Problems and Solutions for Section 4.2 (4.19 through 4.33)

4.19 Calculate the square root of the matrix

$$M = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

$$\left[\text{Hint: Let } M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}; \text{ calculate } \left(M^{1/2}\right)^2 \text{ and compare to } M.\right]$$

Solution: Given:

$$M = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

If

$$M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}, \text{ then}$$

$$M = M^{1/2}M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}\begin{bmatrix} a & -b \\ -b & c \end{bmatrix} = \begin{bmatrix} a^2+b^2 & -ab-bc \\ -ab-bc & b^2+c^2 \end{bmatrix} = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

This yields the 3 nonlinear algebraic equations:

$$a^2 + b^2 = 13$$
$$ab + bc = 10$$
$$b^2 + c^2 = 8$$

There are several possible solutions but only one that makes $M^{1/2}$ positive definite which is $a = 3$, $b = c = 2$ as determined below in Mathcad. Choosing these values results in

$$M^{1/2} = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$a := 1 \qquad b := 1 \qquad c := 1$

given

$$a^2 + b^2 = 13$$

$$a \cdot b + b \cdot c = 10$$

$$b^2 + c^2 = 8$$

$$\text{find}(a, b, c) = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

4.20 Normalize the vectors

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

first with respect to unity (i.e., $x^T x = 1$) and then again with respect to the matrix M (i.e., $x^T Mx = 1$), where

$$M = \begin{bmatrix} 3 & -0.1 \\ -0.1 & 2 \end{bmatrix}$$

Solution:

(a) Normalize the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\alpha_1 = \frac{1}{\sqrt{x^T x}} = \frac{1}{\sqrt{5}}$$

Normalized:

$$\mathbf{x}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.4472 \\ -0.8944 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\alpha_2 = \frac{1}{\sqrt{x^T x}} = \frac{1}{5}$$

Normalized:

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$\alpha_3 = \frac{1}{\sqrt{x^T x}} = \frac{1}{\sqrt{0.02}}$$

Normalized:

$$\mathbf{x}_3 = \sqrt{50}\begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

(b) Mass normalize the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\alpha_1 = \frac{1}{\sqrt{\mathbf{x}^T M\mathbf{x}}} = \frac{1}{\sqrt{11.4}}$$

Mass normalized:

$$\mathbf{x}_1 = \frac{1}{\sqrt{11.4}}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.2962 \\ -.5923 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\alpha_2 = \frac{1}{\sqrt{\mathbf{x}^T M\mathbf{x}}} = \frac{1}{\sqrt{50}}$$

$$\mathbf{x}_2 = \frac{1}{\sqrt{50}}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7071 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$\alpha_3 = \frac{1}{\sqrt{x^T Mx}} = \frac{1}{\sqrt{0.052}}$$

Mass normalized:

$$\mathbf{x}_3 = \frac{1}{\sqrt{0.052}}\begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -0.4385 \\ 0.4385 \end{bmatrix}$$

4.21 For the example illustrated in Figure P4.1 with $c_1 = c_2 = c_3 = 0$, calculate the matrix $\tilde{K}$.

Solution:

From Figure 4.1,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}x = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} m_1^{-1/2} & 0 \\ 0 & m_2^{-1/2} \end{bmatrix}\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} m_1^{-1/2} & 0 \\ 0 & m_2^{-1/2} \end{bmatrix}$$

$$\tilde{K} = \begin{bmatrix} m_1^{-1}\left(k_1 + k_2\right) & -m_1^{-1/2}m_2^{-1/2}k_2 \\ -m_1^{-1/2}m_2^{-1/2}k_2 & m_1^{-1}\left(k_2 + k_3\right) \end{bmatrix}$$

Since $\tilde{K}^T = \tilde{K}, \tilde{K}$ is symmetric.

Using the numbers given in problem 4.2 yields

$$\tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix}$$

This is obviously symmetric.

4.22 Repeat Example 4.2.5 using eight decimal places. Does $P^T P = 1$, and does $P^T \tilde{K} P = \Lambda = \text{ diag } \left[\omega_1^2 \ \ \omega_2^2\right]$ exactly?

