Problems and Solutions for Section 4.3 (4.34 through 4.43)

4.34 Solve Problem 4.11 by modal analysis for the case where the rods have equal stiffness (i.e., $k_1 = k_2$), $J_1 = 3J_2$, and the initial conditions are $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\dot{\mathbf{x}}(0) = \mathbf{0}$.

Solution: From Problem 4.11 and Figure P4.11, with $k = k_1 = k_2$ and $J_1 = 3J_2$:

$$J_2 \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\theta} + k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \theta = 0$$

Calculate eigenvalues and eigenvectors:

$$J^{-1/2} = J_2^{-1/2} \begin{bmatrix} \dfrac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = J^{-1/2} K J^{-1/2} = \frac{k}{J_2} \begin{bmatrix} \dfrac{2}{3} & \dfrac{-1}{\sqrt{3}} \\ \dfrac{-1}{\sqrt{3}} & 1 \end{bmatrix} \Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - \frac{5k}{3J_2}\lambda + \frac{k^2}{3J_2^2} = 0$$

$$\lambda_1 = \frac{\left(5 - \sqrt{13}\right)k}{6J_2} \Rightarrow \quad \omega_1 = \sqrt{\lambda_1}, \text{ and } \quad \frac{\left(5 + \sqrt{13}\right)k}{6J_2} \quad \Rightarrow \omega_2 = \sqrt{\lambda_2}$$

$$\lambda_1 = \frac{\left(5 - \sqrt{13}\right)k}{6J_2} \Rightarrow \begin{bmatrix} \dfrac{\left(5 + \sqrt{13}\right)k}{6J_2} & \dfrac{-k}{\sqrt{3J_2}} \\ \dfrac{-k}{\sqrt{3J_2}} & \dfrac{\left(5 + \sqrt{13}\right)k}{6J_2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\Rightarrow v_{11} = 1.3205 v_{12} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.7992 \\ 0.6011 \end{bmatrix}$$

$$\lambda_2 = \frac{\left(5+\sqrt{13}\right)k}{6J_2} \Rightarrow \begin{bmatrix} \frac{\left(-1-\sqrt{13}\right)k}{6J_2} & \frac{-k}{\sqrt{3J_2}} \\ \frac{-k}{\sqrt{3J_2}} & \frac{\left(1-\sqrt{13}\right)k}{6J_2} \end{bmatrix} \begin{bmatrix} v2_{11} \\ v_{22} \end{bmatrix} = 0$$

$$\Rightarrow v_{21} = -0.7522 v_{22} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.6011 \\ 0.7992 \end{bmatrix}$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.7992 & -0.6011 \\ 0.6011 & 0.7992 \end{bmatrix}$

Calculate S and S^{-1}:

$$S = J^{-1/2}P = \frac{1}{\sqrt{J_2}} \begin{bmatrix} 0.4614 & -0.3470 \\ 0.6011 & 0.7992 \end{bmatrix}$$

$$S^{-1} = P^T J^{1/2} = J_2^{1/2} \begin{bmatrix} 1.3842 & 0.6011 \\ -1.0411 & 0.7992 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\theta(0) = S^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = J_2^{1/2}\begin{bmatrix} 0.6011 \\ 0.7992 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\theta}(0) = 0$$

Modal solution:

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1} \sin\left[\omega_1 t + \tan^{-1}\frac{\omega_1 r_{10}}{\dot{r}_{10}}\right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + \dot{r}_{20}^2}}{\omega_2} \sin\left[\omega_2 t + \tan^{-1}\frac{\omega_2 r_{10}}{\dot{r}_{20}}\right]$$

$$r_1(t) = 0.6011 J_2^{1/2} \sin\left[\omega_1 t + \frac{\pi}{2}\right] = 0.6011 J_2^{1/2}\cos\omega_1 t$$

$$r_2(t) = 0.7992 J_2^{1/2} \sin\left[\omega_2 t + \frac{\pi}{2}\right] = 0.6011 J_2^{1/2}\cos\omega_2 t$$

$$\mathbf{r}(t) = \begin{bmatrix} 0.6011 J_2^{1/2}\cos\omega_1 t \\ 0.7992 J_2^{1/2}\cos\omega_2 t \end{bmatrix}$$

