Problems and Solutions for Section 4.9 (4.80 through 4.90)

4.80 Consider the mass matrix

$$M = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix}$$

and calculate M^{-1}, $M^{-1/2}$, and the Cholesky factor of M. Show that

$$LL^T = M$$
$$M^{-1/2}M^{-1/2} = I$$
$$M^{1/2}M^{1/2} = M$$

Solution: Given

$$M = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix}$$

The matrix, P, of eigenvectors is

$$P = \begin{bmatrix} -0.1091 & -0.9940 \\ -0.9940 & 0.1091 \end{bmatrix}$$

The eigenvalues of M are

$$\lambda_1 = 0.8902$$
$$\lambda_2 = 10.1098$$

From Equation

$$M^{-1} = P\text{diag}\left[\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right]P^T, \quad M^{-1} = \begin{bmatrix} 0.1111 & 0.1111 \\ 0.1111 & 1.1111 \end{bmatrix}$$

From Equation

$$M^{-1/2} = V diag\left[\lambda_1^{-1/2}, \lambda_2^{-1/2}\right]V^T$$

$$M^{-1/2} = \begin{bmatrix} 0.3234 & 0.0808 \\ 0.0808 & 1.0510 \end{bmatrix}$$

The following Mathcad session computes the Cholesky decomposition.

$$M := \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 0.11111 & 0.11111 \\ 0.11111 & 1.11111 \end{bmatrix}$$

$$L := \text{cholesky}(M)$$

$$L = \begin{bmatrix} 3.16228 & 0 \\ -0.31623 & 0.94868 \end{bmatrix} \qquad L \cdot L^T = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix} \qquad L^{-1} \cdot M \cdot \left(L^T\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.81 Consider the matrix and vector

$$A = \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \qquad b = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

use a code to solve $Ax = b$ for $\varepsilon = 0.1, 0.01, 0.001, 10^{-6}$, and 1.

Solution:

The equation is

$$\begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \mathbf{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

The following Mathcad session illustrates the effect of ε on the solution, a entire integer difference. Note that no solution exists for the case ε = 1.

$$b := \begin{bmatrix} 10 \\ 10 \end{bmatrix} \qquad A(\varepsilon) := \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \qquad x(\varepsilon) := A(\varepsilon)^{-1} \cdot b$$

$$x(0.1) = \begin{bmatrix} 22.222 \\ 122.222 \end{bmatrix} \qquad x(0.01) = \begin{bmatrix} 20.202 \\ 1.02 \cdot 10^{3} \end{bmatrix} \qquad x(0.001) = \begin{bmatrix} 20.02 \\ 1.002 \cdot 10^{4} \end{bmatrix}$$

$$x\left(10^{-6}\right) = \begin{bmatrix} 20 \\ 1 \cdot 10^{7} \end{bmatrix}$$

So the solution to this problem is very sensitive, and ill conditioned, because of the inverse.

4.82 Calculate the natural frequencies and mode shapes of the system of Example 4.8.3. Use the undamped equation and the form given by equation (4.161).

Solution:

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.161).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 –30 0;-30 38 –8;0 –8 88]1e4;
[u, d]=eig(k,m);
w=sqrt (d);
```

The matrix `d` contains the square of the natural frequencies, and the matrix `u` contains the corresponding mode shapes.

4.83 Compute the natural frequencies and mode shapes of the undamped version of the system of Example 4.8.3 using the formulation of equation (4.164) and (4.168). Compare your answers.

Solution:

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.161).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 –30 0;-30 38 –8;0 –8 88]1e4;
mi=inv(m);
kt=mi*k;
[u, d]=eig(k,m);
w=sqrt (d);
```

The number of floating point operations needed is 439.
The matrix d contains the square of the natural frequencies, and the matrix u contains the corresponding mode shapes.

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.168).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 –30 0;-30 38 –8;0 –8 88]1e4;
msi=inv(sqrt(m));
kt=msi*k*msi;
[p, d]=eig(kt);
w=sqrt (d);
u=msi*p;
```

The number of floating point operations needed is 461.
The matrix d contains the square of the natural frequencies, and the matrix u contains the corresponding mode shapes.

The method of Equation (4.161) is faster.

4.84 Use a code to solve for the modal information of Example 4.1.5.

Solution: See Toolbox or use the following Mathcad code:

$$\omega := 1$$

Given

$$\left[\omega^4 - \left(6 \cdot \omega^2\right)\right] + 8 = 0$$

$$\text{Find}(\omega) = 1.414$$

$$\omega := 2$$

Given

$$\left[\omega^4 - \left(6 \cdot \omega^2\right)\right] + 8 = 0$$

$$\text{Find}(\omega) = 2$$

4.85 Write a program to perform the normalization of Example 4.4.2 (i.e., calculate α such that the vector $\alpha\mathbf{v}_1$ is normal).

Solution:

The following MATLAB program will perform the normalization of Example 4.4.2.