Solution: From Example 4.2.5,

$$\tilde{K} = \begin{bmatrix} 12 & -1 \\ -1 & 3 \end{bmatrix} \Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 15\lambda + 35 = 0$$
$$\Rightarrow \lambda_1 = 2.89022777, \text{ and } \lambda_2 = 12.10977223$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 2.89022777$$
$$\begin{bmatrix} 9.10977223 & -1 \\ -1 & 0.10977223 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow 9.10977223 v_{11} = v_{12}$$
$$\|v_1\| = \sqrt{v_{11}^2 + v_{12}^2} = \sqrt{v_{11}^2 + (9.10977223)^2 v_{11}^2} = 1$$
$$\Rightarrow v_{11} = 0.10911677 \text{ and } v_{12} = 0.99402894$$
$$\mathbf{v}_1 = \begin{bmatrix} 0.10911677 & 0.99402894 \end{bmatrix}^T$$
$$\lambda_2 = 12.10977223$$
$$\Rightarrow \begin{bmatrix} -0.10977223 & -1 \\ -1 & -9.10977223 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow v_{21} = 9.10977223 v_{22}$$
$$\|v_2\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{(-9.10977223)^2 v_{22}^2 + v_{22}^2} = 1$$
$$v_{21} = -9.10911677, \text{ and } v_{22} = -0.99402894$$
$$\mathbf{v}_2 = \begin{bmatrix} 0.99402894 & 0.10911677 \end{bmatrix}^T$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.10911677 & -0.99402894 \\ 0.99402894 & 0.10911677 \end{bmatrix}$

Check $P^T P = I$

$$P^T P = \begin{bmatrix} 1.00000000 & 0 \\ 0 & 1.00000000 \end{bmatrix} = I \text{ (to 8 decimal places)}$$

Check $P^T \tilde{K} P = \Lambda = \text{diag}\left(\lambda_1, \lambda_2\right)$

$$\Lambda = P^T \tilde{K} P = \begin{bmatrix} 2.89022778 & 0.00000002 \\ 0.00000002 & 12.10977227 \end{bmatrix}$$

$$\operatorname{diag}\left(\lambda_1, \lambda_2\right) = \begin{bmatrix} 2.89022777 & 0 \\ 0 & 12.10977223 \end{bmatrix}$$

This is accurate to 7 decimal places.

4.23 Discuss the relationship or difference between a mode shape of equation (4.54) and an eigenvector of $\tilde{K}$.

Solution:

The relationship between a mode shape, **u**, of $M\ddot{x} + Kx = 0$ and an eigenvector, **v**, of $\tilde{K} = M^{-1/2}KM^{-1/2}$ is given by

$$\mathbf{v}_i = M^{1/2}\mathbf{u}_i \quad \text{or} \quad \mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

If **v** is normalized, then **u** is mass normalized.

This is shown by the relation

$$\mathbf{v}_i^T\mathbf{v}_i = 1 = \mathbf{u}_i^T M\mathbf{u}_i$$

4.24 Calculate the units of the elements of matrix $\tilde{K}$.

Solution:

$\tilde{K} = M^{-1/2}KM^{-1/2}$

$M^{-1/2}$ has units kg$^{-1/2}$

K has units N/m = kg/s^2

So, $\tilde{K}$ has units $\left(\text{kg}^{-1/2}\right)\left(\text{kg/s}^2\right)\left(\text{kg}^{-1/2}\right) = s^{-2}$

4.25 Calculate the spectral matrix Λ and the modal matrix P for the vehicle model of Problem 4.14, Figure P4.14.

Solution: From Problem 4.14:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 0.5 & -3.162 \\ -3.162 & 220 \end{bmatrix}$$

$$\begin{vmatrix} 0.5-\lambda & -3.162 \\ -3.162 & 220-\lambda \end{vmatrix} = \lambda^2 - 220.5\lambda + 100 = 0$$

$$\lambda_{1,2} = 0.454, 220.05$$

The spectral matrix is

$$\Lambda = \operatorname{diag}\left(\lambda_1\right) = \begin{bmatrix} 0.454 & 0 \\ 0 & 220.05 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0.454$$

$$\begin{bmatrix} 0.0455 & -3.162 \\ -3.162 & 219.55 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \Rightarrow v_{11} = 69.426v_{12}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{22}^2} = \sqrt{(69.426)^2 v_{12}^2 + v_{12}^2} = 69.434v_{12} = 1$$

$$\Rightarrow v_{12} = 0.0144, \text{ and } v_{11} = 0.9999$$

$$\Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.9999 \\ 0.0144 \end{bmatrix}$$

$$\lambda_2 = 220.05$$

$$\begin{bmatrix} -219.55 & -3.162 \\ -3.162 & -0.0455 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$v_{21} = 0.0144v_{22}$$

$$\|v_2\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{(-0.0144)^2 v_{22}^2 + v_{22}^2} = 1.0001v_{22} = 1$$

$$\Rightarrow v_{22} = 0.9999, \text{ and } v_{21} = -0.0144$$

$$\Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.0144 \\ 0.9999 \end{bmatrix}$$

