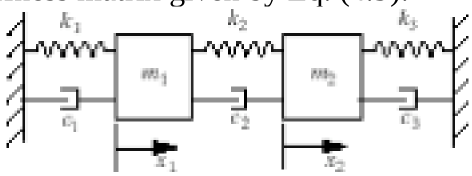
**Problems and Solutions for Section 4.1 (4.1 through 4.16)**

**4.1** Consider the system of Figure P4.1. For $c_1 = c_2 = c_3 = 0,$ derive the equation of motion and calculate the mass and stiffness matrices. Note that setting $k_3 = 0$ in your solution should result in the stiffness matrix given by Eq. (4.9).




**Solution:**

For mass 1:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2\left(x_2 - x_1\right)$$
$$\Rightarrow m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2x_2 = 0$$

For mass 2:

$$m_2\ddot{x}_2 = -k_3x_2 - k_2\left(x_2 - x_1\right)$$
$$\Rightarrow m_2\ddot{x}_2 - k_2x_1 + \left(k_2 + k_3\right)x_2 = 0$$

So, $M\ddot{\mathbf{x}} + K\mathbf{x} = 0$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Thus:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

**4.2** Calculate the characteristic equation from problem 4.1 for the case

$$m_1 = 9 \text{ kg} \quad m_2 = 1 \text{ kg} \quad k_1 = 24 \text{ N/m} \quad k_2 = 3 \text{ N/m} \quad k_3 = 3 \text{ N/m}$$

and solve for the system's natural frequencies.

**Solution:** Characteristic equation is found from Eq. (4.9):

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_1 + k_2 + k_3 \end{vmatrix} = \begin{vmatrix} -9\omega^2 + 27 & -3 \\ -3 & -\omega^2 + 6 \end{vmatrix} = 0$$

$$9\omega^4 - 81\omega^2 + 153 = 0$$

Solving for ω:

$$\left.\begin{array}{l} \omega_1 = \mathbf{1.642} \\ \omega_2 = \mathbf{2.511} \end{array}\right. \text{rad/s}$$

**4.3** Calculate the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ for problem 4.2.

**Solution:** Calculate $\mathbf{u}_1$:

$$\begin{bmatrix} (-2.697)(9) + 27 & -3 \\ -3 & -2.697 + 6 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This yields

$$2.727u_{11} - 3u_{21} = 0$$

$$-3u_{11} + 3.303u_{21} = 0 \qquad \text{or, } u_{21} = 0.909u_{11}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0.909 \end{bmatrix}$$

Calculate $\mathbf{u}_2$:

$$\begin{bmatrix} (-6.303)(9) + 27 & -3 \\ -3 & -6.303 + 6 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-29.727u_{12} - 3u_{22} = 0$$
$$-3u_{12} = 0.303u_{22} = 0 \qquad \text{or, } u_{12} = -0.101u_{22}$$

This yields

$$\mathbf{u}_2 = \begin{bmatrix} -0.101 \\ 1 \end{bmatrix}$$

**4.4** For initial conditions $\mathbf{x}(0) = [1 \;\; 0]^T$ and $\dot{\mathbf{x}}(0) = [0 \; 0]^T$ calculate the free response of the system of Problem 4.2. Plot the response $x_1$ and $x_2$.

**Solution:** Given $\mathbf{x}(0) = [1 \;\; 0]^T$, $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, The solution is

$$\mathbf{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\mathbf{u}_2$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) - 0.101A_2 \sin(\omega_2 t + \phi_2) \\ 0.909A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}$$

Using initial conditions,

$$1 = A_1 \sin\phi_1 - 0.101A_2 \sin\phi_2 \qquad [1]$$
$$0 = 0.909A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = 1.642A_1 \cos\phi_1 - 0.2536A_2 \cos\phi_2 \qquad [3]$$
$$0 = 6.033A_1 \cos\phi_1 + 2.511A_2 \cos\phi_2 \qquad [4]$$

From [3] and [4], $\qquad \phi_1 = \phi_2 = \pi / 2$

From [1] and [2], $\qquad A_1 = 0.916$, and $A_2 = -0.833$

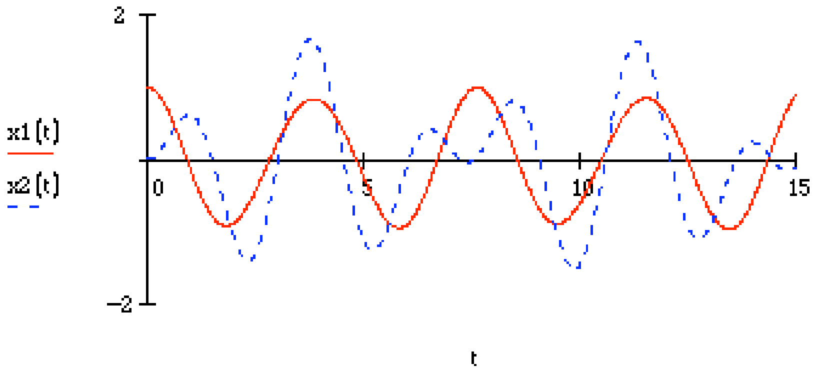
So,

$$x_1(t) = 0.916\sin(1.642t + \pi / 2) + 0.0841\sin(2.511t + \pi / 2)$$
$$x_2(t) = 0.833\sin(1.642t + \pi / 2) - 0.833\sin(2.511t + \pi / 2)$$

