Problems and Solutions Section 3.8 (3.53 through 3.56)

3.53 Show that a critically damped system is BIBO stable.

Solution:

For a critically damped system

$$h(t-\tau) = \frac{1}{m}(t-\tau)e^{-\omega_n(t-\tau)}$$

Let $f(t)$ be bounded by the finite constant M. Using the inequality for integrals and Equation (3.96) yields:

$$|x(t)| \le \int_0^t f(\tau)|h(t-\tau)|\,d\tau = \int_0^t M\frac{1}{m}\left|(t-\tau)e^{-\omega_n(t-\tau)}\right|d\tau$$

The function $h(t-\tau)$ decays exponentially and hence is bounded by some constant times $1/t$, say M_1/t. This is just a statement the exponential decays faster then "one over t" does. Thus the above expression becomes;

$$|x(t)| < M\int_0^t \frac{M_1}{t}d\tau = MM_1$$

This is bounded, so a critically damped system is BIBO stable.

3.54 Show that an overdamped system is BIBO stable.

Solution: For an overdamped system,

$$h(t-\tau)=\frac{1}{2m\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n(t-\tau)}\left(e^{\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}-e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}\right)$$

Let $f(t)$ be bounded by M,

From equation (3.96),

$$\left|x(t)\right|\le M\int_0^t \left|h(t-\tau)\right|d\tau$$

$$\left|x(t)\right|\le M\int_0^t \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left|e^{-\zeta\omega_n(t-\tau)}\left(e^{\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}-e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}\right)\right|d\tau$$

$$\left|x(t)\right|\le\frac{M}{2m\omega_n\sqrt{\zeta^2-1}}\left[\left(\frac{-1}{\omega_n\sqrt{\zeta^2-1}-\zeta\omega}\right)\left(1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}\right)-\left(\frac{-1}{\omega_n\sqrt{\zeta^2-1}+\zeta\omega_n}\right)\left(1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}\right)\right]$$

Since $\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n<0,$ then $1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}$ is bounded.

Also, since $-\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n<0,$ then $1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}$ is bounded.

Therefore, an overdamped system is BIBO stable.

3.55 Is the solution of $2\ddot{x} + 18x = 4\cos 2t + \cos t$ Lagrange stable?

Solution: Given

$$2\ddot{x} + 18x = 4\cos 2t + \cos t$$
$$\omega_n = \sqrt{\frac{k}{m}} = 3$$

The total solution will be

$$x(t) = x_h(t) + x_{P1}(t) + x_{P2}(t)$$

From Eq. (1.3): $$x_h(t) = A\sin(\omega_n t + \phi)$$

From Eq. (2.7): $$x_{P1}(t) = \frac{f_{0_1}}{\omega_n^2 - 2^2}\cos 2t$$

and $$x_{P2}(t) = \frac{f_{0_2}}{\omega_n^2 - 1^2}\cos t$$

Adding the solutions yields

$$|x(t)| = \left| A\sin(3t + \phi) + \frac{f_{0_1}}{3^2 - 2^2}\cos 2t + \frac{f_{0_1}}{3^2 - 1^2}\cos t \right| < M$$

Since $3 \neq 2, 3 \neq 1$, and the homogeneous solution is marginally stable, this system is Lagrange stable.

3.56 Calculate the response of equation (3.99) for $x_0 = 0, v_0 = 1$ for the case that a = 4 and b = 0. Is the response bounded?

Solution: Given: $x_0 = 0, v_0 = 1, a = 4, b = 0$. From Eq. (3.99),

$$\ddot{x} + \dot{x} + 4x = ax + b\dot{x} = 4x$$

So, $\ddot{x} + \dot{x} = 0$

Let

$$x(t) = Ae^{\lambda t}$$
$$\dot{x}(t) = \lambda Ae^{\lambda t}$$
$$\ddot{x}(t) = \lambda^2 Ae^{\lambda t}$$

Substituting,

$$\lambda^2 Ae^{\lambda t} + \lambda Ae^{\lambda t} = 0$$
$$\lambda^2 + \lambda = 0$$

So, $\lambda_{1,2} = 0, -1$

The solution is

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 + A_2 e^{-t}$$
$$\dot{x}(t) = -A_2 e^{-1}$$
$$x(0) = 0 = A_1 + A_2$$
$$\dot{x}(0) = 1 = -A_2$$

So, $A_1 = 1$ and $A_2 = -1$

Therefore,

$$x(t) = 1 - e^{-1}$$

Since $|x(t)| = |1 - e^{-t}| 1$, the response is bounded.