```
x=[.4450 .8019 1];
mag=sqrt(sum(x.^2));
xnorm=x/mag;
```

The variable `mag` is the same as α, and `xnorm` is the normalized vector. The original vector `x` can be any length.

4.86 Use a code to calculate the natural frequencies and mode shapes obtained for the system of Example 4.2.5 and Figure 4.4.

Solution: See Toolbox or use the following Mathcad code:

$$M := \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad K := \begin{bmatrix} 12 & -2 \\ -2 & 12 \end{bmatrix} \qquad Mr := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$Kd := Mr^{-1} \cdot K \cdot Mr^{-1} \qquad Kd = \begin{bmatrix} 12 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\lambda := \text{eigenvals}(Kd) \qquad \lambda = \begin{bmatrix} 12.11 \\ 2.89 \end{bmatrix}$$

$$\omega1 := \sqrt{\lambda_1} \qquad \omega2 := \sqrt{\lambda_0} \qquad \omega1 = 1.7 \qquad \omega2 = 3.48$$

$$v1 := \text{eigenvec}\left(Kd, \lambda_1\right) \qquad v2 := \text{eigenvec}\left(Kd, \lambda_0\right)$$

$$v1 = \begin{bmatrix} 0.109 \\ 0.994 \end{bmatrix} \qquad v2 = \begin{bmatrix} -0.994 \\ 0.109 \end{bmatrix} \qquad v1^T \cdot v2 = 0$$

$$v1^T \; v1 = 1 \qquad v2^T \cdot v2 = 1$$

$$P := \text{augment}(v1, v2) \qquad P^T \cdot Kd \cdot P = \begin{bmatrix} 2.89 & 0 \\ 0 & 12.11 \end{bmatrix} \qquad P^T \cdot P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.87 Following the modal analysis solution of Window 4.4, write a program to compute the time response of the system of Example 4.3.2.

Solution: The following MATLAB program will compute and plot the time response of the system of Example 4.3.2.

```
t=(0:.1:10)';

m=[1 0;0 4];
k=[12 –2;-2 12];
n=max(size(m));

x0=[1 1]';
xd0=[0 0]';

msi=inv(sqrtm(m));
kt=msi*k*msi;

[p, w]=eig(kt);
for i=1: n-1
 for j=1: n-I
     if w(j,j)>w(j+1,j+1)
         dummy=w(j,j);
         w(j,j)=w(j+1,j+1);
         w(j+1,j+1)=dummy;
         dummy=p(:,j);
         p(:,j)=p(:,j+1);
         p(:,j+1)=dummy;
     end
 end
end
pt=p';
s=msi*p;
si=pt*sqrtm(m);

