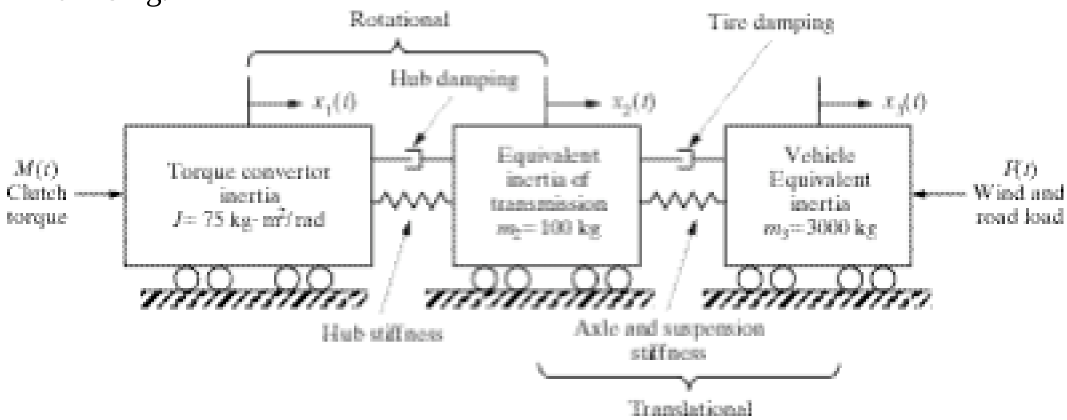
Problems and Solutions for Section 4.4 (4.44 through 4.55)

4.44 A vibration model of the drive train of a vehicle is illustrated as the three-degree-of-freedom system of Figure P4.44. Calculate the undamped free response [i.e. $M(t) = F(t) = 0$, $c_1 = c_2 = 0$] for the initial condition $\mathbf{x}(0) = \mathbf{0}$, $\dot{\mathbf{x}}(0) = [0 \quad 0 \quad 1]^T$. Assume that the hub stiffness is 10,000 N/m and that the axle/suspension is 20,000 N/m. Assume the rotational element J is modeled as a translational mass of 75 kg.



Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness.
The equation of motion is

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix}\ddot{\mathbf{x}} + 10{,}000\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.1155 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.0183 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 133.33 & -115.47 & 0 \\ -115.47 & 300 & -36.515 \\ 0 & -36.515 & 6.6667 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 440\lambda^2 + 28{,}222\lambda = 0$$

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$

$$\lambda_2 = 77.951 \quad \omega_2 = 8.8290 \text{ rad/s}$$

$$\lambda_3 = 362.05 \quad \omega_3 = 19.028 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.1537 \\ 0.1775 \\ 0.9721 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -0.8803 \\ -0.4222 \\ 0.2163 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0.4488 \\ -0.8890 \\ 0.0913 \end{bmatrix}$$

Use the mode summation method to find the solution.
Transform the initial conditions:

$$\mathbf{q}(0) = M^{-1/2}\mathbf{x}(0) = \mathbf{0}, \quad \dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 54.7723 \end{bmatrix}^T$$

The solution is given by:

$$\mathbf{q}(t) = (c_1 + c_4 t)\mathbf{v}_1 + c_2 \sin(\omega_2 t + \phi_2)\mathbf{v}_2 + c_3 \sin(\omega_3 t + \phi_3)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \mathbf{q}(0)}\right) \qquad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}{\omega_i \cos\phi} \qquad i = 2,3$$

Thus,

$$\phi_2 = \phi_3 = 0, c_2 - 1.3417, \text{ and } c_3 = 0.2629$$

So,

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_4\mathbf{v}_1 + \sum_{i=2}^{3} \omega_i c_i \cos\phi_i \mathbf{v}_i$$

Premultiply by v_1^T;

$$\mathbf{v}_1^T \mathbf{q}(0) = 0 = c_1$$

$$\mathbf{v}_1^T \dot{\mathbf{q}}(0) = 53.2414 = c_4$$

So,

$$\mathbf{q}(t) = 53.2414t\mathbf{v}_1 + 1.3417\sin(8.8290t)\mathbf{v}_2 + 0.2629\sin(19.028t)\mathbf{v}_3$$

Change to **q**(t):

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$$

$$\mathbf{x}(t) = 0.9449t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.1364 \\ -0.05665 \\ 0.005298 \end{bmatrix}\sin 8.8290t + \begin{bmatrix} 0.01363 \\ -0.02337 \\ 0.0004385 \end{bmatrix}\sin 19.028t \text{ m}$$

4.45 Calculate the natural frequencies and normalized mode shapes of

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Solution: Given the indicated mass and stiffness matrix, calculate eigenvalues:

$$M^{-1/2} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 1 & -0.3536 & 0 \\ -0.3536 & 1 & -0.7071 \\ 0 & -0.7071 & 1 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 3\lambda^2 + 2.375\lambda - 0.375 = 0$$

$$\lambda_1 = 0.2094, \quad \lambda_2 = 1, \quad \lambda_3 = 1.7906$$

The natural frequencies are:

$$\omega_1 = 0.4576 \text{ rad/s}$$

$$\omega_2 = 1 \text{ rad/s}$$

$$\omega_3 = 1.3381 \text{ rad/s}$$

The corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{bmatrix} -0.3162 \\ -0.7071 \\ -0.6325 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.8944 \\ 0 \\ -0.4472 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.3162 \\ -0.7071 \\ 0.6325 \end{bmatrix}$$