$$\boldsymbol{x}_1(t) = 0.916\cos 1.642t + 0.0841\cos 2.511t$$
$$\boldsymbol{x}_2(t) = 0.833(\cos 1.642t - \cos 2.511t)$$

x1 (t) := 0.916·cos (1.642·t) + 0.0841·cos (2.511·t)

x2 (t) := 0.833·(cos (1.642·t) − cos (2.511·t))




**4.5** Calculate the response of the system of Example 4.1.7 to the initial condition $\mathbf{x}(0) = 0, \dot{\mathbf{x}}(0) = [1 \;\; 0]^T$, plot the response and compare the result to Figure 4.3.

**Solution:** Given: $\mathbf{x}(0) = \mathbf{0}$, $\dot{\mathbf{x}}(0) = [1 \;\; 0]^T$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3} A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

Using initial conditions:

$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$3 = \sqrt{2} A_1 \cos\phi_1 - 2 A_2 \cos\phi_2 \qquad [3]$$
$$0 = \sqrt{2} A_1 \cos\phi_1 + 2 A_2 \cos\phi_2 \qquad [4]$$

From [1] and [2]:

$$\phi_1 = \phi_2 = 0$$

From [3] and [4]:

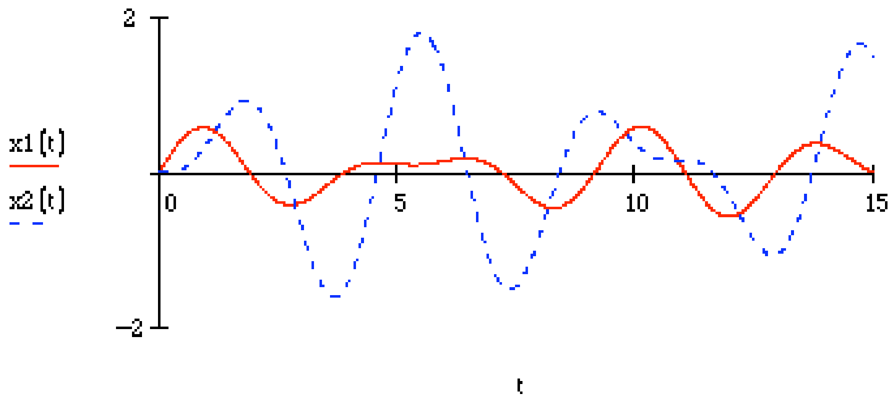
$$A_1 = \frac{3\sqrt{2}}{4}, \text{ and } A_2 = -\frac{3}{4}$$

The solution is

$$x_1(t) = 0.25\left(\sqrt{2}\sin\sqrt{2}t + \sin 2t\right)$$
$$x_2(t) = 0.75\left(\sqrt{2}\sin\sqrt{2}t - \sin 2t\right)$$

As in Fig. 4.3, the second mass has a larger displacement than the first mass.

$$x1(t) := 0.25\cdot\left(\sqrt{2}\cdot\sin\left(\sqrt{2}\cdot t\right) + \sin(2\cdot t)\right) \qquad x2(t) := 0.75\cdot\left(\sqrt{2}\cdot\sin\left(\sqrt{2}\cdot t\right) - \sin(2\cdot t)\right)$$



**4.6** Repeat Problem 4.1 for the case that $k_1 = k_3 = 0$ .

**Solution:**

The equations of motion are

$$m_1\ddot{x}_1 + k_2x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

So, $M\ddot{x} + Kx = 0$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{x} + \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}x = 0$$
$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \text{ and } K = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

**4.7** Calculate and solve the characteristic equation for Problem 4.6 with $m_1 = 9$, $m_2 = 1$, $k_2 = 10$.

**Solution:**

The characteristic equation is found from Eq. (4.19):

$$\det\left(-\omega^2 M + K\right) = 0$$
$$\begin{vmatrix} -9\omega^2 + 10 & -10 \\ -10 & -\omega^2 + 10 \end{vmatrix} = 9\omega^4 - 100\omega^2 = 0$$
$$\omega_{1,2}^2 = 0,11.111$$
$$\omega_1 = 0$$
$$\omega_2 = 3.333$$

**4.8** Compute the natural frequencies of the following system:

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}\ddot{\mathbf{x}}(t) + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x}(t) = 0 .$$

**Solution:**

$$\det(-\omega^2 M + K) = \det\left( -\omega^2 \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right) = 20\omega^4\text{-}22\omega^2\text{+}2\text{=}0,\ \omega^2 = 0.1,\ 1$$

$\underline{\omega_{1,2} = 0.316,\ 1 \text{ rad/s}}$

**4.9** Calculate the solution to the problem of Example 4.1.7, to the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Plot the response and compare it to that of Fig. 4.3.

