Problems and Solutions Section 4.10 (4.91 through 4.98)

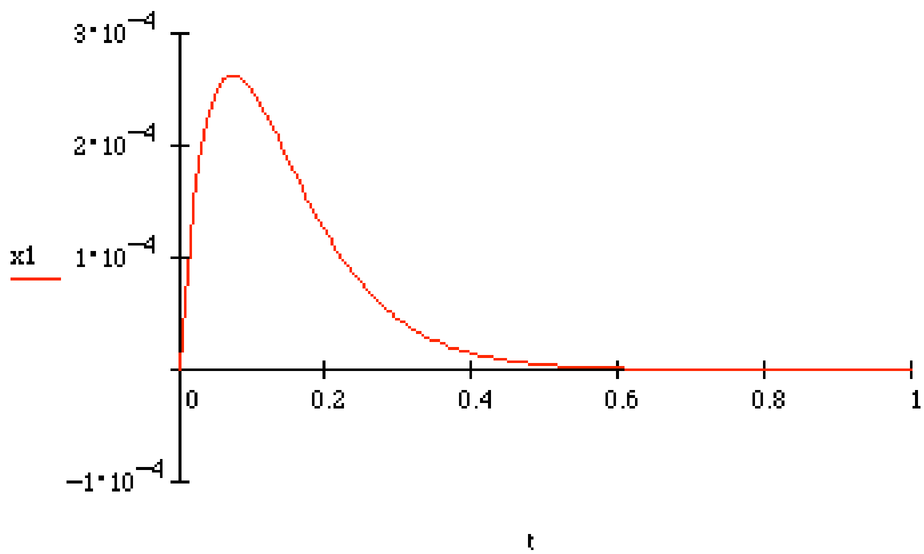
4.91* Solve the system of Example 1.7.3 for the vertical suspension system of a car with m = 1361 kg, k = 2.668 x 10^5 N/m, and c = 3.81 x 10^4 kg/s subject to the initial conditions of $x(0) = 0$ and $v(0) = 0.01$ m/s^2.

Solution: Use a Runge Kutta routine such as the one given in Mathcad here or use the toolbox:

$m := 1361 \qquad k := 2.668\cdot 10^5 \qquad c := 3.81\cdot 10^4$

$$X := \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \qquad A := \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{bmatrix} \qquad D(t,X) := A\cdot X$$

$Z := \mathrm{rkfixed}(X, 0, 20, 3000, D) \qquad t := Z^{<0>} \qquad x1 := Z^{<1>}$



4.92* Solve for the time response of Example 4.4.3 (i.e., the four-story building of Figure 4.9). Compare the solutions obtained with using a modal analysis approach to a solution obtained by numerical integration.

Solution: The following code provides the numerical solution.

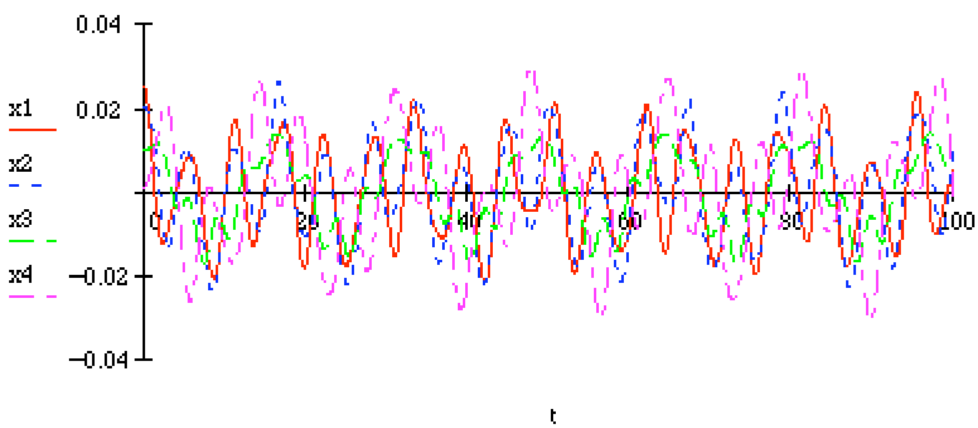
$$I := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M := 4000 \cdot I$$