The relationship between eigenvectors and mode shapes is

$$\mathbf{u} = M^{-1/2}\mathbf{v}$$

The mode shapes are:

$$\mathbf{u}_1 = \begin{bmatrix} -0.1581 \\ -0.5 \\ -0.6325 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0.4472 \\ 0 \\ -0.4472 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0.1581 \\ -0.5 \\ 0.6325 \end{bmatrix}$$

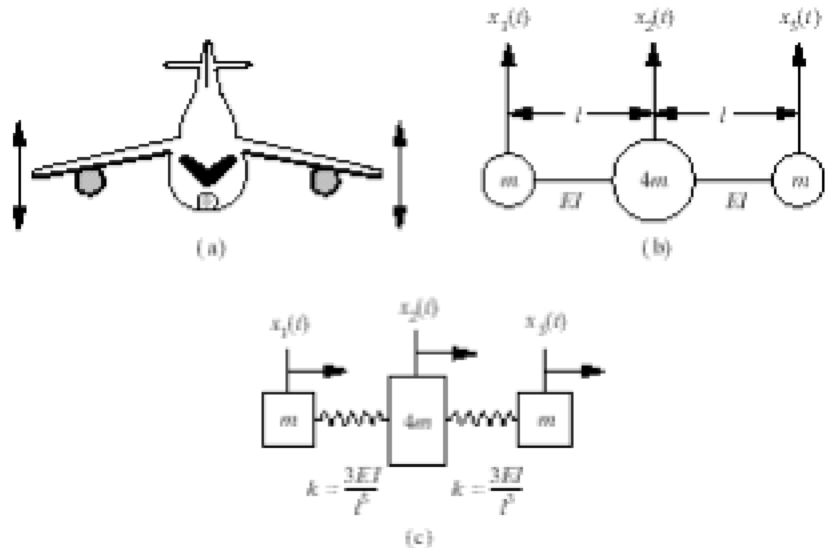
The normalized mode shapes are

$$\hat{\mathbf{u}}_1 = \frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1^T \mathbf{u}_1}} = \begin{bmatrix} 0.192 \\ 0.609 \\ 0.77 \end{bmatrix}, \quad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0.707 \\ 0 \\ -0.707 \end{bmatrix}, \quad \hat{\mathbf{u}}_3 = \begin{bmatrix} 0.192 \\ -0.609 \\ 0.77 \end{bmatrix}.$$

4.46 The vibration is the vertical direction of an airplane and its wings can be modeled as a three-degree-of-freedom system with one mass corresponding to the right wing, one mass for the left wing, and one mass for the fuselage. The stiffness connecting the three masses corresponds to that of the wing and is a function of the modulus E of the wing. The equation of motion is

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The model is given in Figure P4.46. Calculate the natural frequencies and mode shapes. Plot the mode shapes and interpret them according to the airplane's deflection.



Solution: Given the equation of motion indicated above, the mass-normalized stiffness matrix is calculated to be

$$M^{-1/2} = \frac{1}{\sqrt{m}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \;\; \tilde{K} = M^{-1/2}KM^{-1/2} = \frac{EI}{m\ell^3}\begin{bmatrix} 3 & -1.5 & 0 \\ -1.5 & 1.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix}$$

Computing the matrix eigenvalue by factoring out the constant $\frac{EI}{m\ell^3}$ yields

$$\det(\tilde{K} - \lambda I) = 0 \Rightarrow \lambda_1 = 0, \quad \lambda_2 = 3\frac{EI}{m\ell^3}, \quad \lambda_3 = 4.5\frac{EI}{m\ell^3}$$

and eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

The natural frequencies are $\omega_1 = 0$, $\omega_2 = 1.7321\sqrt{\dfrac{EI}{m\ell^3}}$ rad/s, and $\omega_3 =$ $2.1213\sqrt{\dfrac{EI}{m\ell^3}}$ rad/s.

The relationship between the mode shapes and eigenvectors **u** is just $\mathbf{u} = M^{-1/2}\mathbf{v}$. The fist mode shape is the rigid body mode. The second mode shape corresponds to one wing up and one down the third mode shape corresponds to the wings moving up and down together with the body moving opposite. Normalizing the mode shapes yields (calculations in Mathcad):

$$\mathrm{Mh} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{K} := \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \qquad \mathrm{Kh} := \mathrm{Mh}^{-1}\cdot\mathrm{K}\cdot\mathrm{Mh}^{-1}$$

$$\mathrm{Kh} = \begin{bmatrix} 3 & -1.5 & 0 \\ -1.5 & 1.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix}$$

$$\lambda a := \mathrm{eigenvals}(\mathrm{Kh}) \qquad \lambda a = \begin{bmatrix} 4.5 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathrm{v1} := \mathrm{eigenvec}(\mathrm{Kh}, \lambda a_2) \qquad \mathrm{v1} = \begin{bmatrix} 0.408 \\ 0.816 \\ 0.408 \end{bmatrix} \qquad \mathrm{v2} := \mathrm{eigenvec}(\mathrm{Kh}, \lambda a_1) \qquad \mathrm{v2} = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$\mathrm{v3} := \mathrm{eigenvec}(\mathrm{Kh}, \lambda a_0) \qquad \mathrm{v3} = \begin{bmatrix} 0.577 \\ -0.577 \\ 0.577 \end{bmatrix}$$