**Solution:** Given: $\mathbf{x}(0) = \begin{bmatrix} 1/3 & 1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = 0$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3}A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

Using initial conditions:

$$1 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$1 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = \sqrt{2}A_1 \cos\phi_1 - 2A_2 \cos\phi_2 \qquad [3]$$
$$0 = \sqrt{2}A_1 \cos\phi_1 + 2A_2 \cos\phi_2 \qquad [4]$$

From [3] and [4]: $\phi_1 = \phi_2 = \frac{\pi}{2}$

From [1] and [2]: $A_1 = 1$, and $A_2 = 0$

The solution is

$$x_1(t) = \frac{1}{3}\cos\sqrt{2}t$$
$$x_2(t) = \cos\sqrt{2}t$$

In this problem, both masses oscillate at only one frequency.

**4.10** Calculate the solution to Example 4.1.7 for the initial condition

$$x(0) = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \quad \dot{x}(0) = 0$$

**Solution:**

Given: $\mathbf{x}(0) = [-1/3 \quad 1]^T$, $\dot{\mathbf{x}}(0) = 0$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3} A_2 \left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

Using initial conditions:

$$\begin{aligned} -1 &= A_1 \sin\phi_1 - A_2 \sin\phi_2 & [1] \\ 1 &= A_1 \sin\phi_1 + A_2 \sin\phi_2 & [2] \\ 0 &= \sqrt{2} A_1 \cos\phi_1 - 2A_2 \cos\phi_2 & [3] \\ 0 &= \sqrt{2} A_1 \cos\phi_1 + 2A_2 \cos\phi_2 & [4] \end{aligned}$$

From [3] and [4]

$$\phi_1 = \phi_2 = \frac{\pi}{2}$$

From [1] and [2]:

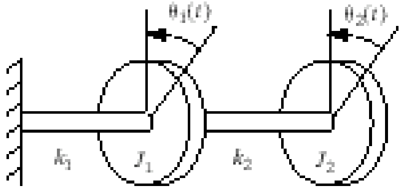
$$\begin{aligned} A_1 &= 0 \\ A_2 &= 1 \end{aligned}$$

The solution is

$$\begin{aligned} x_1(t) &= -\frac{1}{3}\cos 2t \\ x_2(t) &= \cos 2t \end{aligned}$$

In this problem, both masses oscillate at only one frequency (not the same frequency as in Problem 4.9, though.)

**4.11** Determine the equation of motion in matrix form, then calculate the natural frequencies and mode shapes of the torsional system of Figure P4.11. Assume that the torsional stiffness values provided by the shaft are equal $(k_1 = k_2)$ and that disk 1 has three times the inertia as that of disk $2 (J_1 = 3J_2)$.




**Solution:** Let $k = k_1 = k_2$ and $J_1 = 3J_2$. The equations of motion are

$$J_1\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = 0$$
$$J_2\ddot{\theta}_2 - k\theta_1 + k\theta_2 = 0$$

So,

$$J_2\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\theta} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\theta = 0$$

Calculate the natural frequencies:

$$\det\left(-\omega^2 J + K\right) = \begin{vmatrix} -3\omega^2 J_2 + 2k & -k \\ -k & -\omega^2 J_2 + k \end{vmatrix} = 0$$

$$\omega_1 = 0.482\sqrt{\frac{k}{J_2}}$$

$$\omega_2 = 1.198\sqrt{\frac{k}{J_2}}$$

Calculate the mode shapes: mode shape 1:

$$\begin{bmatrix} -3\left(0.2324\right)k + 2k & -k \\ -k & -\left(0.2324\right)k + k \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = 0$$

$$u_{11} = 0.7676u_{12}$$

So, $\mathbf{u}_1 = \begin{bmatrix} 0.7676 \\ 1 \end{bmatrix}$

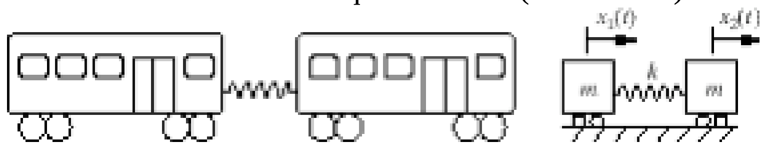
mode shape 2:

$$\begin{bmatrix} -3\left(1.434\right)k + 2k & -k \\ -k & -\left(1.434\right)k + k \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = 0$$

$$u_{21} = -0.434u_{22}$$

So, $\mathbf{u}_2 = \begin{bmatrix} -0.434 \\ 1 \end{bmatrix}$

**4.12** Two subway cars of Fig. P4.12 have 2000 kg mass each and are connected by a coupler. The coupler can be modeled as a spring of stiffness $k$ = 280,000 N/m. Write the equation of motion and calculate the natural frequencies and (normalized) mode shapes.




