**Problems and Solutions for Section 4.6 (4.67 through 4.76)**

**4.67** Calculate the response of the system of Figure 4.16 discussed in Example 4.6.1 if $F_1(t) = \delta(t)$ and the initial conditions are set to zero. This might correspond to a two-degree-of-freedom model of a car hitting a bump.

**Solution:** From example 4.6.1, with $F_1(t) = \delta(t)$, the modal equations are

$$\ddot{r}_1 + 0.2\dot{r}_1 + 2r_1 = 0.7071\delta(t)$$

$$\ddot{r}_2 + 0.4\dot{r}_2 + 4r_2 = 0.7071\delta(t)$$

Also from the example,

$$\omega_{n1} = \sqrt{2}\ \text{rad/s} \qquad \zeta_1 = 0.07071 \qquad \omega_{\mathrm{d}1} = 1.4106\ \text{rad/s}$$

$$\omega_{n2} = 2\ \text{rad/s} \qquad \zeta_2 = 0.1 \qquad \omega_{\mathrm{d}2} = 1.9899\ \text{rad/s}$$

The solution to an impulse is given by equations (3.7) and (3.8):

$$r_i(t) = \frac{\hat{F}}{m_i\omega_{di}} e^{-\zeta_i\omega_{ni}t} \sin\omega_{di}t$$

This yields

$$\mathbf{r}(t) = \begin{bmatrix} 0.5012e^{-0.1t}\sin 1.4106t \\ 0.3553e^{-0.2t}\sin 1.9899t \end{bmatrix}$$

The solution in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t) = \begin{bmatrix} .2357 & -.2357 \\ .7071 & .7071 \end{bmatrix}\begin{bmatrix} 0.167e^{-0.1t}\sin 1.4106t \\ -0.118e^{-.02t}\sin 1.9899t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.0394e^{-0.1t}\sin 1.4106t + 0.0279e^{-0.2t}\sin 1.9899t \\ 0.118e^{-0.1t}\sin 1.4106t - 0.0834e^{-0.2t}\sin 1.9899t \end{bmatrix}$$

**4.68** For an undamped two-degree-of-freedom system, show that resonance occurs at one or both of the system's natural frequencies.

**Solution:**

Undamped two-degree-of-freedom system:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{F}(t)$$

Let $\mathbf{F}(t) = \begin{bmatrix} F_1(t) \\ 0 \end{bmatrix}$

Note: placing $F_1$ on mass 1 is one way to do this. A second force could be placed on mass 2 with or without $F_1$.

Proceeding through modal analysis,

$$I\ddot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Or,

$$\begin{aligned} \ddot{r}_1 + \omega_1^2 r_1 &= b_1 F_1(t) \\ \ddot{r}_2 + \omega_2^2 r_2 &= b_2 F_1(t) \end{aligned}$$

where $b_1$ and $b_2$ are constants from the matrix $P^T M^{-1/2}$.

If $F_1(t) = a \cos \omega t$ and $\omega = \omega_1$ then the solution for $r_1$ is (from Section 2.1),

$$r_1(t) = \frac{\dot{r}_{10}}{\omega_1}\sin\omega_1 t + r_{10}\cos\omega_1 t + \frac{b_1 a}{2\omega_1} t \sin\omega_1 t$$

The solution for $r_2$ is

$$r_2(t) = \frac{\dot{r}_{20}}{\omega_2}\sin\omega_2 t + \left( r_{20} - \frac{b_2 a}{\omega_2^2 - \omega_1^2} \right)\cos\omega_2 t + \frac{b_2 a}{\omega_2^2 - \omega_1^2} t \sin\omega_1 t$$

If the initial conditions are zero,

$$r_1(t) = \frac{b_1 a}{2\omega_1} t \sin \omega_1 t$$

$$r_2(t) = \frac{b_2 a}{\omega_2^2 - \omega_1^2}\left(\cos \omega_1 t - \cos \omega_2 t\right)$$

Converting to physical coordinates $X(t) = M^{-1/2}Pr(t)$ yields

$$x_1(t) = c_1 r_1(t) + c_2 r_2(t)$$

$$x_2(t) = c_3 r_1(t) + c_4 r_2(t)$$

where $c_i$ is a constant from $M^{-1/2}P$.

