Problems and Solutions for Section 4.5 (4.56 through 4.66)

4.56 Consider the example of the automobile drive train system discussed in Problem 4.44. Add 10% modal damping to each coordinate, calculate and plot the system response.

Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness. From Problem 4.44, the equation of motion with damping is

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix}\ddot{\mathbf{x}} + 10{,}000\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s}$$

Other calculations from Problem 4.44 yield:

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$
$$\lambda_2 = 77.951 \quad \omega_2 = 8.8290 \text{ rad/s}$$
$$\lambda_3 = 362.05 \quad \omega_3 = 19.028 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.1537 \\ 0.1775 \\ 0.9721 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.8803 \\ -0.4222 \\ 0.2163 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4488 \\ -0.8890 \\ 0.0913 \end{bmatrix}$$

Use the summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \mathbf{0}$$
$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 54.7723 \end{bmatrix}^T$$

Also, $\zeta_1 = \zeta_2 = \zeta_3 = 0.1$.

$$\omega_{d2} = 8.7848 \text{ rad/s}$$
$$\omega_{d3} = 18.932 \text{ rad/s}$$

The solution is given by

$$\mathbf{q}(t) = (c_1 + c_2 t)\mathbf{v}_1 + \sum_{2}^{3} d_i e^{-\zeta_i \omega_i t}\sin(\omega_{di}t + \phi_i)\mathbf{v}_i$$

where $\phi_i = \tan^{-1}\left(\dfrac{\omega_{di}\mathbf{v}_i^T\mathbf{q}(0)}{\mathbf{v}_i^T\dot{\mathbf{q}}(0) + \zeta_i\omega_i\mathbf{v}_i^T\mathbf{q}(0)}\right) \quad i = 2,3$ Eq. (4.114)

$$d_i = \frac{\mathbf{v}_i^T\dot{\mathbf{q}}(0)}{\omega_{di}\cos\phi_i - \zeta_i\omega_i\sin\phi_i} \qquad i = 2,3$$

Thus,

$$\phi_2 = \phi_3 = 0$$
$$d_2 = 1.3485$$
$$d_3 = 0.2642$$

Now,

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} d_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_2\mathbf{v}_1 + \sum_{i=2}^{3} \left[-\zeta_i\omega_i d_i \sin\phi_i + \omega_{di} d_i \cos\phi_i\right]\mathbf{v}_i$$

Pre-multiply by $\mathbf{v}_1^T$:

$$\mathbf{v}_1^T\mathbf{q}(0) = 0 = c_1$$

$$\mathbf{v}_1^T\dot{\mathbf{q}}(0) = 53.2414 = c_2$$

So,

$$\mathbf{q}(t) = 53.2414\mathbf{v}_1 - 1.3485e^{-0.8829t}\sin(8.7848t)\mathbf{v}_2 + 0.2648te^{-1.9028t}\sin(18.932t)\mathbf{v}_3$$

The solution is given by

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$$

$$\mathbf{x}(t) = 0.9449t\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}-0.1371\\-0.05693\\0.005325\end{bmatrix}e^{-0.8829t}\sin(8.7848t) + \begin{bmatrix}0.01369\\-0.002349\\0.0004407\end{bmatrix}e^{-1.9028t}\sin(18.932t)\ \text{m}$$

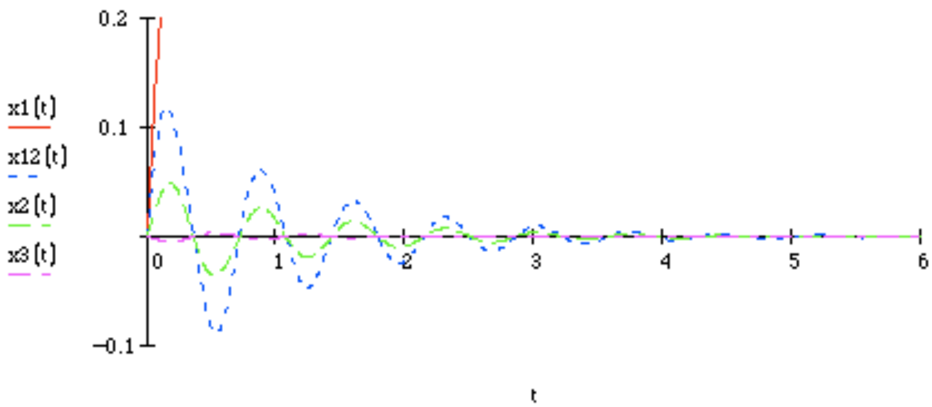
The following Mathcad session illustrates the solution without the rigid body mode (except for x_1 which shows both with and without the rigid mode)