**Solution:** Given: $m_1 = m_2 = m = 2000$ kg $\quad k = 280{,}000$ N/m

The equations of motion are:

$$m\ddot{x}_1 + kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + kx_2 = 0$$

In matrix form this becomes:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\ddot{x} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}x = 0$$
$$\begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix}\ddot{x} + \begin{bmatrix} 280{,}000 & -280{,}000 \\ -280{,}000 & 280{,}000 \end{bmatrix}x = 0$$

Natural frequencies:

$$\det\left(-\omega^2 M + K\right) = 0$$
$$\begin{vmatrix} -2000\omega^2 + 280{,}000 & -280{,}000 \\ -280{,}000 & -2000\omega^2 + 280{,}000 \end{vmatrix} = 0$$
$$4\times10^6\omega^4 - 1.12\times10^9\omega^2 = 0$$
$$\omega^2 = 0, 280 \Rightarrow \omega_1 = 0 \text{ rad/sec and } \omega_2 = 16.73 \text{ rad/sec}$$

Mode shapes:

Mode 1, $\omega_1^2 = 0$

$$\begin{bmatrix} 280{,}000 & -280{,}000 \\ -280{,}000 & 280{,}000 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\therefore u_{11} = u_{12}$$
$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 280$

$$\begin{bmatrix} -280{,}000 & -280{,}000 \\ -280{,}000 & -280{,}000 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore u_{21} = u_{22}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalizing the mode shapes yields

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note that $\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is also acceptable because a mode shape times a constant (-1 in this case) is still a mode shape.

**4.13** Suppose that the subway cars of Problem 4.12 are given the initial position of $x_{10}$ = 0, $x_{20}$ = 0.1 m and initial velocities of $v_{10} = v_{20} = 0$. Calculate the response of the cars.

**Solution:**

Given: $\mathbf{x}(0) = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = \mathbf{0}$

From problem 12,

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\omega_1 = 0 \text{ rad/s and } \omega_2 = 16.73 \text{ rad/s}$$

The solution is

$$\mathbf{x}(t) = (c_1 + c_2 t)\mathbf{u}_1 + A\sin(16.73t + \phi)\mathbf{u}_2$$
$$\Rightarrow \dot{\mathbf{x}}(0) = c_2\mathbf{u}_1 + 16.73A\cos(\phi)\mathbf{u}_2 \text{ and } \mathbf{x}(0) = c_1\mathbf{u}_1 + A\sin(\phi)\mathbf{u}_2$$

Using initial the conditions four equations in four unknowns result:

$$0 = c_1 + A\sin\phi \qquad [1]$$
$$0.1 = c_1 - A\sin\phi \qquad [2]$$
$$0 = c_2 + 16.73A\cos\phi \qquad [3]$$
$$0 = c_2 - 16.73A\cos\phi \qquad [4]$$

From [3] and [4]: $c_2 = 0$, and $\phi = \dfrac{\pi}{2}$ rad

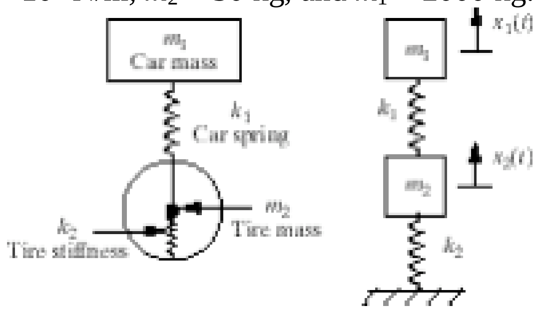
From [1] and [2]: $c_1 = 0.05$ m and $A = -0.05$ m

The solution is

$$x_1(t) = 0.05 - 0.05\cos 16.73t$$
$$x_2(t) = 0.05 + 0.05\cos 16.73t$$

Note that if $\mathbf{u}_2 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is chosen as the second mode shape the answer will remain the same. It might be worth presenting both solutions in class, as students are often skeptical that the two choices will yield the same result.

**4.14** A slightly more sophisticated model of a vehicle suspension system is given in Figure P4.14. Write the equations of motion in matrix form. Calculate the natural frequencies for $k_1 = 10^3$ N/m, $k_2 = 10^4$ N/m, $m_2 = 50$ kg, and $m_1 = 2000$ kg.