The modal matrix is

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.9999 & -0.0144 \\ 0.0144 & 0.9999 \end{bmatrix}$$

4.26 Calculate the spectral matrix Λ and the modal matrix P for the subway car system of Problem 4.12, Figure P4.12.

Solution: From problem 4.12 and Figure P4.12,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 280{,}000 & -280{,}000 \\ -280{,}000 & 280{,}000 \end{bmatrix}\mathbf{x} = 0$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix}$$

$$\begin{vmatrix} 140-\lambda & -140 \\ -140 & 140-\lambda \end{vmatrix} = \lambda^2 - 280\lambda = 0$$

$$\lambda_{1,2} = 0, 280$$

The spectral matrix is

$$\Lambda = \text{diag}\left(\lambda_i\right) = \begin{bmatrix} 0 & 0 \\ 0 & 280 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0$$

$$\begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$v_{11} = v_{12}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{12}^2} = \sqrt{v_{12}^2 + v_{12}^2} = 1.414v_{12} = 1$$

$$v_{12} = 0.7071$$

$$v_{11} = 0.7071$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

$$\lambda_2 = 280$$

$$\begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \Rightarrow v_{21} = v_{22}$$

$$\Rightarrow \|v_2\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{v_{22}^2 + v_{22}^2} = 14.14v_{12} = 1 \Rightarrow v_{22} = 0.7071, v_{21} = -0.7071$$

$$\Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

The modal matrix is $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$

4.27 Calculate $\tilde{K}$ for the torsional vibration example of Problem 4.11. What are the units of $\tilde{K}$?

Solution: From Problem 4.11,

$$J\ddot{\theta} + K\theta = J_2\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\theta} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\theta = 0$$

$$\tilde{K} = J^{-1/2}KJ^{-1/2}$$

$$J^{-1/2} = J_2^{-1/2}\begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = J_2^{-1/2}\begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}J_2^{-1/2}\begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = \frac{k}{J_2}\begin{bmatrix} 0.6667 & -0.5774 \\ -0.5774 & 1 \end{bmatrix}$$

The units of $\tilde{K}$ are

$$\left(\frac{kg \cdot m^2}{\text{rad}}\right)^{-1/2}\left(\frac{N \cdot m}{\text{rad}}\right)\left(\frac{kg \cdot m^2}{\text{rad}}\right)^{-1/2} = s^{-2}$$

4.28 Consider the system in the Figure P4.28 for the case where m_1 = 1 kg, m_2 = 4 kg, k_1 = 240 N/m and k_2=300 N/m. Write the equations of motion in vector form and compute each of the following

a) the natural frequencies
b) the mode shapes
c) the eigenvalues
d) the eigenvectors
e) show that the mode shapes are not orthogonal
f) show that the eigenvectors are orthogonal
g) show that the mode shapes and eigenvectors are related by $M^{-\frac{1}{2}}$
h) write the equations of motion in modal coordinates

Note the purpose of this problem is to help you see the difference between these various quantities.

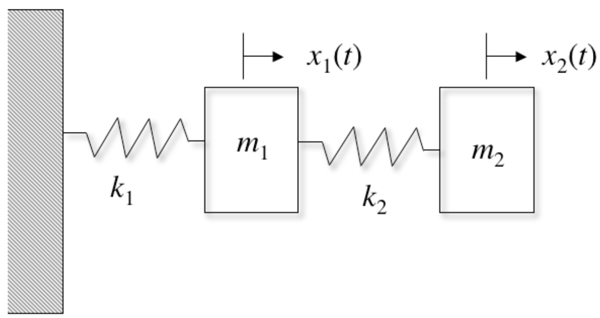


Figure P1.28 A two-degree of freedom system

Solution From a free body diagram, the equations of motion in vector form are

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 540 & -300 \\ -300 & 300 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies can be calculated in two ways. The first is using the determinant following example 4.1.5:

a) $$\det(-\omega^2 M + K) = 0 \Rightarrow \underline{\omega_1 = 5.5509, \omega_2 = 24.1700 \text{ rad/s}}$$

The second approach is to compute the eigenvalues of the matrix $\tilde{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$ following example 4.4.4, which yields the same answers. The mode shapes are calculate following the procedures of example 4.1.6 or numerically using `eig(K,M)` in Matlab

b) $$\mathbf{u}_1 = \begin{bmatrix} 0.5076 \\ 0.8616 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0.9893 \\ -0.1457 \end{bmatrix}$$