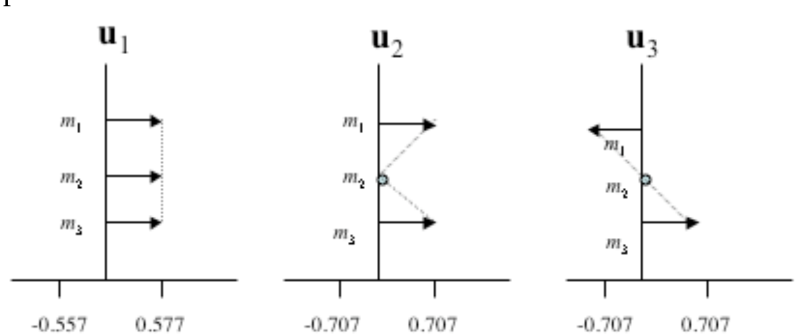
$$\omega := \begin{bmatrix} \sqrt{\lambda a_2} \\ \sqrt{\lambda a_1} \\ \sqrt{\lambda a_0} \end{bmatrix} \qquad \omega = \begin{bmatrix} 2.967\cdot 10^{-8} \\ 1.732 \\ 2.121 \end{bmatrix} \qquad \mathrm{u1} := \mathrm{Mh}^{-1}\cdot\mathrm{v1} \qquad \mathrm{u1} = \begin{bmatrix} 0.408 \\ 0.408 \\ 0.408 \end{bmatrix}$$

$$\mathrm{u1n} := \frac{\mathrm{u1}}{|\mathrm{u1}|} \qquad \mathrm{u1n} = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix} \qquad \mathrm{u2} := \mathrm{Mh}^{-1}\cdot\mathrm{v2} \qquad \mathrm{u2} = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$\mathrm{u3} := \mathrm{Mh}^{-1}\cdot\mathrm{v3} \qquad \mathrm{u3} = \begin{bmatrix} 0.577 \\ -0.289 \\ 0.577 \end{bmatrix} \qquad \mathrm{u2n} := \frac{\mathrm{u2}}{|\mathrm{u2}|} \qquad \mathrm{u2n} = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$\mathrm{u3n} := \frac{\mathrm{u3}}{|\mathrm{u3}|} \qquad \mathrm{u3n} = \begin{bmatrix} 0.667 \\ -0.333 \\ 0.667 \end{bmatrix}$$

These are plotted:



4.47 Solve for the free response of the system of Problem 4.46. Where $E = 6.9 \times 10^9$ N/m^2, l = 2 m, m = 3000 kg, and $I = 5.2 \times 10^{-6}$m^4. Let the initial displacement correspond to a gust of wind that causes an initial condition of $\dot{\mathbf{x}}(0) = \mathbf{0}$, $\mathbf{x}(0) = [0.2 \quad 0 \quad 0]^T$ m. Discuss your solution.

Solution: From problem 4.43 and the given data

$$\begin{bmatrix} 3000 & 0 & 0 \\ 0 & 12{,}000 & 0 \\ 0 & 0 & 3{,}000 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 1.346 & -1.346 & 0 \\ -1.346 & 2.691 & -1.346 \\ 0 & -1.346 & 1.346 \end{bmatrix} \times 10^4 \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0.2 & 0 & 0 \end{bmatrix}^T m$$

$$\dot{\mathbf{x}}(0) = \mathbf{0}$$

Convert to $\mathbf{q}$:

$$I\ddot{\mathbf{q}} + \begin{bmatrix} 4.485 & -2.242 & 0 \\ -2.242 & 2.242 & -2.242 \\ 0 & -2.242 & 4.485 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$\det\left(\tilde{K} - \lambda I\right) = 0 \Rightarrow$$

$$\begin{aligned} \lambda_2 &= 0 & \omega_1 &= 0 \text{ rad/s} \\ \lambda_2 &= 4.485 & \omega_2 &= 2.118 \text{ rad/s} \\ \lambda_3 &= 6.727 & \omega_3 &= 2.594 \text{ rad/s} \end{aligned}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

The solution is given by

$$\mathbf{q}(t) = \left(c_1 + c_4 t\right)\mathbf{v}_1 + c_2 \sin\left(\omega_2 t + \phi_2\right)\mathbf{v}_2 + c_3 \sin\left(\omega_3 t + \phi_3\right)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \qquad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \quad i = 2,3$$

Thus, $\phi_2 = \phi_3 = \dfrac{\pi}{2}, c_2 = -7.7459,$ and $c_3 = 6.3251$

So,

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_4\mathbf{v}_i + \sum_{i=2}^{3} \omega_i c_i \cos\phi_i \mathbf{v}_i$$

Premultiply by $\mathbf{v}_i^T$:

$$\mathbf{v}_i^T \mathbf{q}(0) = 4.4716 = c_1$$

$$\mathbf{v}_i^T \dot{\mathbf{q}}(0) = 0 = c_4$$

So, $\mathbf{q}(t) = 4.4716\mathbf{v}_1 - 7.7459\cos(2.118t)\mathbf{v}_2 + 6.3251\cos(2.594t)\mathbf{v}_3$

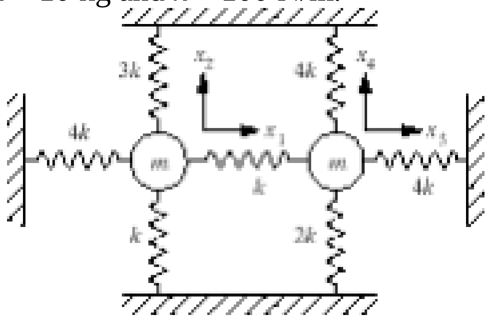
Convert to physical coordinates:

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) \Rightarrow$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.0333 \\ 0.0333 \\ 0.0333 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \\ -0.1 \end{bmatrix} \cos 2.118t + \begin{bmatrix} 0.0667 \\ -0.0333 \\ 0.0667 \end{bmatrix} \cos 2.594t \text{ m}$$

The first term is a rigid body mode, which represents (in this case) a fixed displacement around which the three masses oscillate. Mode two has the highest amplitude (0.1 m).