$$x1(t) := 0.9449\cdot t + 0.1371\cdot e^{-.8829\cdot t}\cdot\sin(8.7848\cdot t) + 0.01369\cdot e^{-1.9028\cdot t}\cdot\sin(18.932\cdot t)$$

$$x12(t) := \left(0.1371\cdot e^{-.8829\cdot t}\cdot\sin(8.7848\cdot t) + 0.01369\cdot e^{-1.9028\cdot t}\cdot\sin(18.932\cdot t)\right)$$

$$x2(t) := 0.05693\cdot e^{-.8829\cdot t}\cdot\sin(8.7848\cdot t) - 0.002349\cdot e^{-1.9028\cdot t}\cdot\sin(18.932\cdot t)$$

$$x3(t) := -0.005325\cdot e^{-.8829\cdot t}\cdot\sin(8.7848\cdot t) + 0.000447\cdot e^{-1.9028\cdot t}\cdot\sin(18.932\cdot t)$$



The read solid line is the first mode with the rigid body mode included.

4.57 Consider the model of an airplane discussed in problem 4.47, Figure P4.46. (a) Resolve the problem assuming that the damping provided by the wing rotation is $\zeta_i = 0.01$ in each mode and recalculate the response. (b) If the aircraft is in flight, the damping forces may increase dramatically to $\zeta_i = 0.1$. Recalculate the response and compare it to the more lightly damped case of part (a).

Solution:

From Problem 4.47, with damping

$$\begin{bmatrix} 3000 & 0 & 0 \\ 0 & 12,000 & 0 \\ 0 & 0 & 3,000 \end{bmatrix}\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + \begin{bmatrix} 13455 & -13455 & 0 \\ -13,455 & 26910 & -13,455 \\ 0 & -13,455 & 13,455 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0.02 & 0 & 0 \end{bmatrix}^T \text{ m}$$
$$\dot{\mathbf{x}}(0) = \mathbf{0}$$
$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$
$$\lambda_2 = 4.485 \quad \omega_2 = 2.118 \text{ rad/s}$$
$$\lambda_3 = 6.727 \quad \omega_3 = 2.594 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.4082 \\ -0.8165 \\ -0.4082 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

The solution is given by

$$\mathbf{q}(t) = (c_1 + c_2 t)\mathbf{v}_1 + \sum_{i=2}^{3} d_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} t + \phi_i)\mathbf{v}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_{di}\mathbf{v}_i^T\mathbf{q}(0)}{\mathbf{v}_i^T\dot{\mathbf{q}}(0) + \zeta_i\omega_i\mathbf{v}_i^T\mathbf{q}(0)}\right) \qquad i = 2,3$$

(Eq. (4.114))

$$d_i = \frac{\mathbf{v}_i^T\mathbf{q}(0)}{\sin\phi_i} \qquad i = 2,3$$

Now,

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} d_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_2\mathbf{v}_1 + \sum_{i=2}^{3}\left[-\zeta_i\omega_i d_i \sin\phi_i + \omega_{di} d_i \cos\phi_i\right]\mathbf{v}_i$$

Premultiply by $\mathbf{v}_1^T$:

$$\mathbf{v}_1^T\mathbf{q}(0) = 4.4721 = c_1$$
$$\mathbf{v}_1^T\dot{\mathbf{q}}(0) = 0 = c_2$$

(a) $\zeta_1 = \zeta_2 = \zeta_3 = 0.01$

$$\omega_{d2} = 2.1177 \text{ rad/s}, \quad \omega_{d3} = 2.593 \text{ rad/s}$$
$$\phi_2 = -1.5808 \text{ rad}, \quad \phi_3 = 1.5608 \text{ rad}$$
$$d_2 = 7.7464, \quad d_3 = 6.3249$$