$$K := \begin{bmatrix} 10000 & -5000 & 0 & 0 \\ -5000 & 10000 & -5000 & 0 \\ 0 & -5000 & 10000 & -5000 \\ 0 & 0 & -5000 & 5000 \end{bmatrix} \qquad C := O \qquad X := \begin{bmatrix} 0.025 \\ 0.02 \\ 0.01 \\ 0.001 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A := \mathrm{augment}\left(\mathrm{stack}\left(O, -M^{-1} \cdot K\right), \mathrm{stack}\left(I, -M^{-1} C\right)\right)$$

$$D(t, X) := A \cdot X \qquad Z := \mathrm{rkfixed}(X, 0, 200, 3000, D)$$

$$t := Z^{\langle 0 \rangle} \qquad x1 := Z^{\langle 1 \rangle} \qquad x2 := Z^{\langle 2 \rangle} \qquad x3 := Z^{\langle 3 \rangle} \qquad x4 := Z^{\langle 4 \rangle}$$



which compares very well with the plots given in Figure 4.11 obtained by plotting the modal equations. One could also plot the modal response and numerical response on the same graph to see a more rigorous comparison.

4.93* Reproduce the plots of Figure 4.13 for the two-degree of freedom system of Example 4.5.1 using a code.

Solution: Use any of the codes. The trick here is to construct the damping matrix from the given modal information by first creating it in modal form and then transforming it back to physical coordinates as indicated in the following Mathcad session:

$$M := \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \qquad P := \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad K := \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

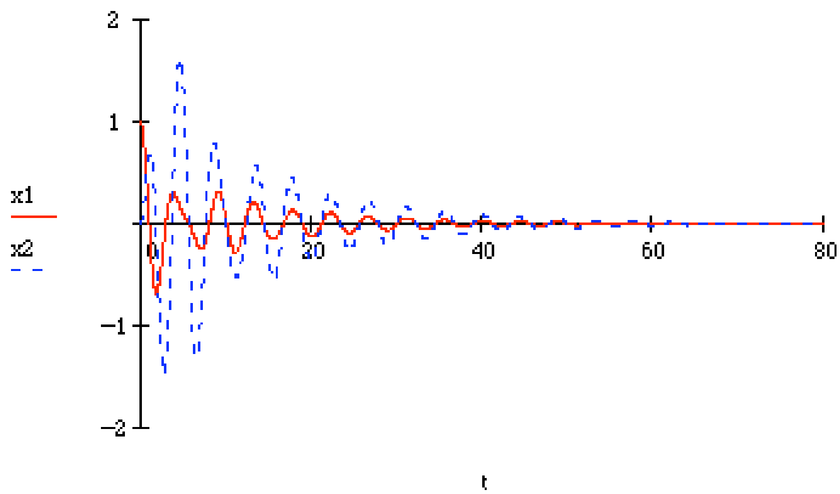
$$Mr := \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \omega 1 := \sqrt{2} \qquad \omega 2 := 2 \qquad \zeta 1 := 0.05 \qquad \zeta 2 := 0.1$$

$$Ac := \begin{bmatrix} 2 \cdot \zeta 1 \cdot \omega 1 & 0 \\ 0 & 2 \cdot \zeta 2 \cdot \omega 2 \end{bmatrix} \qquad C := Mr \cdot P \cdot Ac \cdot P^T \cdot Mr$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} C\right)\right) \qquad X := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(t, X) := A \cdot X$$

$$Z := \text{rkfixed}(X, 0, 80, 4000, D) \qquad t := Z^{\langle 0 \rangle} \qquad x1 := Z^{\langle 1 \rangle} \qquad x2 := Z^{\langle 2 \rangle}$$



4.94*. Consider example 4.8.3 and a) using the damping ratios given, compute a damping matrix in physical coordinates, b) use numerical integration to compute the response and plot it, and c) use the numerical code to design the system so that all 3 physical coordinates die out within 5 seconds (i.e., change the damping matrix until the desired response results).