So, if the driving force contains just one natural frequency, both masses will be excited at resonance. The driving force could contain the other natural frequency ($\omega = \omega_{n2}$), which would cause $r_1$ and $r_2$ to be

$$r_1(t) = \frac{b_1 a}{\omega_1^2 - \omega_2^2}\left(\cos \omega_2 t - \cos \omega_1 t\right)$$

$$r_2(t) = \frac{b_2 a}{2\omega_2} t \sin \omega_2 t$$

and

$$x_1(t) = c_1 r_1(t) + c_2 r_2(t)$$

$$x_2(t) = c_3 r_1(t) + c_4 r_2(t)$$

so both masses still oscillate at resonance.

Also, if $F_1(t) = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t$ where $\omega_1 = \omega_{n1}$ and $\omega_2 = \omega_{n2}$, then both $r_1$ and $r_2$ would be at resonance, so $x_1(t)$ and $x_2(t)$ would also be at resonance.

**4.69** Use modal analysis to calculate the response of the drive train system of Problem 4.44 to a unit impulse on the car body (i.e., and location $q_3$). Use the modal damping of Problem 4.56. Calculate the solution in terms of physical coordinates, and after subtracting the rigid-body modes, compare the responses of each part.

**Solution:**

Let $k_1$ = hub stiffness and $k_2$ = axle and suspension stiffness.

From Problems 41 and 51,

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix}\ddot{\mathbf{q}} + 10{,}000\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}\mathbf{q} = \mathbf{0}$$

$$M^{-1/2} = \begin{bmatrix} .1155 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & .0183 \end{bmatrix}$$

$$P = \begin{bmatrix} .1537 & -.8803 & .4488 \\ .1775 & -.4222 & -.88910 \\ .9721 & .2163 & .0913 \end{bmatrix}$$

$$\lambda_1 = 0 \qquad \omega_{n1} = 0 \text{ rad/s}$$
$$\lambda_2 = 77.951 \qquad \omega_{n2} = 8.8290 \text{ rad/s}$$
$$\lambda_3 = 362.05 \qquad \omega_{n3} = 19.028 \text{ rad/s}$$

The initial conditions are **0**.

Also

$$\zeta_1 = \zeta_2 = \zeta_3 = .1$$
$$\omega_{d1} = 8.7848 \text{ rad/s}$$
$$\omega_{d2} = 18.932 \text{ rad/s}$$

From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} .01775 \\ .003949 \\ .001668 \end{bmatrix}\delta(t)$$

The modal equations are

$$\begin{aligned} &\ddot{r}_1 = .01775\delta(t) \\ &\ddot{r}_2 + 1.7658\dot{r}_2 + 77.951r_2 = .003949\delta(t) \\ &\ddot{r}_3 + 3.8055\dot{r}_3 + 362.05r_3 = .001668\delta(t) \end{aligned}$$

The solution for $r_1$ is

$$r_1(t) = .01775t$$

The solutions for $r_2$ and $r_3$ are given by equations 3.7 and (3.8)

$$r_i(t) = \frac{\hat{F}}{m_i\omega_{di}} e^{-\zeta_i\omega_i t} \sin\omega_{di}t$$

This yields

$$\begin{aligned} &r_2(t) = 4.4949\times10^{-4} e^{-.8829t} \sin 8.7848t \\ &r_3(t) = 8.8083\times10^{-5} e^{-1.9028t} \sin 18.932t \end{aligned}$$

The solution in physical coordinates is

$$\begin{aligned} &\mathbf{q}(t) = M^{-1/2}P\mathbf{r}(t) \\ &\mathbf{q}(t) = 3.1496\times10^{-4}t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4.5691\times10^{-5} \\ -1.8978\times10^{-5} \\ 1.7749\times10^{-6} \end{bmatrix} e^{-.8829t} \sin 8.7848t \\ &\qquad + \begin{bmatrix} 4.5647\times10^{-6} \\ -7.8301\times10^{-6} \\ 1.4689\times10^{-7} \end{bmatrix} e^{-1.9028t} \sin 18.932t \ \text{ m} \end{aligned}$$

The magnitude of the components is much smaller than that in problem 51, but they do oscillate at the same frequencies.