**Solution:** The equations of motion are

$$2000\ddot{x}_1 + 1000x_1 - 1000x_2 = 0$$
$$50\ddot{x}_2 - 1000x_1 + 11{,}000x_2 = 0$$

In matrix form this becomes:

$$\begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Natural frequencies:

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -2000\omega^2 + 1000 & -1000 \\ -1000 & -50\omega^2 + 11{,}000 \end{vmatrix} = 100{,}000\omega^4 - 2.205\times 10^7\omega^2 + 10^7 = 0$$

$$\omega_{1,2}^2 = 0.454,\ \ 220.046 \quad \Rightarrow \omega_1 = 0.674 \text{ rad/s} \quad \text{and} \quad \omega_2 = 14.8 \text{ rad/s}$$

**4.15** Examine the effect of the initial condition of the system of Figure 4.1(a) on the responses $x_1$ and $x_2$ by repeating the solution of Example 4.1.7, first for $x_{10} = 0, x_{20} = 1$ with $\dot{x}_{10} = \dot{x}_{20} = 0$ and then for $x_{10} = x_{20} = \dot{x}_{10} = 0$ and $\dot{x}_{20} = 1$. Plot the time response in each case and compare your results against Figure 4.3.

**Solution:** From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3}A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

(a) $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = \mathbf{0}$. Using the initial conditions:

$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$1 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = \sqrt{2}A_1 \cos\phi_1 - 2A_2 \cos\phi_2 \qquad [3]$$
$$0 = \sqrt{2}A_1 \cos\phi_1 + 2A_2 \cos\phi_2 \qquad [4]$$

From [3] and [4] $\qquad \phi_1 = \phi_2 = \dfrac{\pi}{2}$

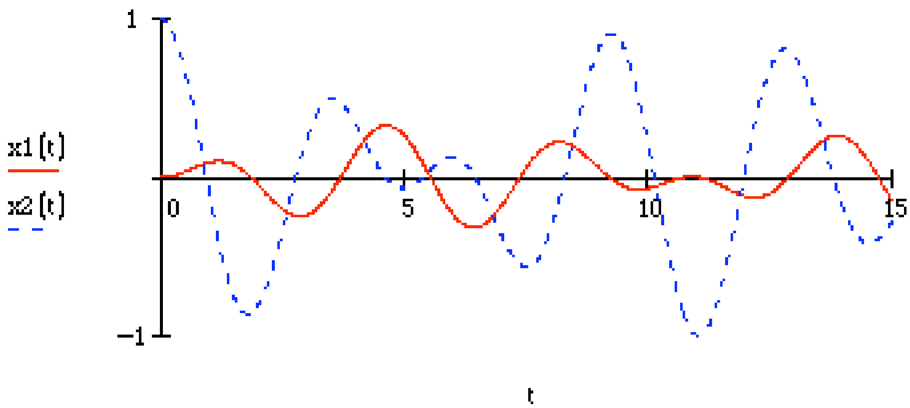
From [1] and [2] $\qquad A_1 = A_2 = \dfrac{1}{2}$

The solution is

$$x_1(t) = \frac{1}{6}\cos\sqrt{2}t - \frac{1}{6}\cos 2t$$
$$x_2(t) = \frac{1}{2}\cos\sqrt{2}t + \frac{1}{2}\cos 2t$$

This is similar to the response of Fig. 4.3

$$x1(t) := \frac{1}{6} \cdot \left(\cos\left(\sqrt{2} \cdot t\right) - \cos(2 \cdot t)\right) \qquad x2(t) := \frac{1}{2} \cdot \left(\cos\left(\sqrt{2} \cdot t\right) + \cos(2 \cdot t)\right)$$




(b) $\mathbf{x}(0) = 0$, $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Using these initial conditions:

$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = \sqrt{2} A_1 \cos\phi_1 - 2A_2 \cos\phi_2 \qquad [3]$$
$$1 = \sqrt{2} A_1 \cos\phi_1 + 2A_2 \cos\phi_2 \qquad [4]$$

From [1] and [2] $\phi_1 = \phi_2 = 0$

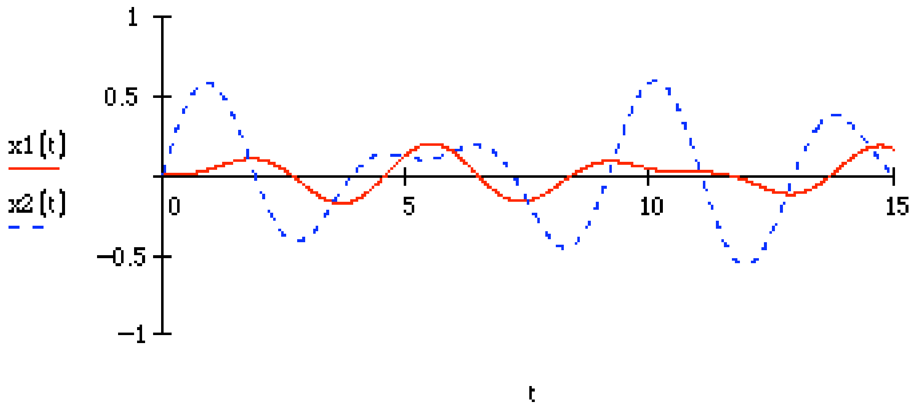
From [3] and [4] $A_1 = \dfrac{\sqrt{2}}{4}$, and $A_2 = \dfrac{1}{4}$