Solution: A Mathcad solution is presented. The damping matrix is found, as in the previous problem, by keeping track of the various transformations. Using the notation of the text, the damping matrix is constructed from:

$$C = M^{1/2} P \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 \\ 0 & 2\zeta_2\omega_2 & 0 \\ 0 & 0 & 2\zeta_3\omega_3 \end{bmatrix} P^T M^{1/2} = \begin{bmatrix} 1.062\times 10^3 & -679.3 & 187.0 \\ -679.3 & 2.785\times 10^3 & 617.8 \\ 187.0 & 617.8 & 2.041\times 10^3 \end{bmatrix}$$

as computed using the code that follows. With this form of the matrix the damping ratios are adjusted until the desired criteria are met:

$$I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \cdot 10^3$$

$$\omega1 := 5.3872 \quad \omega2 := 10.6755 \quad \omega3 := 30.1166$$

$$K := \begin{bmatrix} 30 & -30 & 0 \\ -30 & 38 & -8 \\ 0 & -8 & 88 \end{bmatrix} \cdot 10^4 \quad Mr := \begin{bmatrix} \sqrt{0.4\cdot 10^3} & 0 & 0 \\ 0 & \sqrt{2\cdot 10^3} & 0 \\ 0 & 0 & \sqrt{8\cdot 10^3} \end{bmatrix}$$

$$\zeta1 := 0.2 \quad \zeta2 := 0.05 \quad \zeta3 := 0.05$$

$$P := \begin{bmatrix} -0.4116 & -0.1021 & 0.9056 \\ -0.8848 & -0.1935 & -0.4239 \\ -0.2185 & 0.9758 & 0.0106 \end{bmatrix} \quad Ac := \begin{bmatrix} 2\cdot\omega1\cdot\zeta1 & 0 & 0 \\ 0 & 2\cdot\omega2\cdot\zeta2 & 0 \\ 0 & 0 & 2\cdot\omega3\cdot\zeta3 \end{bmatrix}$$

$$C := Mr\cdot P\cdot Ac\cdot P^T\cdot Mr \qquad A := \mathrm{augment}\left(\mathrm{stack}\left(O, -M^{-1}\cdot K\right), \mathrm{stack}\left(I, -M^{-1}\, C\right)\right)$$

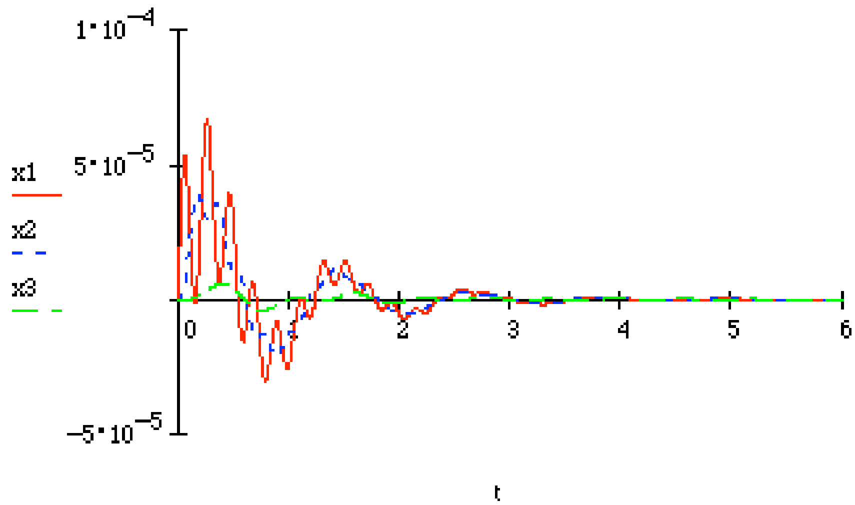
In changing the damping ratios it is best to start with the rubber component which is the first mode-damping ratio. Doubling it nails the first two coordinates but does not affect the third coordinate enough. Hence the second mode-damping ratio must be changed (doubled here) to attack this mode. This could be accomplished by adding a viscoelastic strip as described in Chapter 5 to the metal. Thus the ratios given in the code above do the trick as the following plots show. Note also how much the damping matrix changes.

$$X := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B := \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} \quad f := M^{-1} \cdot B \quad C = \begin{bmatrix} 1.138 \cdot 10^3 & -313.286 & 208.132 \\ -313.286 & 4.536 \cdot 10^3 & 805.984 \\ 208.132 & 805.984 & 8.958 \cdot 10^3 \end{bmatrix}$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_0 \\ f_1 \\ f_2 \end{bmatrix} \cdot \left(\Phi(t) - \Phi(t - 0.001)\right)$$

kfixed (X , 0 , 15 , 4000 , D)

$t := Z^{<0>}$ $\quad x1 := Z^{<1>}$ $\quad x2 := Z^{<2>}$ $\quad x3 := Z^{<3>}$



4.95*. Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{ m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ m/s}$$

Solution: The following Mathcad session illustrates the numerical solution of this problem using a Runge Kutta solver.