**4.70** Consider the machine tool of Figure 4.28. Resolve Ex. 4.8.3 if the floor mass $m$ = 1000 kg, is subject to a force of 10 sin$t$ (in Newtons). Calculate the response. How much does this floor vibration affect the machine's toolhead?

**Solution:**

From example 4.8.3, with $F_3(t)$ = 10 sin$t$ N and $m_3$ = 1000 kg.

$$\left(10^3\right)\begin{bmatrix} .4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \left(10^4\right)\begin{bmatrix} 30 & -30 & 0 \\ -30 & 38 & -8 \\ 0 & -8 & 88 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 10\sin t \end{bmatrix}$$

Calculating the eigenvalues and eigenvectors yields

$$\begin{aligned} \lambda_1 &= 29.980 & \omega_1 &= 5.4761 \text{ rad/s} \\ \lambda_2 &= 868.2743 & \omega_2 &= 29.4665 \text{ rad/s} \\ \lambda_3 &= 921.7378 & \omega_3 &= 30.3601 \text{ rad/s} \end{aligned}$$

And

$$P = \begin{bmatrix} -.4215 & .4989 & .7573 \\ -.9048 & -.1759 & -.3877 \\ -.0602 & -.8486 & .5255 \end{bmatrix}$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -.01904 \\ -.2684 \\ .1662 \end{bmatrix}\sin t$$

Undamped modal equations:

$$\begin{aligned} \ddot{r}_1 + 29.9880 r_1 &= -.01904\sin t \\ \ddot{r}_2 + 868.2743 r_2 &= -.2684\sin t \\ \ddot{r}_3 + 921.7378 r_3 &= .1662\sin t \end{aligned}$$

Inserting the damping terms,

$$\zeta_1 = .1 \qquad 2\zeta_1\omega_1 = 1.0952$$
$$\zeta_2 = .01 \qquad 2\zeta_2\omega_2 = .5893$$
$$\zeta_3 = .05 \qquad 2\zeta_3\omega_3 = 3.0360$$
$$\ddot{r}_1 + 1.0952\dot{r}_1 + 29.9880 r_1 = -.01904 \sin t$$
$$\ddot{r}_2 + .5893\dot{r}_2 + 868.2734 r_2 = -.2684 \sin t$$
$$\ddot{r}_3 + 3.0360\dot{r}_3 + 921.7378 r_3 = .1662 \sin t$$

The damped natural frequencies are

$$\omega_{d1} = \omega_{n1}\sqrt{1-\zeta_1^2} = 5.4487 \text{ rad/s}$$
$$\omega_{d2} = \omega_{n2}\sqrt{1-\zeta_2^2} = 29.4650 \text{ rad/s}$$
$$\omega_{d3} = \omega_{n3}\sqrt{1-\zeta_3^2} = 30.3222 \text{ rad/s}$$

The general solution is

$$r_i(t) = A_i e^{-\zeta_i\omega_{ni}t}\sin(\omega_{di}t - \theta_i) + A_{0i}\sin(\omega t - \phi_i)$$

where

$$A_{0i} = \frac{f_{0i}}{\sqrt{\left(\omega_{ni}^2 - \omega^2\right)^2 + \left(2\zeta_i\omega_{ni}\omega\right)^2}} \quad \text{and} \quad \phi_i = \tan^{-1}\left(\frac{2\zeta_i\omega_{ni}\omega}{\omega_{ni}^2 - \omega}\right)$$

Inserting values,

$$A_{01} = -6.5643 \times 10^{-4} \text{ m} \qquad \phi_1 = 3.7764 \times 10^{-2} \text{ rad}$$
$$A_{02} = -3.0943 \times 10^{-4} \text{ m} \qquad \phi_2 = 6.7952 \times 10^{-4} \text{ rad}$$
$$A_{03} = 1.8049 \times 10^{-4} \text{ m} \qquad \phi_3 = 3.2974 \times 10^{-3} \text{ rad}$$