The solution is

$$x_1(t) = \frac{\sqrt{2}}{12} \sin\sqrt{2}t - \frac{1}{12} \sin 2t$$
$$x_2(t) = \frac{\sqrt{2}}{4} \sin\sqrt{2}t + \frac{1}{4} \sin 2t$$

This is also similar to the response of Fig. 4.3

$$x1(t) := \frac{\sqrt{2}}{12}\cdot\sin\left(\sqrt{2}\cdot t\right) - \frac{1}{12}\cdot\sin(2\cdot t) \qquad x2(t) := \frac{\sqrt{2}}{4}\cdot\sin\left(\sqrt{2}\cdot t\right) + \frac{1}{4}\cdot\sin(2\cdot t)$$



**4.16** Refer to the system of Figure 4.1(a). Using the initial conditions of Example 4.1.7, resolve and plot $x_1$(t) for the cases that $k_2$ takes on the values 0.3, 30, and 300. In each case compare the plots of $x_1$ and $x_2$ to those obtained in Figure 4.3. What can you conclude?

**Solution:** Let $k_2$ = 0.3, 30, 300 for the example(s) in Section 4.1. Given

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \text{ mm, } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

$$m_1 = 9, m_2 = 1, k_1 = 24$$

Equation of motion becomes:

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\ddot{x} + \begin{bmatrix} 24 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} x = 0$$

(a) $k_2$ = 0.3

$$\det(-\omega^2 M + K) = \begin{vmatrix} -9\omega^2 + 24.3 & -0.3 \\ -0.3 & -\omega^2 + 0.3 \end{vmatrix} = 9\omega^4 - 27\omega^2 + 7.2 = 0$$

$$\omega^2 = 0.2598, 2.7042$$

$$\omega_1 = 0.5439$$

$$\omega_2 = 1.6444$$

Mode shapes:

Mode 1, $\omega_1^2 = 0.2958$

$$\begin{bmatrix} 21.6374 & -0.3 \\ -0.3 & 0.004159 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$21.6374u_{11} - 0.3u_{12} = 0$$

$$u_{11} = 0.01386u_{12}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.01386 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 2.7042$

$$\begin{bmatrix} -0.03744 & -0.3 \\ -0.3 & 2.4042 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-0.3u_{21} = 2.4042u_{22}$$

$$u_{22} = -0.1248u_{21}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ -0.1248 \end{bmatrix}$$

The solution is $x(t) = A_1 \sin(\omega_1 t + \phi_1)u_1 + A_2 \sin(\omega_2 t + \phi_2)u_2$

Using initial conditions

$$1 = A_1(0.01386)\sin\phi_1 + A_2 \sin\phi_2 \qquad [1]$$
$$0 = A_1 \sin\phi_1 + A_2(-0.1248)\sin\phi_2 \qquad [2]$$
$$0 = A_1(0.01386)(0.5439)\cos\phi_1 + A_2(1.6444)\cos\phi_2 \qquad [3]$$
$$0 = A_1(0.5439)\cos\phi_1 + A_2(1.6444)(-0.1248)\cos\phi_2 \qquad [4]$$

From [3] and [4],

$$\phi_1 = \phi_2 = \pi / 2$$

From [1] and [2],

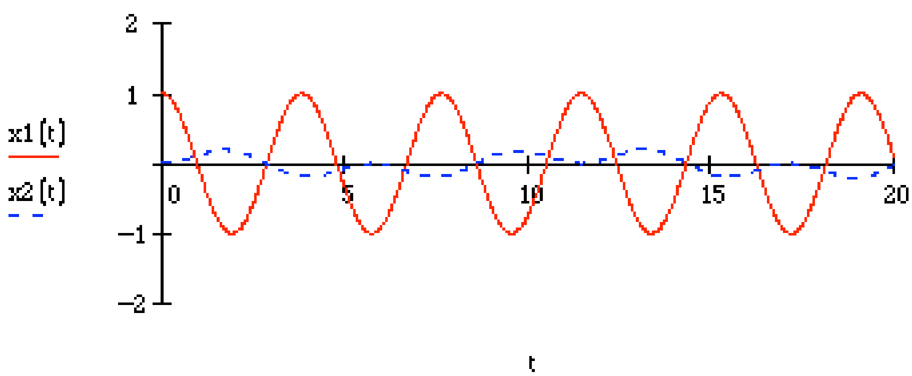
$$A_1 = 0.1246$$
$$A_2 = 0.9983$$

So,

$$x_1(t) = 0.001727\cos(0.5439t) + 0.9983\cos(1.6444t) \text{ mm}$$
$$x_2(t) = 0.1246\left[\cos(0.5439t) - \cos(1.6444t)\right] \text{ mm}$$

```
x1 (t) := 0.001727·cos (0.5439·t) + 0.9983·cos (1.644·t)

x2 (t) := 0.1246· (cos (0.5439·t) − cos (1.644·t))
```