4.48 Consider the two-mass system of Figure P4.48. This system is free to move in the $x_1 - x_2$ plane. Hence each mass has two degrees of freedom. Derive the linear equations of motion, write them in matrix form, and calculate the eigenvalues and eigenvectors for m = 10 kg and k = 100 N/m.



Solution: Given: $m = 10\,\text{kg}, k = 100$ N/m

<u>Mass 1</u>

$$x_1 - \text{direction:} \quad m\ddot{x}_1 = -4kx_1 + k\left(x_3 - x_1\right) = -5kx_1 + kx_3$$

$$x_2 - \text{direction:} \quad m\ddot{x}_2 = -3kx_2 - kx_2 = -4kx_2$$

<u>Mass 2</u>

$$x_3 - \text{direction:} \quad m\ddot{x}_3 = -4kx_3 - k\left(x_3 - x_1\right) = -kx_1 - 5kx_3$$

$$x_4 - \text{direction:} \quad m\ddot{x}_4 = -4kx_4 - 2kx_4 = -6kx_4$$

In matrix form with the values given:

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 500 & 0 & -100 & 0 \\ 0 & 400 & 0 & 0 \\ -100 & 0 & 500 & 0 \\ 0 & 0 & 0 & 600 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 50 & 0 & -10 & 0 \\ 0 & 40 & 0 & 0 \\ -10 & 0 & 50 & 0 \\ 0 & 0 & 0 & 60 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^4 - 200\lambda^3 + 14{,}800\lambda^2 - 480{,}000\lambda + 5{,}760{,}000 = 0$$

$$\Rightarrow \lambda_1 = 40, \quad \lambda_2 = 40, \quad \lambda_3 = 60, \quad \lambda_4 = 60$$

The corresponding eigenvectors are found from solving $(\tilde{K} - \lambda_i)\mathbf{v}_i = 0$ for each value of the index and normalizing:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

These are not unique.

4.49 Consider again the system discussed in Problem 4.48. Use modal analysis to calculate the solution if the mass on the left is raised along the x_2 direction exactly 0.01 m and let go.

Solution: From Problem 4.48:

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 500 & 0 & -100 & 0 \\ 0 & 400 & 0 & 0 \\ -100 & 0 & 500 & 0 \\ 0 & 0 & 0 & 600 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$M^{-1/2} = 0.3162 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 50 & 0 & -10 & 0 \\ 0 & 40 & 0 & 0 \\ -10 & 0 & 50 & 0 \\ 0 & 0 & 0 & 60 \end{bmatrix}$$

$$\lambda_1 = 40 \quad \omega_1 = 6.3246 \text{ rad/s}$$
$$\lambda_2 = 40 \quad \omega_2 = 6.3246 \text{ rad/s}$$
$$\lambda_3 = 60 \quad \omega_3 = 7.7460 \text{ rad/s}$$
$$\lambda_4 = 60 \quad \omega_4 = 7.7460 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Also, $\mathbf{x}(0) = [0 \quad 0.01 \quad 0 \quad 0]^T$ m and $\dot{\mathbf{x}}(0) = \mathbf{0}$

Use the mode summation method to find the solution.
Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 0.003162 & 0 & 0 \end{bmatrix}^T$$
$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.103),

$$\mathbf{x}(t) = \sum_{i=1}^{4} d_i \sin(\omega_i t + \phi_i)\mathbf{u}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \quad i = 1,2,3,4 \quad \text{(Eq.(4.97))}$$
$$\mathrm{d}_\mathrm{i} = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \qquad i = 1,2,3,4 \quad \left(\text{Eq. }(4.98)\right)$$
$$\mathbf{u}_\mathrm{i} = M^{-1/2}\mathbf{v}_i$$

Substituting known values yields

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = \frac{\pi}{2} \text{ rad}$$

$$\mathrm{d}_1 = 0.003162$$
$$d_2 = d_3 = d_4 = 0$$

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0.3162 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.2236 \\ 0 \\ 0.2236 \\ 0 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.2236 \\ 0 \\ -0.2236 \\ 0 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.3162 \end{bmatrix}$$

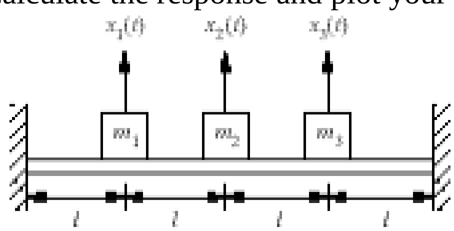
The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0 \\ 0.001\cos 6.3246t \\ 0 \\ 0 \end{bmatrix}$$

4.50 The vibration of a floor in a building containing heavy machine parts is modeled in Figure P4.50. Each mass is assumed to be evenly spaced and significantly larger than the mass of the floor. The equation of motion then becomes $(m_1 = m_2 = m_3 = m)$.

$$m I \ddot{\mathbf{x}} + \frac{EI}{l^3}\begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

Calculate the natural frequencies and mode shapes. Assume that in placing box m_2 on the floor (slowly) the resulting vibration is calculated by assuming that the initial displacement at m_2 is 0.05 m. If l = 2 m, m = 200 kg, $E = 0.6 \times 10^9$ N/m^2, $I = 4.17 \times 10^{-5}$ m^4. Calculate the response and plot your results.