Mode shapes:

$$\mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$
$$\mathbf{u}_1 = \begin{bmatrix} -0.007454 \\ -0.007454 \\ -0.007454 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.01291 \\ 0 \\ -0.01291 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.01054 \\ -0.005270 \\ 0.01054 \end{bmatrix}$$

The solution is given by

$$\mathbf{x}(t) = (c_1 + c_2 t)\mathbf{u}_1 + \sum_{i=2}^{3} d_i e^{-\zeta_i \omega_i t}\sin(\omega_{di}t + \phi_i)\mathbf{u}_i$$

$$x(t) = 0.0333\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.100 \\ 0 \\ 0.100 \end{bmatrix} e^{-0.0212t}\sin(2.1178t - 1.5808)$$
$$+ \begin{bmatrix} 0.0667 \\ -0.0333 \\ 0.0677 \end{bmatrix} e^{-0.0259t}\sin(2.5937t + 1.5608)$$

b) $\zeta_1 = \zeta_2 = \zeta_3 = 0.1$

Same thing as part (a), but now the following values are obtained

$$\omega_{d2} = 2.1072 \text{ rad/sec} \quad \omega_{d3} = 2.5807 \text{ rad/sec}$$
$$\phi_2 = -1.6710 \text{ rad} \quad \phi_3 = 1.4706 \text{ rad}$$
$$d_2 = 7.7850 \quad d_3 = 6.3564$$

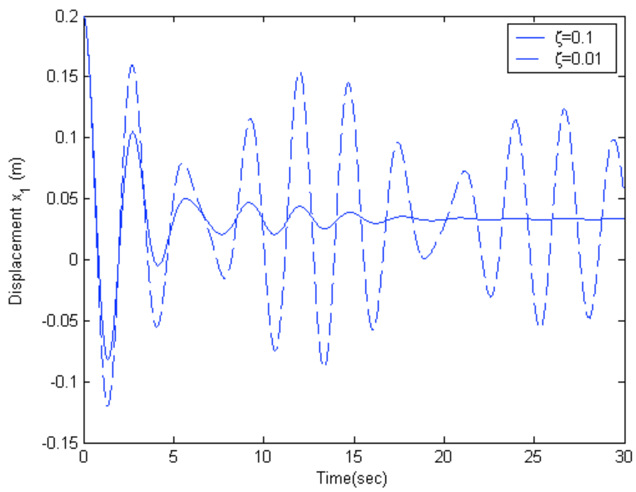
Notice that the rigid mode is not effected by changing the damping ratio, and hence

$c = 4.4721$

Consequently, the solution becomes

$$x(t) = 0.0333\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}-0.1005\\0\\0.1005\end{bmatrix}e^{-0.2118t}\sin(2.1072t - 1.6710)$$
$$+\begin{bmatrix}0.0670\\-0.0335\\0.0670\end{bmatrix}e^{-0.2594t}\sin(2.5807t + 1.4706)$$

Below is the plot of the displacement of the left wing



4.58 Repeat the floor vibration problem of Problem 4.50 using modal damping ratios of

$$\zeta_1 = 0.01 \quad \zeta_2 = 0.1 \quad \zeta_3 = 0.2$$

Solution: The equation of motion will be of the form:

$$200\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + 3.197\times10^{-4}\begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 & 0.05 & 0 \end{bmatrix}^T \text{ m and } \dot{\mathbf{x}}(0) = \mathbf{0}.$$

$$M^{-1/2} = 0.7071$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 2.2482 & 2.6645 & 1.0825 \\ 2.6645 & 5.3291 & 2.6645 \\ 1.0825 & 2.6645 & 2.2482 \end{bmatrix}\times10^{-7}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255\times10^{-7}\lambda^2 + 1.3645\times10^{-13}\lambda - 4.1382\times10^{-21} = 0$$

$$\lambda_1 = 4.3142\times10^{-8} \quad \omega_1 = 2.0771\times10^{-4} \text{ rad/s}$$

$$\lambda_2 = 1.1657\times10^{-7} \quad \omega_2 = 3.34143\times10^{-4} \text{ rad/s}$$

$$\lambda_3 = 8.2283\times10^{-7} \quad \omega_3 = 9.0710\times10^{-4} \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.5604 \\ -0.6098 \\ 0.5604 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{bmatrix}$$

