Problems and Solutions for Section 4.7 (4.76 through 4.79)

4.76 Use Lagrange's equation to derive the equations of motion of the lathe of Fig. 4.21 for the undamped case.

Solution: Let the generalized coordinates be θ_1, θ_2 and θ_3.

The kinetic energy is

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} J_3 \dot{\theta}_3^2$$

The potential energy is

$$U = \frac{1}{2} k_1 \left(\theta_2 - \theta_2\right)^2 + \frac{1}{2} k_2 \left(\theta_3 - \theta_2\right)^2$$

There is a nonconservative moment *M(t)* on inertia 3. The Lagrangian is

$$L = T - U = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} J_3 \dot{\theta}_3^2 - \frac{1}{2} k_1 \left(\theta_2 - \theta_1\right)^2 - \frac{1}{2} k_2 \left(\theta_3 - \theta_2\right)^2$$

Calculate the derivatives from Eq. (4.136):

$$\frac{\partial L}{\partial \dot{\theta}_1} = J_1 \dot{\theta}_1 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = J_1 \ddot{\theta}_1$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = J_2 \dot{\theta}_2 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = J_2 \ddot{\theta}_2$$

$$\frac{\partial L}{\partial \dot{\theta}_3} = J_3 \dot{\theta}_3 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) = J_3 \ddot{\theta}_3$$

$$\frac{\partial L}{\partial \theta_1} = -k_1 \theta_1 + k_1 \theta_2$$

$$\frac{\partial L}{\partial \theta_2} = -k_1 \theta_1 - \left(k_1 + k_2\right) \theta_2 + k_2 \theta_3$$

$$\frac{\partial L}{\partial \theta_3} = -k_2 \theta_2 - k_2 \theta_3$$

Using Eq. (4.136) yields

$$J_1 \ddot{\theta}_1 + k_1 \theta_1 - k_2 \theta_2 = 0$$

$$J_2 \ddot{\theta}_2 - k_1 \theta_1 + \left(k_1 + k_2\right) \theta_2 - k_2 \theta_3 = 0$$

$$J_3 \ddot{\theta}_3 - k_2 \theta_2 + k_2 \theta_3 = M\left(t\right)$$

In matrix form this yields

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \ddot{\theta} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \theta = \begin{bmatrix} 0 \\ 0 \\ M(t) \end{bmatrix}$$

4.77 Use Lagrange's equations to rederive the equations of motion for the automobile of Example 4.8.2 illustrated in Figure 4.25 for the case $c_1 = c_2 = 0$.

Solution: Let the generalized coordinates be x and θ.
The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1\left(x - l_1\theta\right)^2 + \frac{1}{2}k_2\left(x + l_2\theta\right)^2$$

The Lagrangian is

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}k_1\left(x - l_1\theta\right)^2 - \frac{1}{2}k_2\left(x + l_2\theta\right)^2$$

Calculate the derivatives from Eq. (4.136):

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial \dot{\theta}} = J\dot{\theta} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = J\ddot{\theta}$$

$$\frac{\partial L}{\partial x} = -\left(k_1 + k_2\right)x + \left(k_1 l_1 - k_1 l_2\right)\theta$$

$$\frac{\partial L}{\partial \theta} = \left(k_1 l_1 - k_2 l_2\right)x - \left(k_1 l_2^2\right)\theta$$

Using Eq. (4.136) yields

$$m\ddot{x} + \left(k_1 + k_2\right)x + \left(k_1 l_1 - k_2 l_2\right)\theta = 0$$

$$J\ddot{\theta} + \left(k_1 l_2 - k_1 l_1\right)x - \left(k_1 l_1^2 + k_2 l_2^2\right)\theta = 0$$

In matrix form this yields

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \mathbf{0}$$

4.78 Use Lagrange's equations to rederive the equations of motion for the building model presented in Fig. 4.9 of Ex. 4.4.3 for the undamped case.

Solution:

Let the generalized coordinates be x_1, x_2, x_3 and x_4.
The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}m_4\dot{x}_4^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left(x_2 - x_1\right)^2 + \frac{1}{2}k_3\left(x_3 - x_2\right)^2 + \frac{1}{2}k_4\left(x_4 - x_3\right)^2$$

The Lagrangian is

$$\begin{aligned} L = T - U = &\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 + \frac{1}{2}m\dot{x}_4^2 \\ &- \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2\left(x_2 - x_1\right)^2 \frac{1}{2}k_3\left(x_3 - x_2\right)^2 - \frac{1}{2}k_4\left(x_4 - x_3\right)^2 \end{aligned}$$

Calculate the derivatives from Eq. (4.136):

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_1} &= m_1\dot{x}_1 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1 \\ \frac{\partial L}{\partial \dot{x}_1} &= m_2\dot{x}_2 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_2\ddot{x}_2 \\ \frac{\partial L}{\partial \dot{x}_1} &= m_3\dot{x}_3 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_3\ddot{x}_3 \\ \frac{\partial L}{\partial \dot{x}_1} &= m_4\dot{x}_4 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_4\ddot{x}_4 \end{aligned}$$

$$\frac{\partial L}{\partial x_1} = -\left(k_1 + k_2\right)x_1 + k_2 x_2$$
$$\frac{\partial L}{\partial x_2} = k_2 x_1 - \left(k_2 + k_3\right)x_2 + k_3 x_3$$
$$\frac{\partial L}{\partial x_3} = k_2 x_2 - \left(k_2 + k_4\right)x_3 - k_4 x_4$$
$$\frac{\partial L}{\partial x_4} = k_4 x_3 - k_4 x_4$$