Solution:

The equations of motion can be written as

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = 0$$

or $mI\ddot{x} + Kx = 0$ where I is the 3x3 identity matrix.

The natural frequencies of the system are obtained using the characteristic equation

$$\left|K - \omega^2 M\right| = 0$$

Using the given mass and stiffness matrices yields the following characteristic equation

$$m^3\omega^6 - \frac{59EI\,m^3}{96l^3}\omega^4 + \frac{41(EI)^2\,m}{768l^6}\omega^2 - \frac{7(EI)^3}{6912l^9} = 0$$

Substituting for *E*, *I*, *m*, and *l* yields the following answers for the natural frequency

$$\omega_1 = \pm\sqrt{\frac{\left(13-\sqrt{137}\right)EI}{ml^3}}\ ,\ \omega_2 = \pm\sqrt{\frac{7EI}{96ml^3}}\ ,\ \omega_3 = \pm\sqrt{\frac{\left(13+\sqrt{137}\right)EI}{48ml^3}}$$

The plus minus sign shown above will cause the exponential terms to change to trigonometric terms using Euler's formula. Hence, the natural frequencies of the system are 0.65 rad/sec, 1.068 rad/sec and 2.837 rad/sec.

Let the mode shapes of the system be $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. The mode shapes should satisfy the following equation

$$\left[K - \omega_1^2 M\right]\begin{Bmatrix} u_{i1} \\ u_{i2} \\ u_{i3} \end{Bmatrix} = 0, i = 1,2,3$$

Notice that the system above does not have a unique solution for $\mathbf{u}_1$ since $\left[K - \omega_1^2 M\right]$ had to be singular in order to solve for the natural frequency ω. Solving the above equation yields the following relations

$$\frac{u_{i2}}{u_{i3}} = \frac{1}{3}\frac{96m\omega_i^2 l^3 - 7EI}{13m\omega_i^2 l^3 + EI}, i = 1,2,3$$

and $u_{i1} = u_{i3}, i = 1,3$ but for the second mode shape this is different $u_{21} = u_{23}$

Substituting the values given yields

$$\frac{u_{12}}{u_{13}} = \frac{1}{3}\frac{96m\omega_1^2 l^3 - 7EI}{13m\omega_1^2 l^3 + EI} = -1.088$$

$$\frac{u_{22}}{u_{23}} = \frac{1}{3}\frac{96m\omega_2^2 l^3 - 7EI}{13m\omega_2^2 l^3 + EI} = 0$$

$$\frac{u_{32}}{u_{33}} = \frac{1}{3}\frac{96m\omega_3^2 l^3 - 7EI}{13m\omega_3^2 l^3 + EI} = 1.838$$

If we let $u_{i3} = 1, i = 1,2,3$, then

$$u_1 = \begin{Bmatrix} 1 \\ -1.088 \\ 1 \end{Bmatrix}, u_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}, u_3 = \begin{Bmatrix} 1 \\ 1.838 \\ 1 \end{Bmatrix}$$

These mode shapes can be normalized to yield

$$u_1 = \begin{Bmatrix} 0.5604 \\ -0.6098 \\ 0.5604 \end{Bmatrix}, u_2 = \begin{Bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{Bmatrix}, u_3 = \begin{Bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{Bmatrix}$$

This solution is the same if obtained using MATLAB

$$u_1 = \begin{Bmatrix} -0.5604 \\ 0.6098 \\ -0.5604 \end{Bmatrix},\ u_2 = \begin{Bmatrix} -0.7071 \\ 0.0000 \\ 0.7071 \end{Bmatrix},\ u_3 = \begin{Bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{Bmatrix}$$

The second box, m_2, is placed slowly on the floor; hence, the initial velocity can be safely assumed zero. The initial displacement at m_2 is given to be 0.05 m.

Hence, the initial conditions in vector form are given as

$$x(0) = \begin{Bmatrix} 0 \\ -0.05 \\ 0 \end{Bmatrix} \text{ and } \dot{x}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The equations of motion given by $M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$ can be transformed into the modal coordinates by applying the following transformation

$\mathbf{x}(t) = S\mathbf{r}(t) = M^{-\frac{1}{2}}P\mathbf{r}(t)$ where P is the basis formed by the mode shapes of the system, given by

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$$

Hence, the transformation S is given by

$$S = \begin{bmatrix} -0.04 & -0.05 & 0.03 \\ 0.043 & 0 & 0.056 \\ -0.04 & 0.05 & 0.03 \end{bmatrix}$$

The initial conditions will be also transformed

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \begin{Bmatrix} -0.431 \\ 0 \\ -0.56 \end{Bmatrix}$$

Hence, the modal equations are

with the above initial conditions.

The solution will then be

$$\mathbf{r}(t) = \begin{Bmatrix} 0.431\cos(0.65t) \\ 0 \\ 0.56\cos(2.837t) \end{Bmatrix}$$

The solution can then be determined by

$$\mathbf{x}(t) = \begin{Bmatrix} 0.0172\cos(0.65t) - 0.0168\cos(2.837t) \\ -0.0185\cos(0.65t) - 0.0313\cos(2.837t) \\ 0.0172\cos(0.65t) - 0.0168\cos(2.837t) \end{Bmatrix}$$

The equations of motion can be also be solved using MATLAB to yield the following response.