Use the mode summation method to find the solution. First transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 0.7071 & 0 \end{bmatrix}^T$$

$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.115):

$$\mathbf{x}(t) = \sum_{i=1}^{3} d_i e^{-\zeta_i\omega_i t}\sin\left(\omega_{di}t + \phi_i\right)\mathbf{u}_i$$

where $$\phi_i = \tan^{-1}\left(\frac{\omega_{di}\mathbf{v}_i^T\mathbf{q}(0)}{\mathbf{v}_i^T\dot{\mathbf{q}}(0) + \zeta_i\omega_i\mathbf{v}_i^T\mathbf{q}(0)}\right) \quad i = 1,2,3$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}'(0)}{\sin\phi_i} \qquad i = 1,2,3, \quad \mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

$$\zeta_1 = 0.01, \quad \zeta_2 = 0.1, \qquad \zeta_3 = 0.2$$

Substituting

$$\omega_{d1} = 2.0770 \times 10^{-4} \text{ rad/s}, \; \omega_{\text{d2}} = 3.3972 \times 10^{-4} \text{ rad/s} \;\; \omega_{\text{d3}} = 8.8877 \times 10^{-4} \text{ rad/s}$$

$$\phi_1 = 1.5808 \text{ rad}, \quad \phi_2 = 1.6710 \text{ rad}, \quad \phi_3 = 1.3694 \text{ rad}$$

$$d_1 = 0.4312, \quad d_2 = 0, \qquad d_3 = 0.5720$$

The mode shapes are

$$\mathbf{u}_1 = \begin{bmatrix} 0.03963 \\ -0.04312 \\ 0.03963 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -0.05 \\ 0 \\ 0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.03049 \\ 0.05604 \\ 0.03049 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0.01709 \\ -0.01859 \\ 0.01709 \end{bmatrix} e^{-2.0771\times10^{-4}\,t} \sin\left(2.0770 \times 10^{-4} t - 1.5808\right)$$

$$+ \begin{bmatrix} 0.01744 \\ 0.03206 \\ 0.01744 \end{bmatrix} e^{-2.0771\times10^{-4}\,t} \sin\left(8.8877 \times 10^{-4} t + 1.3694\right) \text{m}$$

4.59 Repeat Problem 4.58 with constant modal damping of ζ_1, ζ_2, ζ_3 = 0.1 and compare this with the solution of Problem 4.58.

Solution: Use the equations of motion and initial conditions from Problem 4.58. The mode shapes, natural frequencies and transformed initial conditions remain the same. However the constants of integration are effected by the damping ratio so the solution

$$\mathbf{x}(t) = \sum_{i=1}^{3} d_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} t + \phi_i)\mathbf{u}_i$$

has new constants determined by $\phi_i = \tan^{-1}\left(\dfrac{\omega_{di}\mathbf{v}_i^T\mathbf{q}(0)}{\mathbf{v}_i^T\dot{\mathbf{q}}(0) + \zeta_i\omega_i\mathbf{v}_i^T\mathbf{q}(0)}\right) \quad i = 1,2,3$

$$d_i = \frac{\mathbf{v}_i^T\mathbf{q}'(0)}{\sin\phi_i} \qquad i = 1,2,3$$

$$\mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

$$\zeta_1 = \zeta_2 = \zeta_3 = 0.1$$

Substituting yields

$$\omega_{d1} = 2.0667\times10^{-4} \text{ rad/s}, \ \ \omega_{d2} = 3.3972\times10^{-4} \text{ rad/s}, \ \ \omega_{d3} = 9.0255\times10^{-4} \text{ rad/s}$$

$$\phi_1 = -1.6710 \text{ rad}, \ \ \phi_2 = -1.6710 \text{ rad}, \ \ \phi_3 = 1.4706 \text{ rad}$$

$$d_1 = 0.4334, \quad d_2 = 0.0, \quad d_3 = 0.5633$$

Mode shapes:

$$\mathbf{u}_1 = \begin{bmatrix} 0.03963 \\ -0.04312 \\ 0.03963 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -0.05 \\ 0 \\ 0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.03049 \\ 0.05604 \\ 0.03049 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0.01717 \\ -0.01869 \\ 0.01717 \end{bmatrix} e^{-2.0771\times10^{-4}t}\sin(2.0667\times10^{-4}t - 1.6710)$$
$$+ \begin{bmatrix} 0.01717 \\ 0.03157 \\ 0.01717 \end{bmatrix} e^{-9.0710\times10^{-4}t}\sin(9.0255\times10^{-4}t + 1.4706)\text{m}$$

The primary difference between problems 4.58 and 4.59 is the settling time; the responses in Problem 4.59 decay faster than those of Problem 4.58.