Convert to physical coordinates:

$$\theta(t) = S\mathbf{r}(t) = J_2^{1/2}\begin{bmatrix} 0.4614 & -0.3470 \\ 0.6011 & 0.7992 \end{bmatrix}\begin{bmatrix} 0.6011 J_2^{1/2}\cos\omega_1 t \\ 0.7992 J_2^{1/2}\cos\omega_2 t \end{bmatrix}$$

$$\theta(t) = \begin{bmatrix} 0.2774\cos\omega_1 t - 0.2774\cos\omega_2 t \\ 0.3613\cos\omega_1 t + 0.6387\cos\omega_2 t \end{bmatrix}$$

where $\omega_1 = 0.4821\sqrt{\dfrac{k}{J_2}}$ and $\omega_2 = 1.1976\sqrt{\dfrac{k}{J_2}}$,

4.35 Consider the system of Example 4.3.1. Calculate a value of $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ such that both masses of the system oscillate with a single frequency of 2 rad/s.

Solution:

From Example 4.3.1,

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1/3 & 1/3 \\ 1 & -1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

From Equations (4.67) and (4.68),

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1}\sin\left[\omega_1 t + \tan^{-1}\frac{\omega_1 r_{10}}{\dot{r}_{10}}\right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + \dot{r}_{20}^2}}{\omega_2}\sin\left[\omega_2 t + \tan^{-1}\frac{\omega_2 r_{20}}{\dot{r}_{20}}\right]$$

Choose $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ so that $r_1(t) = 0$. This will cause the frequency $\sqrt{2}$ to drop out. For $r_1(t) = 0$, its coefficient must be zero.

$$\frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1} = 0 \quad \text{or} \quad \omega_1^2 r_{10}^2 + \dot{r}_{10}^2 = 0$$

Choose $r_{10} = \dot{r}_{20} = 0$.

Let $r_{20} = 3/\sqrt{2}$ and $\dot{r}_{20} = 0$ as calculated in Example 4.3.1.

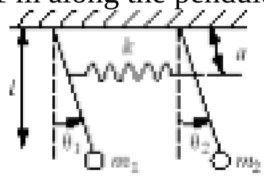
So, $\mathbf{r}(0) = \begin{bmatrix} 0 & 3/\sqrt{2} \end{bmatrix}^T$ and $\dot{\mathbf{r}}(0) = \mathbf{0}$.

Solve for $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$:

$$\mathbf{x}(0) = S\mathbf{r}(0) = \frac{1}{\sqrt{12}}\begin{bmatrix} 1/3 & 1/3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1.5 \end{bmatrix}$$

$$\dot{\mathbf{x}}(0) = S\dot{\mathbf{r}}(0) = \mathbf{0}$$

4.36 Consider the system of Figure P4.36 consisting of two pendulums coupled by a spring. Determine the natural frequency and mode shapes. Plot the mode shapes as well as the solution to an initial condition consisting of the first mode shape for k = 20 N/m, l = 0.5 m and $m_1 = m_2$ = 10 kg, a = 0.1 m along the pendulum.



Solution: Given:

$$k = 20 \text{ N/m} \quad m_1 = m_2 = 10 \text{ kg}$$
$$a = 0.1 \text{ m} \qquad l = 0.5 \text{ m}$$

For gravity use $g = 9.81$ m/s^2. For a mass on a pendulum, the inertia is: $I = ml^2$