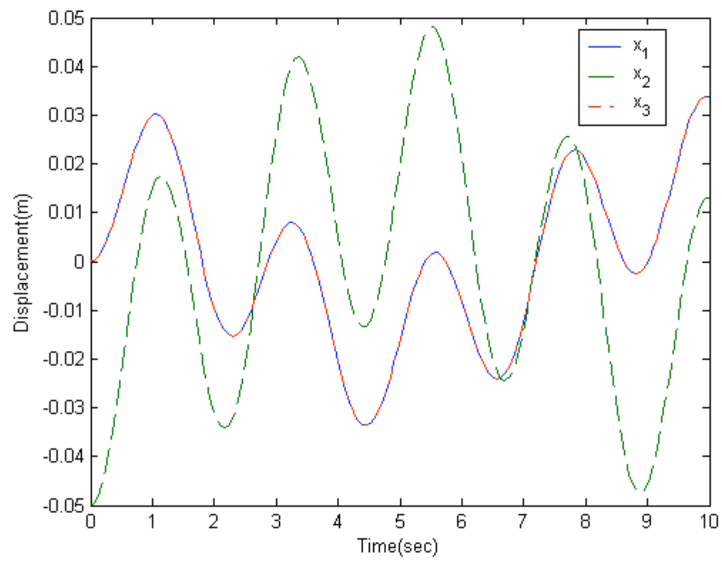


Figure 1 Numerical response due to initial deflection at m_2

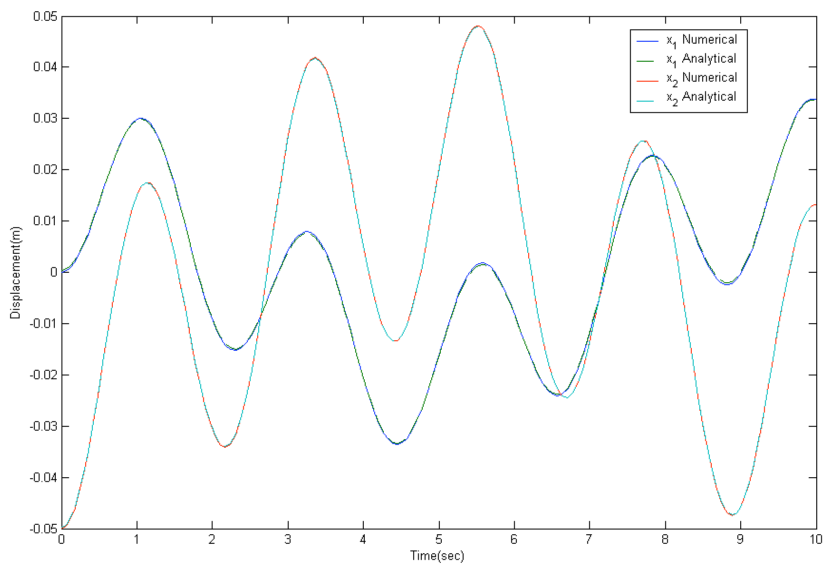


Figure 2 Numerical vs. Analytical Response (shown for x_1 and x_2 only)

The MATLAB code is attached below

```
% Set the values of the physical parameters
% ***********************************************************************
*

% Declare global variables to be used in the differential equation file
global M K
```

```
% Define the mass of the each box
m=200;

% Define the distance l
l=2;

% Define the area moment of inertia
I=4.17*10^-5;

% Define the modulus of elasticity
E=0.6*10^9;

% Define the flexural rigidity
EI=E*I;

% Define the system matrices
%
***********************************************************************
*

% Define the mass matrix
M=m*eye(3,3);

% Define the stiffness matrix
K=EI/l^3*[9/64 1/6 13/192;1/6 1/3 1/6;13/192 1/6 9/64];

% Solve the eigen value problem
[u,lambda]=eig(M\K);

% Simulate the response of the system to the given initial conditions
% The states are arranges as: [x1;x2;x3;x1_dot;x2_dot;x3_dot]
[t,xn]=ode45('sys4p47',[0 10],[0 ; -0.05 ; 0 ; 0 ; 0 ; 0]);

% Plot the results
plot(t,xn(:,1),t,xn(:,2),'--',t,xn(:,3),'-.');
set(gcf,'Color','White');
xlabel('Time(sec)');
ylabel('Displacement(m)');
legend('x_1','x_2','x_3');

% Analytical solution

for i=1:length(t)
   xa(:,i)=[0.0172*cos(0.65*t(i))-0.0168*cos(2.837*t(i));
          -0.0185*cos(0.65*t(i))-0.0313*cos(2.837*t(i)) ;
```

```
            0.0172*cos(0.65*t(i))-0.0168*cos(2.837*t(i))];
end;

% Camparison
figure;
plot(t,xn(:,1),t,xa(1,:),'--',t,xn(:,2),t,xa(2,:),'--');
set(gcf,'Color','White');
xlabel('Time(sec)');
ylabel('Displacement(m)');
legend('x_1 Numerical','x_1 Analytical','x_2 Numerical','x_2 Analytical');
```

4.51 Recalculate the solution to Problem 4.50 for the case that m_2 is increased in mass to 2000 kg. Compare your results to those of Problem 4.50. Do you think it makes a difference where the heavy mass is placed?