4.60 Consider the damped system of Figure P4.1. Determine the damping matrix and use the formula of Eq. (4.119) to determine values of the damping coefficient c_I for which this system would be proportionally damped.

Solution:

From Fig. 4.29,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

From Eq. (4.119)

$$C = \alpha M + \beta K$$

$$\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} \alpha m_1 + \beta\left(k_1 + k_2\right) & -\beta k_2 \\ -\beta k_2 & \alpha m_1 + \beta\left(k_2 + k_3\right) \end{bmatrix}$$

To be proportionally damped,

$$\begin{aligned} c_2 &= \beta k_2 \\ c_1 &= \alpha m_1 + \beta k_1 \\ c_3 &= \alpha m_2 + \beta k_3 \end{aligned}$$

Alternately, compute $KM^{-1}C$ symbolically and show that the condition for symmetry:

$$\begin{bmatrix} k1 + k2 & -k2 \\ -k2 & k2 + k3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{m1} & 0 \\ 0 & \frac{1}{m2} \end{bmatrix} \cdot \begin{bmatrix} c1 + c2 & -c2 \\ -c2 & c2 + c3 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{(m2 \cdot k1 \cdot c1 + m2 \cdot k1 \cdot c2 + m2 \cdot k2 \cdot c1 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1)}{(m1 \cdot m2)} & \dfrac{-(m2 \cdot k1 \cdot c2 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + k2 \cdot m1 \cdot c3)}{(m1 \cdot m2)} \\ \dfrac{-(m2 \cdot k2 \cdot c1 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + c2 \cdot m1 \cdot k3)}{(m1 \cdot m2)} & \dfrac{(m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + k2 \cdot m1 \cdot c3 + c2 \cdot m1 \cdot k3 + m1 \cdot k3 \cdot c3)}{(m1 \cdot m2)} \end{bmatrix}$$

Requiring the off diagonal elements to be equal enforces symmetry. This requires

$$m_1 k_2 c_3 = m_2 k_2 c_1 + (m_2 k_1 - m_1 k_3) c_2$$

4.61 Let $k_3 = 0$ in Problem 4.60. Also let $m_1 = 1, m_2 = 4, k_1 = 2, k_2 = 1$ and calculate c_1, c_2 and c_3 such that $\zeta_1 = 0.01$ and $\zeta_2 = 0.1$.

Solution:

From Figure P4.1 the equation of motion is,

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_1 = 0.1619 \quad \omega_1 = 0.4024 \text{ rad/s}$$

$$\lambda_2 = 3.0881 \quad \omega_2 = 1.7573 \text{ rad/s}$$

From Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega}{2}$$

So, $0.01 = \dfrac{\alpha}{2(0.4024)} + \dfrac{\beta(0.4024)}{2}$

and $0.1 = \dfrac{\alpha}{2(1.7573)} + \dfrac{\beta(1.7573)}{2}$

Solving for α and β yields

$$\alpha = -0.01096$$

$$\beta = 0.1174$$

From Eq. (4.119),

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \alpha M + \beta K = \begin{bmatrix} 0.3411 & -0.1174 \\ -0.1174 & 0.07354 \end{bmatrix}$$

Thus,

$$c_1 = 0.2238$$

$$c_2 = 0.1174$$

$$c_3 = -0.04382$$

Since negative damping is not usually possible, this design would not work.

4.62 Calculate the constants α and β for the two-degree-of-freedom system of Problem 4.29 such that the system has modal damping of $\zeta_1 = \zeta_2 = 0.3$.