So,

$$r_1(t) = A_1 e^{-.5476t}\sin(5.4487t - \theta_1) - 6.543 \times 10^{-4}\sin(t - 3.7764 \times 10^{-2})$$
$$r_2(t) = A_2 e^{-.2947t}\sin(29.4650t - \theta_2) - 3.0943 \times 10^{-4}\sin(t - 6.7952 \times 10^{-4})$$
$$r_3(t) = A_3 e^{-1.5180t}\sin(30.3222t - \theta_3) + 1.8049 \times 10^{-4}\sin(t - 3.2974 \times 10^{-3})$$

With zero initial conditions:

$$A_1 = 1.2047\times 10^{-4} \ \text{m} \qquad \theta_1 = .2072 \ \text{rad}$$
$$A_2 = 1.0502\times 10^{-5} \ \text{m} \qquad \theta_2 = .02002 \ \text{rad}$$
$$A_3 = -5.9524\times 10^{-6} \ \text{m} \qquad \theta_3 = .1002 \ \text{rad}$$

Now,

$$r_1(t) = 1.2047\times 10^{-4} e^{-.5476t}\sin(5.4487t - .2027) - 6.543\times 10^{-4}\sin(t - 3.7764\times 10^{-2})$$
$$r_2(t) = 1.0502\times 10^{-5} e^{-.2947t}\sin(29.4650t - .02002) - 3.0943\times 10^{-4}\sin(t - 6.7952\times 10^{-4})$$
$$r_3(t) = -5.9524\times 10^{-6} e^{-1.5180t}\sin(30.3222t - .1002) + 1.8049\times 10^{-4}\sin(t - 3.2974\times 10^{-3})$$

Convert to physical coordinates:

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t) = \begin{bmatrix} -.02108 & .02494 & .03786 \\ -.02023 & -.003993 & -.008670 \\ -.001904 & -.02684 & .01662 \end{bmatrix}\mathbf{r}(t)$$

Therefore

$$x_1(t) = -.02108r_1 + .02494r_2 + .03786r_3$$
$$x_2(t) = -.02023r_1 - .003933r_2 - .008670r_3$$
$$x_3(t) = -.001904r_1 - .02684r_2 + .01662r_3$$

**4.71** Consider the airplane of Figure P4.46 with damping as described in Problem 4.57 with $\zeta_1 = 0.1$. Suppose that the airplane hits a gust of wind, which applies an impulse of $3\delta(t)$ at the end of the left wing and $\delta(t)$ at the end of the right wing. Calculate the resulting vibration of the cabin [$x_2(t)$].

**Solution:** From Problems 4.46 and 4.57

$$M^{-1/2} = \begin{bmatrix} .01826 & 0 & 0 \\ 0 & .009129 & 0 \\ 0 & 0 & .01826 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.4082 & -0.7071 & 0.5774 \\ 0.8165 & 0 & -0.5774 \\ 0.4082 & 0.7071 & 0.5774 \end{bmatrix}$$

$$\lambda_1 = 0 \qquad \omega_{n1} = 0 \text{ rad/s}$$

$$\lambda_2 = 4.485 \qquad \omega_{n2} = 2.118 \text{ rad/s}$$

$$\lambda_3 = 6.727 \qquad \omega_{n3} = 2.594 \text{ rad/s}$$

Also:

$$\zeta_1 = \zeta_2 = \zeta_3 = 0.1$$

$$\mathbf{F}(t) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \delta(t)$$

$$\omega_{d1} = 0 \text{ rad/s}, \ \ \omega_{d2} = 2.1072 \text{ rad/s}, \ \omega_{d3} = 2.5807 \text{ rad/s}$$

From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -0.0298 \\ 0.0258 \\ 0.0422 \end{bmatrix} \delta(t)$$