The eigenvectors are vectors that satisfy $\tilde{K}\mathbf{v} = \lambda\mathbf{v}$, where λ are the eigenvalues. These can be computed following example 4.2.2, or using `[V,Dv]=eig(Kt)` in Matlab. The eigenvalues and eigenvectors are

c) $$\lambda_1 = 30.8120, \quad \lambda_2 = 584.1880,$$

d) $$\mathbf{v}_1 = \begin{bmatrix} 0.2826 \\ 0.9592 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -0.9592 \\ 0.2826 \end{bmatrix}$$

To show that the mode shapes are not orthogonal, show that $\mathbf{u}_1^T\mathbf{u}_2 \neq 0$:

e) $\mathbf{u}_1^T\mathbf{u}_2 = (0.5076)(0.9893) + (0.8616)(-0.1457) = 0.3767 \neq 0$

To show that the eigenvectors are orthogonal, compute the inner product to show that $\mathbf{v}_1^T\mathbf{v}_2 = 0$:

f) $\mathbf{v}_1^T\mathbf{v}_2 = (0.2826)(-0.9592) + (0.9592)(0.2826) = 0$

To solve the next part merely compute $M^{-1/2}\mathbf{v}_2$ and show that it is equal to $\mathbf{u}_2$ (see the discussion at the top of page 262.

g) $M^{-1/2}\mathbf{v}_2 = \begin{bmatrix} 0.9592 \\ -0.1413 \end{bmatrix}$, normalize to get $\begin{bmatrix} -0.9893 \\ 0.1457 \end{bmatrix} = -\mathbf{u}_2$

Likewise, $M^{-1/2}\mathbf{v}_1 = \mathbf{u}_1$. Note that if you use Matlab you'll automatically get normalized vectors. But the product $M^{-1/2}\mathbf{v}_2$ will not be normalized, so it must be normalized before comparing it to $\mathbf{u}_2$.

h) We can write down the modal equations, just as soon as we know the eigenvalues (squares of the frequencies). They are:

$$\ddot{r}_1(t) + 30.812 r_1(t) = 0$$
$$\ddot{r}_2(t) + 583.189 r_2(t) = 0$$

4.29 Consider the following system:

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

where M is in kg and K is in N/m. (a) Calculate the eigenvalues of the system. (b) Calculate the eigenvectors and normalize them.

Solution: Given:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -0.5 \\ -0.5 & 0.25-\lambda \end{vmatrix} = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_{1,2} = 0.162, 3.088$$

The spectral matrix is

$$\Lambda = \text{diag}\left(\lambda_i\right) = \begin{bmatrix} 0.162 & 0 \\ 0 & 3.088 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0.162$$

$$\begin{bmatrix} 2.838 & -0.5 \\ -0.5 & 0.088 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \Rightarrow v_{11} = 1.762 v_{21}$$

$$\left\| v_1 \right\| = \sqrt{v_{11}^2 + v_{21}^2} = \sqrt{\left(0.1762\right)^2 v_{21}^2 + v_{21}^2} = 1.015 v_{21} = 1$$

$$v_{21} = 0.9848 \text{ and } v_{11} = 0.1735 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.1735 \\ 0.9848 \end{bmatrix}$$

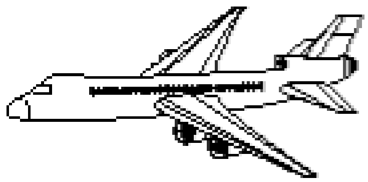
$$\lambda_2 = 3.088$$

$$\begin{bmatrix} -0.088 & -0.5 \\ -0.5 & -2.838 \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \Rightarrow v_{12} = 1.762 v_{22}$$

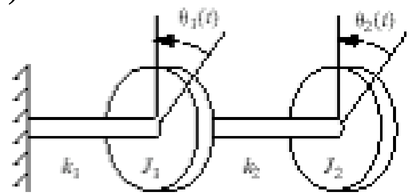
$$\left\| v_2 \right\| = \sqrt{v_{12}^2 + v_{22}^2} = \sqrt{\left(-5.676\right)^2 v_{22}^2 + v_{22}^2} = 5.764 v_{22} = 1$$

$$\Rightarrow v_{22} = 0.1735 \text{ and } v_{12} = -0.9848 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.9848 \\ 0.1735 \end{bmatrix}$$

4.30 The torsional vibration of the wing of an airplane is modeled in Figure P4.30. Write the equation of motion in matrix form and calculate the natural frequencies in terms of the rotational inertia and stiffness of the wing (See Figure 1.22).