(b) $k_2 = 30$

$$\det(-\omega^2 M + K) = \begin{vmatrix} -9\omega^2 + 54 & -30 \\ -30 & -\omega^2 + 30 \end{vmatrix} = 9\omega^4 - 32\omega^2 + 720 = 0$$

$$\omega^2 = 2.3795, 33.6205$$
$$\omega_1 = 1.5426$$
$$\omega_2 = 5.7983$$

Mode shapes:

Mode 1, $\omega_1^2 = 2.3795$

$$\begin{bmatrix} 32.5845 & -30 \\ -30 & 27.6205 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$30u_{11} = 27.6205u_{12}$$

$$u_{11} = 0.9207u_{12}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.9207 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 33.6205$

$$\begin{bmatrix} -248.5845 & -30 \\ -30 & -3.6205 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$30u_{21} = -3/6205u_{22}$$

$$u_{21} = -0.1207u_{22}$$

$$\mathbf{u}_2 = \begin{bmatrix} -0.1207 \\ 1 \end{bmatrix}$$

The solution is

$$x(t) = A_1 \sin(\omega_1 t + \phi_1)u_1 + A_2 \sin(\omega_2 t + \phi_2)u_2$$

Using initial conditions,

$$1 = A_1(0.9207)\sin\phi_1 + A_2(-0.1207\sin\phi_2 \qquad [1]$$

$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$

$$0 = A_1(0.9207)(1.5426)\cos\phi_1 + A_2(-0.1207)(5.7983)\cos\phi_2 \qquad [3]$$

$$0 = A_1(1.5426)\cos\phi_1 + A_2(5.7983)\cos\phi_2 \qquad [4]$$

From [3] and [4]

$$\phi_1 = \phi_2 = \pi/2$$

From [1] and [2]

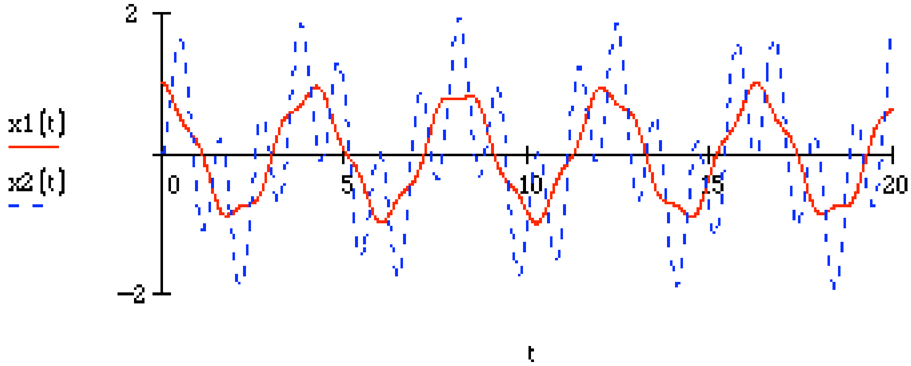
$$A_1 = 0.9602$$
$$A_2 = -0.9602$$

So,

$$x_1(t) = 0.8841\cos(1.5426t) + 0.1159\cos(5.7983t) \text{ mm}$$
$$x_2(t) = 0.9602[\cos(1.5426t) - \cos(5.7983t)] \text{ mm}$$

```
x1(t) := 0.8841·cos(1.5426·t) + 0.1159·cos(5.7983·t)

x2(t) := 0.9602·(cos(1.5426·t) − cos(5.7983·t))
```




(c) $k_2 = 300$

$$\det\left(-\omega^2 M + K\right) = \begin{vmatrix} -9\omega^2 + 324 & -300 \\ -300 & -\omega^2 + 300 \end{vmatrix} = 9\omega^4 - 3024\omega^2 + 7200 = 0$$
$$\omega^2 = 2.3981, 333.6019$$
$$\omega_1 = 1.5486$$
$$\omega_2 = 18.2648$$

Mode shapes:

Mode 1, $\omega_1^2 = 2.3981$

$$\begin{bmatrix} 302.4174 & -300 \\ -300 & 297.6019 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$302.4174 u_{11} = 300 u_{12}$$

$$u_{11} = 0.9920 u_{12}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.9920 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 333.6019$

$$\begin{bmatrix} -2678.4174 & -300 \\ -300 & -33.6019 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$300 u_{21} = 33.6019 u_{22}$$

$$u_{21} = -0.1120 u_{22}$$

$$\mathbf{u}_2 = \begin{bmatrix} -0.1120 \\ 1 \end{bmatrix}$$

The solution is

$$x(t) = A_1 \sin(\omega_1 t + \phi_1) u_1 + A_2 \sin(\omega_2 t + \phi_2) u_2$$

Using initial conditions

$$1 = A_1 (0.9920) \sin\phi_1 + A_2 (-0.1120) \sin\phi_2 \qquad [1]$$

$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$

$$0 = A_1 (0.9920)(1.5486) \cos\phi_1 + A_2 (-0.1120)(18.2648) \qquad [3]$$

$$0 = A_1 (1.5486) \cos\phi_1 + A_2 (18.2648) \cos\phi_2 \qquad [4]$$

From [3] and [4] $\qquad \phi_1 = \phi_2 = \pi / 2$

From [1] and [2], $\qquad A_1 = 0.9058$ and $A_2 = -0.9058$.