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad X := \begin{bmatrix} 0 \\ 0.1 \\ 1 \\ 0 \end{bmatrix}$$

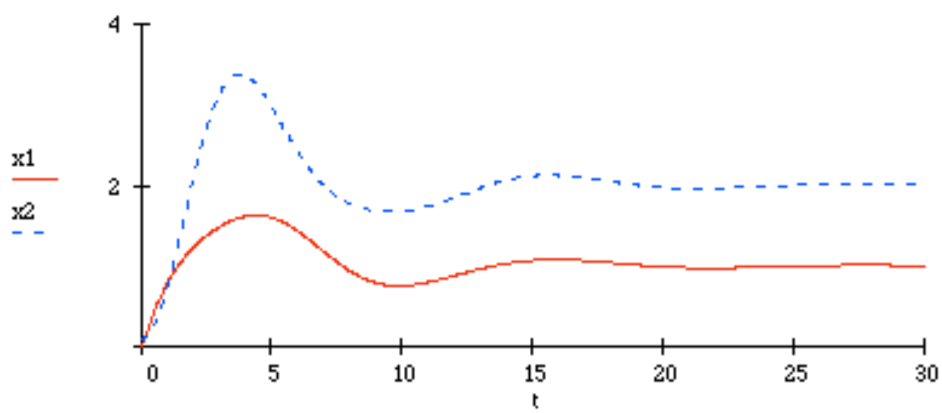
$$C := \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1}\cdot K\right), \text{stack}\left(I, -M^{-1}\, C\right)\right)$$

$$D(t, X) := A\cdot X + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

$$Z := \text{rkfixed}(X, 0, 100, 3000, D)$$

$$t := Z^{<0>} \qquad x1 := Z^{<1>} \qquad x2 := Z^{<2>}$$



4.96* Consider the following system excited by a pulse of duration 0.1 s (in Newtons):

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0.3 & -0.05 \\ -0.05 & 0.05 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[\Phi(t-1) - \Phi(t-3)]$$

and subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \text{m}, \quad \mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{m/s}$$

Compute and plot the response of the system. Here Φ indicates the Heaviside Step Function introduced in Section 3.2.

Solution: The following Mathcad solution (see example4.10.3 for the other codes) gives the solution:

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad X := \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ 0 \end{bmatrix}$$

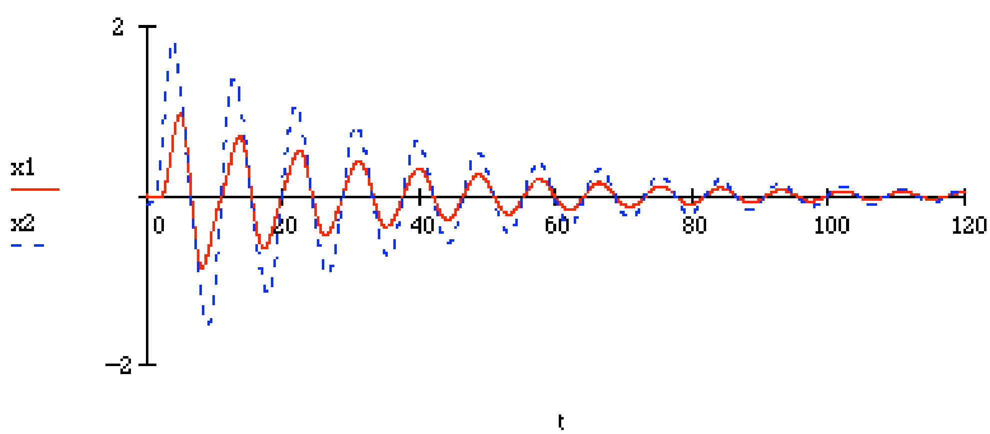
$$C := \begin{bmatrix} 0.33 & -0.05 \\ -0.05 & 0.05 \end{bmatrix} \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad f := M^{-1} \cdot B$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1}\,C\right)\right)$$