The modal equations are

$$\ddot{r}_1 = -0.02981\delta(t)$$

$$\ddot{r}_2 + 0.424\dot{r}_2 + 4.485 r_2 = 0.0258\delta(t)$$

$$\ddot{r}_3 + 0.519\dot{r}_3 + 6.727 r_3 = 0.0422\delta(t)$$

The solution for $r_1$ is

$$r_1(t) = -0.02981t$$

The solutions for $r_2$ and $r_3$ are given by equations (3.7) and (3.8)

$$r_i(t) = \frac{\hat{F}}{m_i \omega_{di}} e^{-\zeta_i \omega_i t} \sin \omega_{di} t$$

This yields

$$r_2(t) = 1.2253 \times 10^{-2} e^{-0.212t} \sin 2.107t$$
$$r_3(t) = 1.6338 \times 10^{-2} e^{-0.259t} \sin 2.581t$$

The solution in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2} P \mathbf{r}(t)$$

For $x_2$:

$$x_2(t) = 2.221 \times 10^{-4} t + 8.06 \times 10^{-5} e^{-0.259t} \sin 2.581t$$

**4.72** Consider again the airplane of Figure P4.46 with the modal damping model of Problem 4.57 ($\zeta_i = 0.1$). Suppose that this is a propeller-driven airplane with an internal combustion engine mounted in the nose. At a cruising speed the engine mounts transmit an applied force to the cabin mass (4m at $x_2$) which is harmonic of the form 50 sin 10*t*. Calculate the effect of this harmonic disturbance at the nose and on the wind tips after subtracting out the translational or rigid motion.

**Solution:** From Problems 4.47 and 4.57

$$M^{-1/2} = \begin{bmatrix} .01826 & 0 & 0 \\ 0 & .009129 & 0 \\ 0 & 0 & .01826 \end{bmatrix}, \quad P = \begin{bmatrix} -.4082 & .7071 & .5774 \\ -.8165 & 0 & -.5774 \\ -.4082 & -.7071 & .5774 \end{bmatrix}$$

$$\lambda_1 = 0 \qquad \omega_{n1} = 0 \text{ rad/s}$$
$$\lambda_2 = 17.94 \qquad \omega_{n2} = 4.2356 \text{ rad/s}$$
$$\lambda_3 = 26.91 \qquad \omega_{n3} = 5.1875 \text{ rad/s}$$

Also,

$$\zeta_1 = \zeta_2 = \zeta_3 = 0.1, \Rightarrow \omega_{d1} = 0 \text{ rad/s}, \;\; \omega_{d2} = 4.2143 \text{ rad/s}, \;\; \omega_{d3} = 5.1615 \text{ rad/s}$$

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 50\sin 10t \\ 0 \end{bmatrix}$$

The initial conditions are **0**. From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -.3727 \\ 0 \\ -.2635 \end{bmatrix} \sin 10t$$

The modal equations are

$$\ddot{r}_1 = -.3727\sin 10t$$
$$\ddot{r}_2 + .8471\dot{r}_2 + 17.94 r_2 = 0$$
$$\ddot{r}_3 + 1.0375\dot{r}_3 + 26.91 r_3 = -.2635\sin 10t$$

The solutions are

$$r_1(t) = .003727\sin 10$$
$$r_2(t) = 0$$
$$r_3(t) = -.006915e^{-.5188t}\sin(5.1615t + .0726) + .003569\sin(10t + .141)$$

The solutions in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t)$$

The wing tips are $x_1$ and $x_3$, so

$$\begin{aligned} x_1(t) = x_3(t) = {} & 2.7780\times 10^{-5}\sin 10t - 7.2891\times 10^{-5}e^{-.5188t}\sin(5.1615t + .0726) \\ & + 3.7621\times 10^{-5}\sin(10t + .141) \end{aligned}$$

**4.73** Consider the automobile model of Problem 4.14 illustrated in Figure P4.14. Add modal damping to this model of $\zeta_1 = 0.01$ and $\zeta_2 = 0.2$ and calculate the response of the body [$x_2(t)$] to a harmonic input at the second mass of 10 sin3$t$ N.