Solution: Given the data indicated the equation of motion becomes:

$$\begin{bmatrix} 200 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 200 \end{bmatrix} \ddot{\mathbf{x}} + 3.197\times 10^{-4} \begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = [0 \quad 0.05 \quad 0]^T, \dot{\mathbf{x}}(0) = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.07071 & 0 & 0 \\ 0 & 0.02246 & 0 \\ 0 & 0 & 0.07071 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 2.2482 & 0.8246 & 1.0825 \\ 0.8246 & 0.5329 & 0.8246 \\ 1.0825 & 0.8246 & 2.2482 \end{bmatrix} \times 10^{-7}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255\times 10^{-7}\lambda^2 + 1.3645\times 10^{-14}\lambda - 4.1382\times 10^{-22} = 0$$

$$\lambda_1 = 4.3142\times 10^{-9} \quad \omega_1 = 2.0771\times 10^{-5} \text{ rad/s}$$

$$\lambda_2 = 1.1657\times 10^{-7} \quad \omega_2 = 3.4143\times 10^{-4} \text{ rad/s}$$

$$\lambda_3 = 8.2283\times 10^{-7} \quad \omega_3 = 9.0710\times 10^{-4} \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.2443 \\ -0.9384 \\ 0.2443 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.6636 \\ 0.3455 \\ 0.6636 \end{bmatrix}$$

Use the mode summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 2.2361 & 0 \end{bmatrix}^T$$
$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.103),

$$\mathbf{x}(t) = \sum_{i=1}^{4} d_i \sin(\omega_i t + \phi_i)\mathbf{u}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \quad i = 1,2,3 \quad (\text{Eq. } (4.97))$$
$$d_{\mathrm{i}} = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \quad i = 1,2,3 \quad (\text{Eq. } (4.98))$$
$$\mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

Substituting known values yields

$$\phi_1 = \phi_2 = \phi_3 = \frac{\pi}{2} \text{ rad}$$
$$d_1 = -2.0984$$
$$d_2 = 0$$
$$d_3 = 0.7726$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.0178 \\ -0.02098 \\ 0.01728 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.05 \\ 0 \\ -0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.04692 \\ 0.007728 \\ 0.04692 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} -0.03625 \\ 0.04403 \\ -0.03625 \end{bmatrix} \cos\left(9.7044 \times 10^{-5} t\right) + \begin{bmatrix} 0.03625 \\ 0.005969 \\ 0.0325 \end{bmatrix} \cos\left(6.1395 \times 10^{-4} t\right) \text{ m}$$

The results are very similar to Problem 50. The responses of mass 1 and 3 are the same for both problems, except the amplitudes and frequencies are changed due to the increase in mass 2. There would have been a greater change if the heavy mass was placed at mass 1 or 3.

4.52 Repeat Problem 4.46 for the case that the airplane body is 10 m instead of 4 m as indicated in the figure. What effect does this have on the response, and which design (4m or 10 m) do you think is better as to vibration?

Solution: Given:

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \frac{EI}{l^3}\begin{bmatrix} 3 & -3 & - \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = m^{-1/2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.3612 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \frac{EI}{ml^3}\begin{bmatrix} 3 & -0.9487 & 0 \\ -0.9487 & 0.6 & -0.9487 \\ 0 & -0.9487 & 3 \end{bmatrix}$$

Again choose the parameters so that the coefficient is 1 and compute the eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 6.6\lambda^2 + 10.8\lambda = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = 3$$

$$\lambda_3 = 3.6$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.2887 \\ -0.9129 \\ -0.2887 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.6455 \\ -0.4082 \\ 0.6455 \end{bmatrix}$$

The natural frequencies are

$$\omega_1 = 0 \text{ rad/s}$$

$$\omega_2 = 1.7321 \text{ rad/s}$$

$$\omega_3 = 1.8974 \text{ rad/s}$$

The relationship between eigenvectors and mode shapes is

$$\mathbf{u} = M^{-1/2}\mathbf{v}$$

$$\mathbf{u}_1 = m^{-1/2}\begin{bmatrix} -0.2887 \\ -0.2887 \\ -0.2887 \end{bmatrix} \quad \mathbf{u}_2 = m^{-1/2}\begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.6455 \\ -0.1291 \\ 0.6455 \end{bmatrix}$$

It appears that the mode shapes contain less "amplitude" for the wing masses. This seems to be a better design from a vibration standpoint.

4.53 Often in the design of a car, certain parts cannot be reduced in mass. For example, consider the drive train model illustrated in Figure P4.44. The mass of the torque converter and transmission are relatively the same from car to car. However, the mass of the car could change as much as 1000 kg (e.g., a two-seater sports car versus a family sedan). With this in mind, resolve Problem 4.44 for the case that the vehicle inertia is reduced to 2000 kg. Which case has the smallest amplitude of vibration?

Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness. From Problem 4.44, the equation of motion becomes

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 2000 \end{bmatrix} \ddot{\mathbf{x}} + 10{,}000 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s.}$$

Calculate eigenvalues and eigenvectors.

$$M^{-1/2} = \begin{bmatrix} 0.1155 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.0224 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 133.33 & -115.47 & 0 \\ -115.47 & 300 & -44.721 \\ 0 & -44.721 & 10 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 443.33\lambda^2 + 29{,}000\lambda = 0$$

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$

$$\lambda_2 = 70.765 \qquad \omega_2 = 8.9311 \text{ rad/s}$$

$$\lambda_3 = 363.57 \qquad \omega_3 = 19.067 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.1857 \\ -0.2144 \\ -0.9589 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.8758 \\ 0.4063 \\ -0.2065 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4455 \\ -0.8882 \\ 0.1123 \end{bmatrix}$$

Use the mode summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = 0$$

$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 44.7214 \end{bmatrix}^T$$

The solution is given by

$$\mathbf{q}(t) = (c_1 + c_4 t)\mathbf{v}_1 + c_2 \sin(\omega_2 t + \phi_2)\mathbf{v}_2 + c_3 \sin(\omega_3 t + \phi_3)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right), \quad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\omega_i \cos\phi_i}, \quad i = 2,3$$

Thus $\phi_2 = \phi_3 = 0$, $c_2 = -1.3042$ and $c_3 = 0.2635$. Next apply the initial conditions:

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i \ \text{ and } \ \dot{\mathbf{q}}(0) = c_4\mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i$$

Pre multiply each of these by $\mathbf{v}_1{}^T$ to get:

$$c_1 = 0 = \mathbf{v}_1^T \mathbf{q}(0) \ \text{ and } \ c_4 = -42.8845 = \mathbf{v}_1^T \dot{\mathbf{q}}(0)$$

So

$$\mathbf{q}(t) = -42.8845t\mathbf{v}_1 - 1.3042\sin(8.9311t)\mathbf{v}_2 + 0.2635\sin(19.067t)\mathbf{v}_3$$

Next convert back to the physical coordinates by

$$\begin{aligned}\mathbf{x}(t) &= M^{-1/2}\mathbf{q}(t)\\ &= 0.9195t\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}-0.1319\\-0.05299\\0.007596\end{bmatrix}\sin 8.9311t + \begin{bmatrix}0.01355\\-0.02340\\0.0006620\end{bmatrix}\sin 19.067t \text{ m}\end{aligned}$$

Comparing this solution to problem 4.44, the car will vibrate at a slightly higher amplitude when the mass is reduced to 2000 kg.

4.54 Use *mode summation method* to compute the analytical solution for the response of the 2-degree-of-freedom system of Figure P4.28 with the values where $m_1 = 1$ kg, $m_2 = 4$ kg, $k_1 = 240$ N/m and k_2=300 N/m, to the initial conditions of

$$\mathbf{x}_0 = \begin{bmatrix}0\\0.01\end{bmatrix}, \quad \dot{\mathbf{x}}_0 = \begin{bmatrix}0\\0\end{bmatrix}.$$

Solution: Following the development of equations (4.97) through (4.103) for the mode summation for the free response and using the values of computed in problem 1, compute the initial conditions for the "**q**" coordinate system:

$$M^{1/2} = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} \Rightarrow \mathbf{q}(0) = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}0\\0.01\end{bmatrix} = \begin{bmatrix}0\\0.02\end{bmatrix}, \mathbf{q}(0) = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

From equation (4.97):

$$\phi_1 = \tan^{-1}\left(\frac{x}{0}\right) = \phi_2 = \tan^{-1}\left(\frac{x}{0}\right) = \frac{\pi}{2}$$

From equation (4.98):

$$d_1 = \frac{\mathbf{v}_1^T\mathbf{q}(0)}{\sin\left(\pi/2\right)} = \mathbf{v}_1^T\mathbf{q}(0), d_2 = \frac{\mathbf{v}_2^T\mathbf{q}(0)}{\sin\left(\pi/2\right)} = \mathbf{v}_2^T\mathbf{q}(0)$$

Next compute $\mathbf{q}(t)$ from (4.92) and multiply by $M^{1/2}$ to get $\mathbf{x}(t)$ or use (4.103) directly to get

$$\mathbf{q}(t) = d_1\cos(\omega_1 t)\mathbf{v}_1 + d_2\cos(\omega_2 t)\mathbf{v}_2 = \cos(\omega_1 t)\mathbf{v}_1^T\mathbf{q}(0)\mathbf{v}_1 + \cos(\omega_2 t)\mathbf{v}_1^T\mathbf{q}(0)\mathbf{v}_1$$

$$= \cos(5.551t)\begin{bmatrix} 0.0054 \\ 0.0184 \end{bmatrix} + \cos(24.170t)\begin{bmatrix} -0.0054 \\ 0.0016 \end{bmatrix}$$

Note that as a check, substitute $t = 0$ in this last line to recover the correct initial condition $\mathbf{q}(0)$. Next transform the solution back to the physical coordinates

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = \cos(5.551t)\begin{bmatrix} 0.0054 \\ 0.0092 \end{bmatrix} + \cos(24.170t)\begin{bmatrix} -0.0054 \\ 0.0008 \end{bmatrix} \text{ m}$$

4.55 For a zero value of an eigenvalue and hence frequency, what is the corresponding time response? Or asked another way, the form of the modal solution for a non-zero frequency is $A\sin(\omega_n t + \phi)$, what is the form of the modal solution that corresponds to a zero frequency? Evaluate the constants of integration if the modal initial conditions are: $r_1(0) = 0.1$, and $\dot{r}_1(0) = 0.01$.

Solution: A zero eigenvalue corresponds to the modal equation:

$$\ddot{r}_1(t) = 0 \Rightarrow r_1(t) = a + bt$$

Applying the given initial conditions:

$$r_1(0) = a + b(0) = 0.1 \Rightarrow a = 0.1$$
$$\dot{r}_1(0) = b = 0.01$$
$$\Rightarrow r_1(t) = 0.1 + 0.01t$$