Calculate mass and stiffness matrices (for small θ). The equations of motion are:

$$\begin{aligned} I_1\ddot{\theta}_1 &= ka^2\left(\theta_2 - \theta_1\right) - m_1 gl\theta_1 \\ I_2\ddot{\theta}_2 &= -ka^2\left(\theta_2 - \theta_1\right) - m_2 gl\theta_2 \end{aligned} \Rightarrow ml^2\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} mgl + ka^2 & -ka^2 \\ -ka^2 & mgl + ka^2\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Substitution of the given values yields:

$$\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} 49.05 & -0.2 \\ -0.2 & 49.05 \end{bmatrix}\theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 19.7 & -0.08 \\ -0.08 & 19.7 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 19.54 \text{ and } \lambda_2 = 19.7 \Rightarrow \omega_1 = 4.42 \text{ rad/s and } \omega_2 = 4.438 \text{ rad/s}$$

Eigenvectors:

$$\lambda_1 = 19.54$$

$$\begin{bmatrix} 0.08 & -0.08 \\ -0.08 & 0.08 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 19.7$$

$$\begin{bmatrix} -0.08 & -0.08 \\ -0.08 & -0.08 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

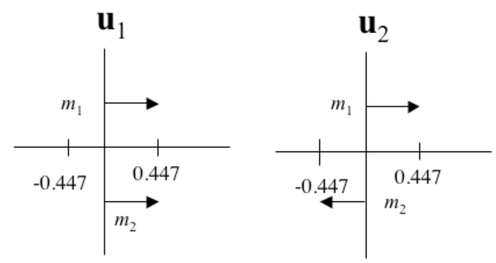
Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Mode shapes:

$$\mathbf{u}_1 = M^{-1/2}\mathbf{v}_1 = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix}$$

$$\mathbf{u}_2 = M^{-1/2}\mathbf{v}_2 = \begin{bmatrix} 0.4472 \\ -0.4472 \end{bmatrix}$$

A plot of the mode shapes is simply



This shows the first mode vibrates in phase and in the second mode the masses vibrate out of phase.

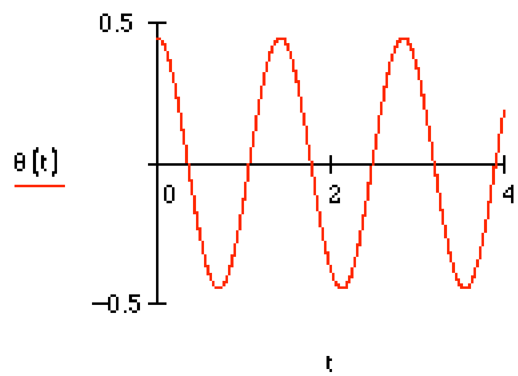
$$\theta(0) = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix} \qquad \dot{\theta}(0) = 0,\ \ S = M^{-1/2}P = \begin{bmatrix} 0.4472 & 0.4472 \\ 0.4472 & -0.4472 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 1.118 & 1.118 \\ 1.118 & -1.118 \end{bmatrix},\ \ \mathbf{r}(0) = S^{-1}\theta(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \dot{\mathbf{r}}(0) = 0$$

From Eq. (4.67) and (4.68): $r_1(t) = \sin\left(4.42t + \dfrac{\pi}{2}\right) = \cos 4.45t,\ \ r_2(t) = 0$

Convert to physical coordinates: $\theta(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.4472\cos 4.42t \\ 0.4472\cos 4.42t \end{bmatrix}$ rad

$\theta(t) := 0.4472\cdot\cos(4.429\cdot t)$



4.37 Resolve Example 4.3.2 with m_2 changed to 10 kg. Plot the response and compare the plots to those of Figure 4.6.

Solution: From examples 4.3.2 and 4.2.5, with m_2 = 10 kg,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 12 & -2 \\ -2 & 12 \end{bmatrix}\mathbf{x} = 0$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 12 & -0.6325 \\ -0.6325 & 1.2 \end{bmatrix}$$

$$\det(\tilde{K} - \lambda I) = \lambda^2 - 13.2\lambda + 14 = 0$$

$$\lambda_1 = 1.163 \quad \omega_1 = 1.078 \text{ rad/s}$$

$$\lambda_2 = 12.04 \quad \omega_2 = 3.469 \text{ rad/s}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.9983 & 0.0583 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.9983 & 0.0583 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 0.0583 & 3.1569 \\ -0.9983 & 0.1842 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2152 \\ -0.8141 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = 0$$