**Solution:** From problem 4.14

$$M = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix},\ K = \begin{bmatrix} 1000 & -1000 \\ -1000 & 11000 \end{bmatrix},\ P = \begin{bmatrix} .9999 & -.1044 \\ .1044 & .9999 \end{bmatrix}$$

$\lambda_1 = 0.4545 \qquad \omega_1 = 0.6741$ rad/s, and $\lambda_2 = 220.05 \qquad \omega_2 = 14.834$ rad/s

Also,

$$\zeta_1 = .01,\ \ \zeta_2 = 0.2,\ \ \omega_{d1} = 0.6741 \text{ rad/s},\ \ \omega_{d2} = 14.534 \text{ rad/s}$$

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 10\sin 3t \end{bmatrix}$$

The initial conditions are all **0**. From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} 0.02036 \\ 1.4141 \end{bmatrix}\sin 3t$$

The modal equations are

$$\ddot{r}_1 + 0.01348\dot{r}_1 + 0.454r_1 = 0.02036\sin 3t$$

$$\ddot{r}_2 + 5.9336\dot{r}_2 + 220.046r_2 = 1.4141\sin 3t$$

The solutions are

$$r_1(t) = -0.1088e^{-0.006741t}\sin(0.6741t + 1.0914\times 10^{-4}) + .002445\sin(3t - .004857)$$

$$r_2(t) = -0.07500e^{-2.9668t}\sin(14.534t + 1.3087) + .07586\sin(3t + 1.26947)$$

The solutions in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t)$$

The response of the body is

$$\begin{aligned} x_1(t) = &-.002433e^{-0.006741t}\sin(.6471t - 1.0914\times 10^{-4}) \\ &+ 5.4665\times 10^{-5}\sin(3t - .004857) \\ &+ 2.4153\times 10^{-5}e^{-2.9668t}\sin(14.534t - 1.3087) \\ &- 2.4430\times 10^{-5}\sin(3t + 1.2694) \end{aligned}$$

**4.74** Determine the *modal equations* for the following system and comment on whether or not the system will experience resonance.

$$\ddot{\mathbf{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(0.618t)$$

**Solution:** Here $M = I$ so that the eigenvectors and mode shapes are the same. Computing the natural frequencies from $\det(-\omega^2 I + K) = 0$ yields:

$$\omega_1 = 0.618 \text{ rad/s and } \omega_2 = 1.681 \text{ rad/s}$$

Next solve for the mode shapes and normalize them to get

$$P = \begin{bmatrix} 0.526 & -0.851 \\ 0.851 & 0.526 \end{bmatrix}, \quad \text{so that } P^T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.526 \\ -0.851 \end{bmatrix}$$

The modal equations then become:

$$\ddot{r}_1 + (0.618)^2 r_1 = \ddot{r}_1 + 0.3819 r_1 = 0.526\sin(0.618t)$$

$$\ddot{r}_2 + (1.618)^2 r_2 = \ddot{r}_2 + 2.6179 r_2 = -0.851\sin(0.618t)$$

The driving frequency is equal to the natural frequency of mode one so the system exhibits resonance.

**4.75** Consider the following system and compute the solution using the mode summation method.

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \; K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}, \; \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

**Solution:** From Example 4.2.4

$$M^{\frac{1}{2}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, M^{-\frac{1}{2}} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \text{ Also } \omega_1 = \sqrt{2}, \omega_2 = 2 \text{ rad/s}$$

$$\text{Appropriate IC are } \mathbf{q}_0 = M^{\frac{1}{2}}\mathbf{x}_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \; \dot{\mathbf{q}}_0 = M^{\frac{1}{2}}\mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\phi_i = \tan^{-1}\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)} = \tan^{-1}\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{0} \Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2}/2 \\ 3\sqrt{2}/2 \end{bmatrix}$$

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \frac{3\sqrt{2}}{2}\sin\left(\sqrt{2}t + \frac{\pi}{2}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3\sqrt{2}}{2}\sin\left(2t + \frac{\pi}{2}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} \tfrac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} \tfrac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} 1/3 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}\cos\left(\sqrt{2}t\right) + \frac{1}{2}\cos(2t) \\ \frac{3}{2}\cos\left(\sqrt{2}t\right) - \frac{3}{2}\cos\left(2t\right) \end{bmatrix}$$