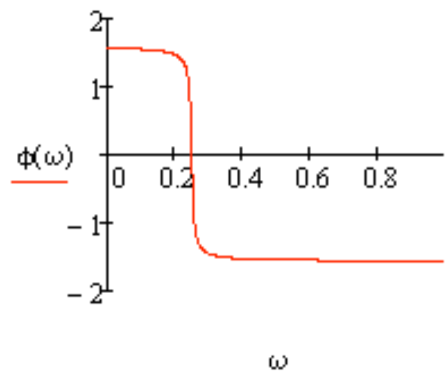
$$(j\omega)H(j\omega) = \frac{j\omega\left(k - m\omega^2\right) + c\omega^2}{\left(k - m\omega^2\right)^2 + (c\omega)^2}$$

So, $\phi = \tan^{-1}\left(\frac{\omega\left(k - m\omega^2\right)}{c\omega^2}\right) = \tan^{-1}\left(\frac{0.5 - 8\omega^2}{0.087\omega}\right)$

The magnitude and phase plots are shown on a semilog scale.

$$\phi(\omega) := \operatorname{atan}\left(\frac{-8\cdot\omega^2 + 0.5}{0.087\cdot\omega}\right) \qquad H(\omega) := \frac{\omega^2}{\sqrt{\left(0.5^2 - 8\cdot\omega^2\right)^2 + (0.087\cdot\omega)^2}}$$





3.49 Calculate the compliance transfer function for a system described by

$$a\ddot{x} + b\ddot{x} + c\ddot{x} + d\dot{x} + ex = f(t)$$

where $f(t)$ is the input force and $x(t)$ is a displacement.

Solution:

The compliance transfer function is $\dfrac{X(s)}{F(s)}$.

Taking the Laplace Transform yields

$$\left(as^4 + bs^3 + cs^2 + ds + e\right)X(s) = F(s)$$

So, $\dfrac{X(s)}{F(s)} = \dfrac{1}{as^4 + bs^3 + cs^2 + ds + e}$

3.50 Calculate the frequency response function for the compliance of Problem 3.49.

Solution: From problem 3.49,

$$H(s) = \frac{1}{as^4 + bs^3 + cs^2 + ds + e}$$

Substitute $s = j\omega$ to get the frequency response function:

$$H(j\omega) = \frac{1}{a(j\omega)^4 + b(j\omega)^3 + c(j\omega)^2 + d(j\omega) + e}$$

$$H(j\omega) = \frac{a\omega^4 - c\omega^2 + e - j(-b\omega^3 + d\omega)}{(a\omega^4 - c\omega^2 + e)^2 + (-b\omega^3 + d\omega)^2}$$

3.51 Plot the magnitude of the frequency response function for the system of Problem 3.49 for $a = 1, b = 4, c = 11, d = 16,$ and $e = 8.$

Solution: From Problem 3.50

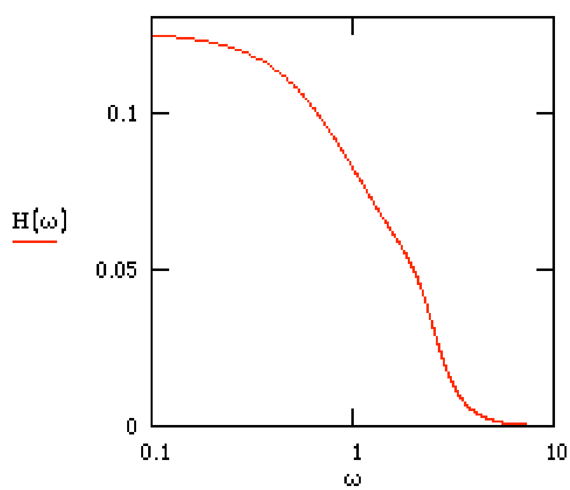
$$H(j\omega) = \frac{a\omega^4 - c\omega^2 + e - j(-b\omega^3 + d\omega)}{(a\omega^4 - c\omega^2 + e)^2 + (-b\omega^3 + d\omega)^2}$$

The magnitude is

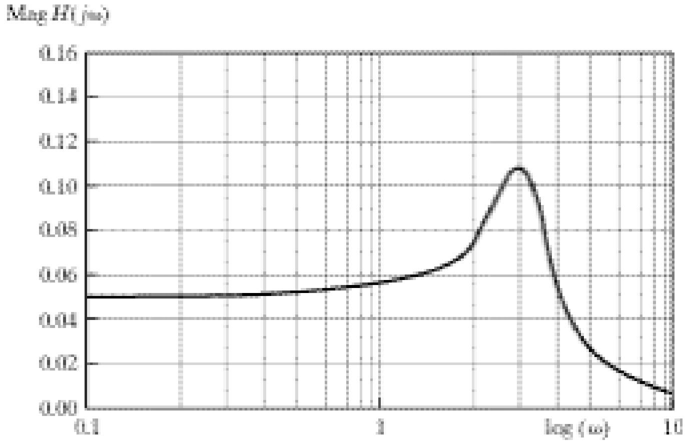
$$|H(j\omega)| = \frac{1}{\sqrt{(\omega^4 - 11\omega^2 + 8)^2 + (-4\omega^3 + 16\omega)^2}}$$

This is plotted in the following Mathcad session:

$$H(\omega) := \frac{1}{\sqrt{\left(\omega^4 - 11\cdot\omega^2 + 8\right)^2 + \left(16\cdot\omega - 4\cdot\omega^3\right)^2}}$$



3.52 An experimental (compliance) magnitude plot is illustrated in Fig. P3.52. Determine $\omega, \zeta, c, m,$ and k. Assume that the units correspond to m/N along the vertical axis.



Solution: Referring to the plot, it starts at

$$\left|H(\omega j)\right| = \frac{1}{k}$$

Thus: $$0.05 = \frac{1}{k} \Rightarrow k = 20 \text{ N/m}$$

At the peak, $\omega_n = \omega = 3$ rad/s. Thus the mass can be determined by

$$m = \frac{k}{\omega_n^2} \Rightarrow m = 2.22 \text{ kg}$$

The damping is found from

$$\frac{1}{c\omega} = 0.11 \Rightarrow c = 3.03 \text{ kg/s} \Rightarrow \zeta = \frac{c}{2\sqrt{km}} = \frac{3.03}{2\sqrt{20 \cdot 2.22}} = 0.227$$