Modal solution (from Eqs. (4.67) and (4.68):

$$r_1(t) = 3.2152\sin\left[1.078t + \frac{\pi}{2}\right] = 3.2152\cos 1.078t$$

$$r_2(t) = -0.8141\cos 3.469t$$

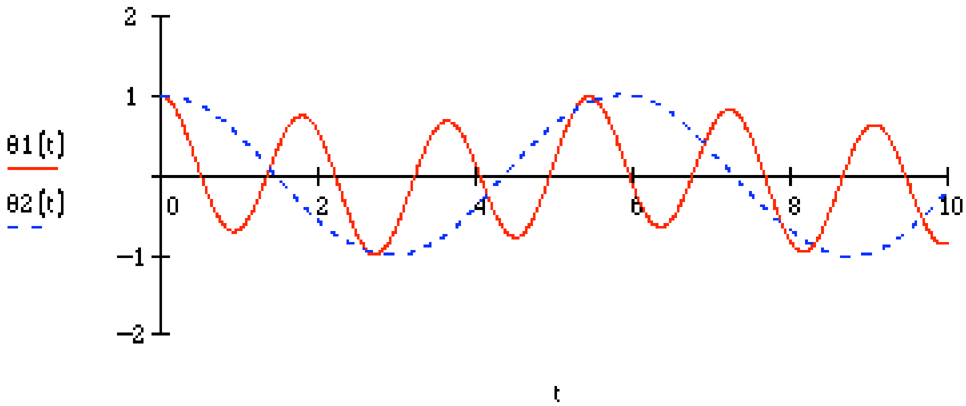
Covert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.3157 & 0.0184 \end{bmatrix}\begin{bmatrix} 3.2152\cos 1.078t \\ -0.8141\cos 3.469t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.1873\cos 1.078t + 0.8127\cos 3.469t \\ 1.015\cos 1.078t - 0.0150\cos 3.469t \end{bmatrix}$$

$$\theta 1(t) := 0.1873 \cdot \cos(1.078 \cdot t) + 0.8127 \cdot \cos(3.469 \cdot t)$$

$$\theta 2(t) := 1.015 \cdot \cos(1.078 \cdot t) - 0.0150 \cdot \cos(3.469 \cdot t)$$



These figures are similar to those of Figure 4.6, except the responses are reversed (θ_2 looks like x_2 in Figure 4.6, and θ_1 looks like x_1 in Figure 4.6)

4.38 Use modal analysis to calculate the solution of Problem 4.29 for the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T (\text{mm}) \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \ (\text{mm/s})$$

Solution: From Problem 4.29,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$
$$\omega_1 = \sqrt{\lambda_1} = 0.4024 \text{ rad/s}$$
$$\omega_2 = \sqrt{\lambda_2} = 1.7573 \text{ rad/s}$$
$$\text{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.9848 & 0.1735 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.4924 & 0.0868 \end{bmatrix}$$
$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 0.1735 & 1.9697 \\ -0.9848 & 0.3470 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.9697 \\ 0.3470 \end{bmatrix}$$
$$\mathbf{r}(0) = S^{-1}\dot{\mathbf{x}}(0) = 0$$

Modal solution (from Eqs. (4.67) and (4.68):

$$r_1(t) = 1.9697\cos 0.4024t$$
$$r_2(t) = -.3470\cos 1.7573t$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.4924 & 0.0868 \end{bmatrix}\begin{bmatrix} 1.9697\cos 0.4024t \\ 0.3470\cos 1.7573t \end{bmatrix}$$
$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 0.3417\cos 0.4024t - 0.3417\cos 1.7573t \\ 0.9699\cos 0.4024t + 0.0301\cos 1.7573t \end{bmatrix} \text{mm}$$

4.39 For the matrices

$$M^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

calculate $M^{-1/2}P$, $\left(M^{-1/2}P\right)^T$, and $P^T M^{-1/2}$ and hence verify that the computations in Eq. (4.70) make sense.