Airplane wing with engines



Wing modeled as two shafts and two disks for torsional vibration

Solution: From Figure 1.22,

$$k_1 = \frac{GJ_p}{l_1} \text{ and } k_2 = \frac{GJ_p}{l_2}$$

Equation of motion:

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\theta = 0$$

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} GJ_p\left(\frac{1}{l_1} + \frac{1}{l_2}\right) & \frac{-GJ_p}{l_2} \\ \frac{-GJ_p}{l_2} & \frac{GJ_p}{l_2} \end{bmatrix}\theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} \frac{GJ_p}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) & \frac{-GJ_p}{l_2\sqrt{J_1J_2}} \\ \frac{-GJ_p}{l_2\sqrt{J_1J_2}} & \frac{GJ_p}{J_2l_2} \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \begin{bmatrix} \frac{GJ_p}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) - \lambda & \frac{-GJ_p}{l_2\sqrt{J_1J_2}} \\ \frac{-GJ_p}{l_2\sqrt{J_1J_2}} & \frac{GJ_p}{J_2l_2} - \lambda \end{bmatrix}$$

Solving for λ yields

$$\lambda_{1,2} = \frac{GJ_p}{2}\left[\frac{1}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + \frac{1}{J_2l_2}\right] \pm \frac{GJ_p}{2}\sqrt{\left[\frac{1}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + \frac{1}{J_2l_2}\right]^2 - \frac{4}{J_1J_2l_1l_2}}$$

The natural frequencies are

$$\omega_1 = \sqrt{\lambda_1} \text{ and } \omega_2 = \sqrt{\lambda_2}$$

4.31 Calculate the value of the scalar a such that $\mathbf{x}_1 = [a \quad -1 \quad 1]^T$ and $\mathbf{x}_2 = [1 \quad 0 \quad 1]^T$ are orthogonal.

Solution: To be orthogonal, $\mathbf{x}_1^T\mathbf{x}_2 = 0$

So, $\mathbf{x}_1^T\mathbf{x}_2 = \begin{bmatrix} a & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = a + 1 = 0$. Therefore, a = -1.

4.32 Normalize the vectors of Problem 4.31. Are they still orthogonal?

Solution: From Problem 4.31, with a = -1,

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Normalize $\mathbf{x}_1$:

$$(\alpha x_1)^T(\alpha x_1) = 1$$

$$a^2\begin{bmatrix} -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 3\alpha^2 = 1$$

$$\alpha = 0.5774$$

So, $\mathbf{x}_1 = 0.5774\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Normalize $\mathbf{x}_2$:

$$(\alpha x_2)^T(\alpha x_2) = 1$$

$$a^2\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2\alpha^2 = 1$$

$$\alpha = 0.7071$$

So, $\mathbf{x}_2 = 0.7071\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Check orthogonality:

$$\mathbf{x}_1^T\mathbf{x}_2 = (0.5774)(0.7071)\begin{bmatrix} -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \text{ Still orthogonal}$$

4.33 Which of the following vectors are normal? Orthogonal?

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \end{bmatrix}$$

Solution:

Check vectors to see if they are normal:

$$\|x_1\| = \sqrt{1/2 + 0 + 1/2} = \sqrt{1} = 1 \qquad \text{Normal}$$

$$\|\mathrm{x}_2\| = \sqrt{.1^2 + .2^2 + .3^2} = \sqrt{.14} = 0.3742 \qquad \text{Not normal}$$

$$\|\mathrm{x}_3\| = \sqrt{.3^2 + .4^2 + .3^2} = \sqrt{.34} = 0.5831 \qquad \text{Not normal}$$

Check vectors to see if they are orthogonal:

$$\mathbf{x}_1^T \mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} .1 \\ .2 \\ .3 \end{bmatrix} = .2828 \qquad \text{Not orthogonal}$$

$$\mathbf{x}_2^T \mathbf{x}_3 = \begin{bmatrix} .1 & .2 & .3 \end{bmatrix} \begin{bmatrix} .3 \\ .4 \\ .3 \end{bmatrix} = 0.2 \qquad \text{Not orthogonal}$$

$$\mathbf{x}_3^T \mathbf{x}_1 = \begin{bmatrix} .3 & .4 & .3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = 0.4243 \qquad \text{Not orthogonal}$$

$\therefore$ Only $\mathbf{x}_1$ is normal, and none are orthogonal.