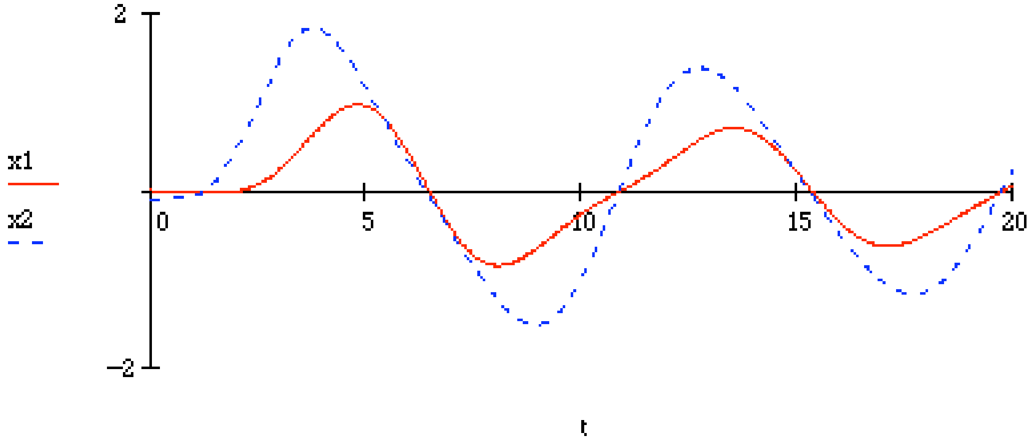
$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ f_0 \\ f_1 \end{bmatrix} \cdot \left(\Phi(t-1) - \Phi(t-3)\right)$$

$$Z := \text{rkfixed}(X, 0, 120, 3000, D)$$

$$t := Z^{<0>} \quad x1 := Z^{<1>} \quad x2 := Z^{<2>}$$



It is also interesting to examine the first 20 seconds more closely to see the effect of the impact:



Note that the impact has much more of an effect on the response than does the initial condition.

4.97.* Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 30 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{m/s}$$

Solution: Following the codes of Example 4.10.2 yields the solution directly.

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M := \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \qquad K := \begin{bmatrix} 30 & -1 \\ -1 & 1 \end{bmatrix} \qquad X := \begin{bmatrix} 0 \\ 0.1 \\ 1 \\ 0 \end{bmatrix}$$

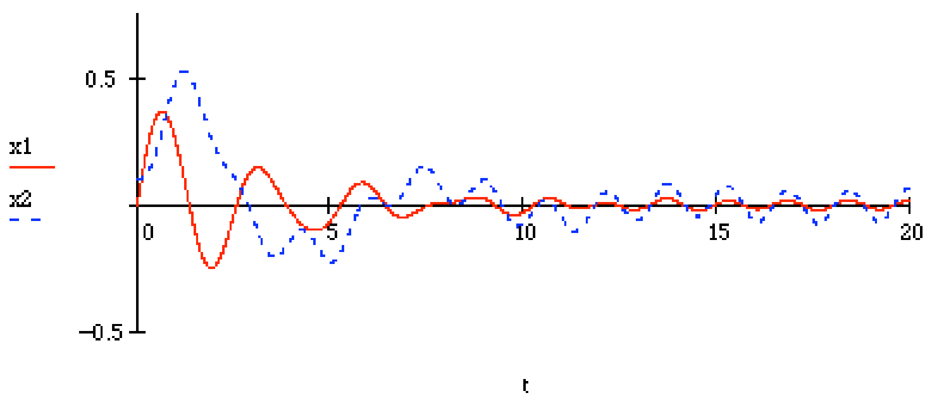
$$C := \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \qquad B := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad f := M^{-1} \cdot B$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} C\right)\right)$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ f_0 \\ f_1 \end{bmatrix} \cdot \sin(4 \cdot t)$$

$$Z := \text{rkfixed}(X, 0, 20, 3000, D)$$

$$t := Z^{\langle 0 \rangle} \qquad x1 := Z^{\langle 1 \rangle} \qquad x2 := Z^{\langle 2 \rangle}$$