Solution:

Given

$$M^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Now

$$M^{-1/2}P = \begin{bmatrix} 0.5 & 0.5 \\ -2\sqrt{2} & -2\sqrt{2} \end{bmatrix}$$

So

$$\left(M^{-1/2}P\right)^T = \begin{bmatrix} 0.5 & -2\sqrt{2} \\ 0.5 & -2\sqrt{2} \end{bmatrix}$$

$$P^T M^{-1/2} = \begin{bmatrix} 0.5 & -2\sqrt{2} \\ 0.5 & -2\sqrt{2} \end{bmatrix}$$

Thus, $\left(M^{-1/2}P\right)^T = P^T M^{-1/2}$ [Equation (4.71)]

4.40 Consider the 2-degree-of-freedom system defined by:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}.$$

Calculate the response of the system to the initial condition

$$\mathbf{x}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \quad \dot{\mathbf{x}}_0 = \mathbf{0}$$

What is unique about your solution compared to the solution of Example 4.3.1.

Solution: Following the calculations made for this system in Example 4.3.1,

$$\omega_1 = \sqrt{\lambda_1} = 1.414 \text{ rad/s}, \quad \omega_2 = \sqrt{\lambda_2} = 2 \text{ rad/s}$$

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow S = M^{-1/2}P = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \text{ and } S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Next compute the modal initial conditions

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution (from Eqs. (4.67) and (4.68)):

$$\mathbf{r}(t) = \begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

Note that the second coordinate modal coordinate has zero initial conditions and is hence not vibrating. Convert this solution back into physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 0.236\cos 1.414t \\ 0.707\cos 1.414t \end{bmatrix}$$

The unique feature about the solution is that both masses are vibrating at only one frequency. That is the frequency of the first mode shape. This is because the system is excited with a position vector equal to the first mode of vibration.

4.41 Consider the 2-degree-of-freedom system defined by:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}.$$

Calculate the response of the system to the initial condition

$$\mathbf{x}_0 = \mathbf{0}, \quad \text{and } \dot{\mathbf{x}}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} \\ -1 \end{bmatrix}$$

What is unique about your solution compared to the solution of Example 4.3.1 and to Problem 4.40, if you also worked that?

Solution: From example 4.3.1,

$$\omega_1 = \sqrt{\lambda_1} = 1.414 \text{ rad/s}, \ \omega_2 = \sqrt{\lambda_2} = 2 \text{ rad/s}, \ P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow S = M^{-1/2} P = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix}, \text{ and } S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \mathbf{0}, \text{ and } \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Modal solution (from Eqs. (4.67) and (4.68)):

$$\mathbf{r}(t) = \begin{bmatrix} 0 \\ \frac{1}{\omega_2} \cos 2t \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \cos 2t \end{bmatrix}$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \cos 2t \end{bmatrix} = \begin{bmatrix} 0.118 \cos 2t \\ -0.354 \cos 2t \end{bmatrix}$$

Compared to Example 4.3.1, only the second mode is excited, because the initial velocity is proportional to the second mode shape, and the displacement is zero. Compared to the previous problem, here it is the second mode rather then the first mode that is excited.