Solution:

From Problem 4.29 with proportional damping added,

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + (\alpha M + \beta K)\dot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$
$$\det(\tilde{K} - \lambda I) = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_1 = 0.1619 \quad \omega_1 = 0.4024 \text{ rad/s}$$
$$\lambda_2 = 3.0881 \quad \omega_2 = 1.7573 \text{ rad/s}$$

From Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}$$

So, $0.3 = \dfrac{\alpha}{2(0.4024)} + \dfrac{\beta(0.4024)}{2}$

and $0.3 = \dfrac{\alpha}{2(1,7573)} + \dfrac{\beta(1.7573)}{2}$

Solving for α and β yields

$$\alpha = 0.1966$$
$$\beta = 0.2778$$

4.63 Equation (4.124) represents *n* equations in only two unknowns and hence cannot be used to specify all the modal damping ratios for a system with $n > 2$. If the floor vibration system of Problem 4.51 has measured damping of $\zeta_1 = 0.01$ and $\zeta_2 = 0.05$, determine ζ_3.

Solution:

From Problem 4.51

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255\times10^{-7}\lambda^2 + 1.3645\times10^{-14}\lambda - 4.1382\times10^{-22} = 0$$
$$\lambda_1 = 4.3142\times10^{-9} \quad \omega_1 = 2.0771\times10^{-5} \text{ rad/s}$$
$$\lambda_2 = 1.1657\times10^{-7} \quad \omega_2 = 3.4143\times10^{-4} \text{ rad/s}$$
$$\lambda_3 = 8.2283\times10^{-7} \quad \omega_3 = 9.0710\times10^{-4} \text{ rad/s}$$

Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}$$

Since the problem contains three modes only, and since the first and second modal damping ratios are give as $\zeta_1 = 0.01$ and $\zeta_2 = 0.05$ then the following linear system can be set up

$$\frac{\alpha}{2\left(2.0771\times10^{-5}\right)} + \frac{\beta\left(2.0771\times10^{-5}\right)}{2} = 0.01$$
$$\frac{\alpha}{2\left(3.4143\times10^{-4}\right)} + \frac{\beta\left(3.4143\times10^{-4}\right)}{2} = 0.05$$

which can be solve to yield $\alpha = 2.9\times10^{-7}$ and $\beta = 290.397$. Hence, the modal damping of the third mode can be obtained using 4.124

$$\zeta_3 = \frac{\alpha}{2\omega_3} + \frac{\beta\omega_3}{2} = 0.132$$

4.64 Does the following system decouple? If so, calculate the mode shapes and write the equation in decoupled form.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Solution:

The system will decouple if

$$C = \alpha M + \beta K$$
$$\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} \alpha + 5\beta & -\beta \\ -\beta & \alpha + \beta \end{bmatrix}$$

Clearly the off-diagonal terms require

$$\beta = 3$$

Therefore, the diagonal terms require

$$5 = \alpha + 15$$
$$3 = \alpha + 3$$

These yield different values of α, so the system does not decouple. An easier approach is to compute $CM^{-1}K$ to see if it is symmetric:

$$CM^{-1}K = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -12 & 6 \end{bmatrix}$$

Since this is not symmetric, the system cannot be decoupled.

4.65 Calculate the damping matrix for the system of Problem 4.63. What are the units of the elements of the damping matrix?

Solution:

From Problem 4.58,

$$\alpha = -8.8925 \times 10^{-7}$$
$$\beta = 3.0052 \times 10^{2}$$

From Problem 4.48

$$M = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 200 \end{bmatrix}$$

$$K = 3.197 \times 10^{-4} \begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}$$

So,

$$C = \alpha M + \beta K$$

$$C = \begin{bmatrix} 0.01334 & 0.01602 & 0.006506 \\ 0.01602 & 0.03025 & 0.01602 \\ 0.006506 & 0.01602 & 0.01334 \end{bmatrix}$$

The units are kg/s

4.66 Show that if the damping matrix satisfies $C = \alpha M + \beta K$, then the matrix $CM^{-1}K$ is symmetric and hence that $CM^{-1}K = KM^{-1}C$.

Solution: Compute the product $CM^{-1}K$ where C has the form: $C = \alpha M + \beta K$.

$$CM^{-1} = (\alpha M + \beta K)M^{-1} = \alpha I + \beta KM^{-1} \Rightarrow CM^{-1}K = \alpha K + \beta KM^{-1}K$$
$$KM^{-1}C = KM^{-1}(\alpha M + \beta K) = \alpha K + \beta KM^{-1}K$$
$$\Rightarrow KM^{-1}C = CM^{-1}K$$