4.98.* Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} 500 & -100 & 0 & 0 \\ -100 & 200 & -100 & 0 \\ 0 & -100 & 200 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \end{bmatrix} \text{m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{m/s}$$

Solution: Again follow Example 4.10.2 for the various codes. Mathcad is given.

$$I := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$K := \begin{bmatrix} 500 & -100 & 0 & 0 \\ -100 & 200 & -100 & 0 \\ 0 & -100 & 200 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \quad C := \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad X := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad f := M^{-1} \cdot B$$

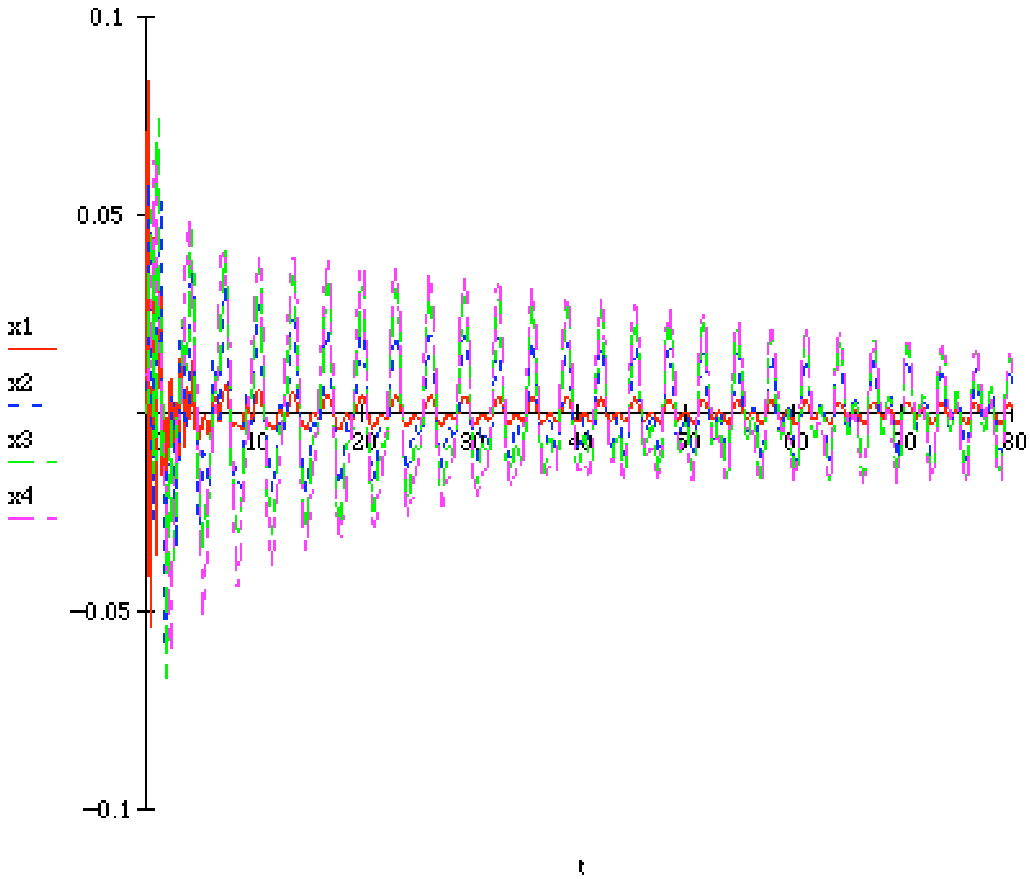
$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1}\,C\right)\right)$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \cdot \sin(4 \cdot t)$$

$$Z := \text{rkfixed}(X, 0, 200, 3000, D)$$

$$x2 := Z^{\langle 2 \rangle} \quad x3 := Z^{\langle 3 \rangle} \quad x4 := Z^{\langle 4 \rangle}$$

$$t := Z^{\langle 0 \rangle} \quad x1 := Z^{\langle 1 \rangle}$$



0.1
0.05
−0.05
−0.1
x1
x2
x3
x4
10
20
30
40
50
60
70
80
t