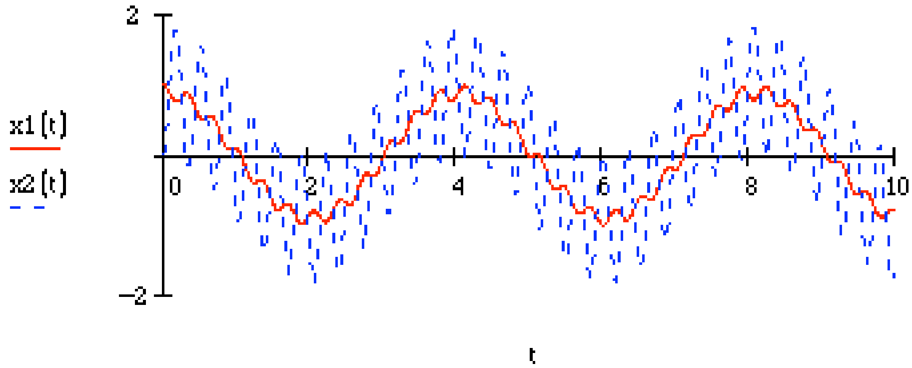
So,

$$x_1(t) = 0.8986 \cos(1.5486t) + 0.1014 \cos(18.2648t) \text{ mm}$$

$$x_2(t) = 0.9058 \left[ \cos(1.5486t) - \cos(18.2648t) \right] \text{ mm}$$

$$x1(t) := 0.8986 \cdot \cos(1.5486 \cdot t) + 0.1014 \cdot \cos(18.2648 \cdot t)$$

$$x2(t) := 0.9052 \cdot (\cos(1.5486 \cdot t) - \cos(18.2648 \cdot t))$$




As the value of $k_2$ increases the effect on mass 1 is small, but mass 2 oscillates similar to mass 1 with a superimposed higher frequency oscillation.

**4.17** Consider the system of Figure 4.1(a) described in matrix form by Eqs. (4.11), (4.9), and (4.6). Determine the natural frequencies in terms of the parameters $m_1$, $m_2$, $k_1$ and $k_2$. How do these compare to the two single-degree-of-freedom frequencies $\omega_1 = \sqrt{k_1 / m_1}$ and $\omega_2 = \sqrt{k_2 / m_2}$ ?

**Solution:**

The equation of motion is

$$M\ddot{x} + Kx = 0$$
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}x = 0$$

The characteristic equation is found from Eq. (4.19):

$$\det\left(-\omega^2 M + K\right) = 0$$
$$\begin{vmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{vmatrix}$$
$$m_1 m_2 \omega^4 - \left(k_1 m_2 + k_2\left(m_1 + m_2\right)\right)\omega^2 + k_1 k_2 = 0$$

$$\omega_{1,2}^2 = \frac{k_1 m_2 + k_2\left(m_1 + m_2\right) \pm \sqrt{\left[k_1 m_2 + k_2\left(m_1 + m_2\right)\right]^2 - 4m_1 m_2 k_1 k_2}}{2m_1 m_2}$$

So,

$$\omega_{1,2} = \sqrt{\frac{k_1 m_2 + k_2\left(m_1 + m_2\right) \pm \sqrt{\left[k_1 m_2 + k_2\left(m_1 + m_2\right)\right]^2 - 4m_1 m_2 k_1 k_2}}{2m_1 m_2}}$$

In two-degree-of-freedom systems, each natural frequency depends on all four parameters ($m_1$, $m_2$, $k_1$, $k_2$), while a single-degree-of-freedom system's natural frequency depends only on one mass and one stiffness.

**4.18** Consider the problem of Example 4.1.7 and use a trig identity to show the $x_1(t)$ experiences a beat. Plot the response to show the beat phenomena in the response.

**Solution** Applying the trig identity of Example 2.2.2 to $x_1$ yields

$$x_1(t) = (\cos\sqrt{2}t + \cos 2t) = \cos(\frac{\sqrt{2}-2}{2}t)\cos(\frac{\sqrt{2}+2}{2}t) = \cos 0.586t \cos 3.414t$$

Plotting $x_1$ and $\cos(0.586t)$ yields the clear beat:

$x(t) := \cos(0.586 \cdot t) \cos(3.414 \cdot t)$

$y(t) := \cos(0.586 \cdot t)$