4.42 Consider the system of Problem 4.1. Let k_1 = 10,000 N/m, k_2 = 15,000 N/m, and k_3 = 10,000 N/m. Assume that both masses are 100 kg. Solve for the free response of this system using modal analysis and the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Solution: Given:

$$\begin{aligned} k_1 &= 10{,}000 \text{ N/m} \quad & m_1 &= m_2 = 100 \text{ kg} \\ k_2 &= 15{,}000 \text{ N/m} \quad & \mathbf{x}(0) &= \begin{bmatrix} 1 & 0 \end{bmatrix}^T \\ k_3 &= 10{,}000 \text{ N/m} \quad & \dot{\mathbf{x}}(0) &= \mathbf{0} \end{aligned}$$

Equation of motion:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 25{,}000 & -15{,}000 \\ -15{,}000 & 25{,}000 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 250 & -150 \\ -150 & 250 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 500\lambda + 40{,}000 = 0$$

$$\lambda_1 = 100 \quad \omega_1 = 10 \text{ rad/s}$$
$$\lambda_2 = 400 \quad \omega_2 = 20 \text{ rad/s}$$

$$\lambda_1 = 100$$

$$\begin{bmatrix} 150 & -150 \\ -150 & 150 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 400$

$$\begin{bmatrix} -150 & -150 \\ -150 & -150 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Calculate S and S^{-1}:

$$S = M^{-1/2} P = \frac{1}{\sqrt{2}} \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 10 & 10 \\ 10 & -10 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solutions:

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1} \sin\left[\omega_1 t + \tan^{-1} \frac{\omega_1 r_{10}}{\dot{r}_{10}} \right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + \dot{r}_{20}^2}}{\omega_2} \sin\left[\omega_2 t + \tan^{-1} \frac{\omega_2 r_{20}}{\dot{r}_{20}} \right]$$

So

$$r_1(t) = 7.071 \sin(10t + \pi/2) = 7.071 \cos 10t$$

$$r_2(t) = 7.071 \sin(20t + \pi/2) = 7.071 \cos 20t$$

$$\mathbf{r}(t) = \begin{bmatrix} 7.071 \cos 10t \\ 7.071 \cos 20t \end{bmatrix}$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}\begin{bmatrix} 7.071\cos 10t \\ 7.7071\cos 20t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.5(\cos 10t + \cos 20t) \\ 0.5(\cos 10t - \cos 20t) \end{bmatrix}$$

4.43 Consider the model of a vehicle given in Problem 4.14 and illustrated in Figure P4.14. Suppose that the tire hits a bump which corresponds to an initial condition of

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Use modal analysis to calculate the response of the car $x_1(t)$. Plot the response for three cycles.

Solution: From Problem 4.14,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate the eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.0224 & 0 \\ 0 & 0.1414 \end{bmatrix}, \ \tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 0.5 & -3.1623 \\ -3.1623 & 0.1414 \end{bmatrix}$$

$$\Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 220.05\lambda + 100 = 0 \Rightarrow \begin{matrix} \lambda_1 = 0.4545 & \omega_1 = 0.6741 \text{ rad/s} \\ \lambda_2 = 220.05 & \omega_2 = 14.834 \text{ rad/s} \end{matrix}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.9999 & -0.0144 \\ 0.0144 & 0.9999 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.0224 & -0.003 \\ 0.0020 & 0.1414 \end{bmatrix}, \ S^{-1} = P^T M^{1/2} = \begin{bmatrix} 44.7167 & 0.1018 \\ -0.6441 & 7.0703 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.001018 \\ 0.07070 \end{bmatrix}, \ \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution (from equations (4.67) and (4.68)): $\mathbf{r}(t) = \begin{bmatrix} 0.001018\cos 0.6741t \\ 0.07070\cos 14.834t \end{bmatrix}$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.0224 & -0.0003 \\ 0.0020 & 0.1414 \end{bmatrix}\begin{bmatrix} 0.001018\cos 0.6741t \\ 0.07070\cos 14.834t \end{bmatrix} = \begin{bmatrix} 2.277\times10^{-5}\cos 0.6741t - 2.277\times10^{-5}\cos 14.834t \\ 2.074\times10^{-6}\cos 0.6741t + 9.998\times10^{-3}\cos 14.834t \end{bmatrix}$$

$$x1(t) := 2.277\cdot10^{-5}\cdot\cos(0.674\cdot t) - 2.277\cdot10^{-5}\cdot\cos(14.834\cdot t)$$