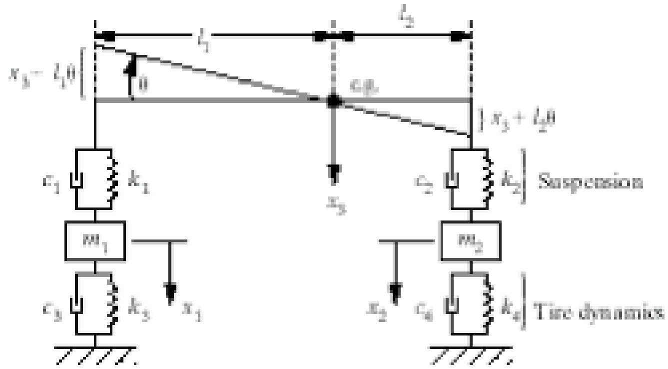
Using Eq. (4.136) yields

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 - k_3 x_3 = 0$$
$$m_3\ddot{x}_3 - k_3 x_2 + \left(k_3 + k_4\right)x_3 - k_4 x_4 = 0$$
$$m_4\ddot{x}_4 - k_4 x_3 + k_4 x_4 = 0$$

In matrix form this yields

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

4.79 Consider again the model of the vibration of an automobile of Fig. 4.25. In this case include the tire dynamics as indicated in Fig. P4.79. Derive the equations of motion using Lagrange formulation for the undamped case. Let m_3 denote the mass of the car acting at c.g.



Solution:

Let the generalized coordinates be x_1, x_2, x_3 and θ. The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}J\dot{\theta}^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1\left(x_3 - \ell_1\theta - x_1\right)^2 + \frac{1}{2}k_2(x_3 - \ell_2\theta - x_2)^2 + \frac{1}{2}k_3x_1^2 + \frac{1}{2}k_4x_2^2$$

The Lagrangian is thus:

$$L = T - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}k_1\left(x_3 - l_1\theta - x_1\right)^2$$
$$-\frac{1}{2}k_2\left(x_3 + l_2\theta - x_2\right)^2 - \frac{1}{2}k_3x_1^2 - \frac{1}{2}k_4x_2^2$$

Calculate the derivatives indicated in Eq. (4.146):

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$
$$\frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_1\ddot{x}_2$$
$$\frac{\partial L}{\partial \dot{x}_3} = m_3\dot{x}_3 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = m_3\ddot{x}_3$$
$$\frac{\partial L}{\partial \theta} = J\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{\theta}}\right) = J\ddot{\theta}$$

$$\frac{\partial L}{\partial x_1} = -\left(k_1 + k_3\right)x_1 + k_1 x_3 - k_1 l_1 \theta$$
$$\frac{\partial L}{\partial x_2} = -\left(k_2 + k_4\right)x_2 + k_2 x_3 - k_2 l_2 \theta$$
$$\frac{\partial L}{\partial x_3} = k_1 x_1 + k_2 x_2 - \left(k_1 + k_2\right)x_3 + \left(k_1 l_1 + k_2 l_2\right)\theta$$
$$\frac{\partial L}{\partial \theta} = -k_1 l_1 x_1 - k_2 l_2 x_2 + \left(k_1 l_1 + k_2 l_2\right)x_3 - \left(k_1 l_1^2 + k_2 l_2^2\right)\theta$$

Using Eq. (4.146) yields

$$m_1 \ddot{x}_1 + \left(k_3 + k_1\right)x_1 - k_1 x_3 + k_1 l_1 \theta = 0$$
$$m_2 \ddot{x}_2 + \left(k_4 + k_2\right)x_2 - k_2 x_3 - k_2 l_2 \theta = 0$$
$$m_3 \ddot{x}_3 - k_1 x_1 - k_2 x_2 + \left(k_1 + k_2\right)x_3 - \left(k_1 l_1 - k_2 l_2\right)\theta = 0$$
$$J\ddot{\theta} + k_1 l_1 x_1 - k_2 l_2 x_2 - \left(k_1 l_1 - k_2 l_2\right)x_3 + \left(k_1 l_1^2 + k_2 l_2^2\right) = 0$$

in matrix form

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \left(k_3 + k_1\right) & 0 & -k_1 & k_1 l_1 \\ 0 & \left(k_4 + k_2\right) & -k_2 & k_2 l_2 \\ -k_1 & -k_2 & \left(k_1 + k_2\right) & -\left(k_2 l_2 + k_1 l_1\right) \\ k_1 l_1 & k_2 l_2 & -\left(k_2 l_2 + k_1 l_1\right) & \left(k_1 l_1^2 + k_2 l_2^2\right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \theta \end{Bmatrix} = 0$$