r0=si*x0;
rd0=si*xd0;
r=[];
for i=1: n,
 wi=sqrt(w(i,i));
 rcol=(swrt((wi*r0(i))^2+rd0(i)^2/wi)*…
     sin(wi*t+atan2(wi*r0(i),rd0(i)));
 r(:,i)=rcol;
end
x=s*r;
plot(t,x);
end
```

4.88 Use a code to solve the damped vibration problem of Example 4.6.1 by calculating the natural frequencies, damping ratios, and mode shapes.

Solution: See Toolbox or use the following Mathcad code (all will do this)

$M := \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$ $K := \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$ $Mr := \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $C := \begin{bmatrix} 2.7 & -.3 \\ -.3 & 0.3 \end{bmatrix}$

$Kd := Mr^{-1} \cdot K \cdot Mr^{-1}$ $Kd = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ $Cd := Mr^{-1} \cdot \left(C \cdot Mr^{-1}\right)$

$Cd = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.3 \end{bmatrix}$

$\lambda := \text{eigenvals}(Kd)$ $\lambda = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\omega 1 := \sqrt{\lambda_1}$ $\omega 2 := \sqrt{\lambda_0}$ $\omega 1 = 1.414$ $\omega 2 = 2$

$v1 := \text{eigenvec}\left(Kd, \lambda_1\right)$ $v2 := \text{eigenvec}\left(Kd, \lambda_0\right)$

$v1 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$ $v2 = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}$ $v1^T \cdot v2 = 0$

$v1^T \ v1 = 1$ $v2^T \cdot v2 = 1$

$P := \text{augment}(v1, v2)$ $P^T \cdot Kd \cdot P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ $P^T \cdot P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$ $Cz := P^T \cdot Cd \cdot P$ $Cz = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$

$\zeta 1 := \dfrac{Cz_{0,0}}{2 \cdot \omega 1}$ $\zeta 1 = 0.071$ $\zeta 2 := \dfrac{Cz_{1,1}}{2 \cdot \omega 2}$ $\zeta 2 = 0.1$

$\omega d1 := \omega 1 \cdot \sqrt{1 - \zeta 1^2}$ $\omega d2 := \omega 2 \cdot \sqrt{1 - \zeta 2^2}$ $\omega d2 = 1.99$

$\omega d1 = 1.411$

4.89 Consider the vibration of the airplane of Problems 4.46 and 4.47 as given in Figure P4.46. The mass and stiffness matrices are given as

$$M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K = \frac{EI}{l^3}\begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

where $m = 3000$ kg, $l = 2$ m, $I = 5.2 \times 10^{-6}$ m^4, $E = 6.9 \times 10^9$ N/m^2, and the damping matrix C is taken to be $C = (0.002)K$. Calculate the natural frequencies, normalized mode shapes, and damping ratios.

Solution: Use the Toolbox or use a code directly such as the following Mathcad session:

$E := 6.9\cdot 10^9 \qquad I := 5.2\cdot 10^{-6} \qquad m := 3000 \qquad L := 2$

$$M := m\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K := \frac{E\cdot I}{L^3}\cdot\begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \qquad C := 0.002\cdot K$$

$$M = \begin{bmatrix} 3\cdot 10^3 & 0 & 0 \\ 0 & 1.2\cdot 10^4 & 0 \\ 0 & 0 & 3\cdot 10^3 \end{bmatrix} \qquad K = \begin{bmatrix} 1.346\cdot 10^4 & -1.346\cdot 10^4 & 0 \\ -1.346\cdot 10^4 & 2.691\cdot 10^4 & -1.346\cdot 10^4 \\ 0 & -1.346\cdot 10^4 & 1.346\cdot 10^4 \end{bmatrix}$$

$i := 0, 1 .. 2 \qquad j := 0, 1 .. 2$

$Mr_{i,j} := \sqrt{M_{i,j}}$

$$Kh := Mr^{-1}\cdot K\cdot Mr^{-1} \qquad Kh = \begin{bmatrix} 4.485 & -2.242 & 0 \\ -2.242 & 2.242 & -2.242 \\ 0 & -2.242 & 4.485 \end{bmatrix} \qquad Ch := Mr^{-1}\cdot\left(C\cdot Mr^{-1}\right)$$

$$\text{eigenvals}(Kh) = \begin{bmatrix} 6.727 \\ 4.485 \\ 0 \end{bmatrix} \qquad \lambda_1 := 0 \qquad \omega_1 := \sqrt{\lambda_1}$$

$\lambda_2 := 4.485 \quad \omega_2 := \sqrt{\lambda_2} \qquad \omega_2 = 2.118$

$\lambda_3 := 6.727 \quad \omega_3 := \sqrt{\lambda_3} \qquad \omega_3 = 2.594$

$v_1 := \mathrm{eigenvec}(Kh, \lambda_1) \qquad v_1 = \begin{bmatrix} 0.408 \\ 0.816 \\ 0.408 \end{bmatrix}$

$v_2 := \mathrm{eigenvec}(Kh, \lambda_2) \qquad v_2 = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix} \qquad v_3 := \mathrm{eigenvec}(Kh, \lambda_3) \qquad v_3 = \begin{bmatrix} 0.577 \\ -0.577 \\ 0.577 \end{bmatrix}$

$P1 := \mathrm{augment}(v_1, v_2) \quad P := \mathrm{augment}(P1, v_3) \qquad Ac := P^T \cdot Ch \cdot P$

$$Ac = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8.97 \cdot 10^{-3} & 0 \\ 0 & 0 & 0.013 \end{bmatrix} \qquad \zeta 2 := \frac{Ac_{1,1}}{2 \cdot \omega_2} \quad \zeta 2 = 2.118 \cdot 10^{-3}$$

$$\zeta 3 := \frac{Ac_{2,2}}{2 \cdot \omega_3} \qquad \zeta 3 = 2.594 \cdot 10^{-3}$$

The normalized mode shapes are

$$u1 := Mr^{-1} \cdot v_1 \qquad u1n := \frac{u1}{|u1|} \qquad u1n = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}$$

$$u2 := Mr^{-1} \cdot v_2 \qquad u2n := \frac{u2}{|u2|} \qquad u2n = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$u3 := Mr^{-1} \cdot v_3 \qquad u3n := \frac{u3}{|u3|} \qquad u3n = \begin{bmatrix} 0.667 \\ -0.333 \\ 0.667 \end{bmatrix}$$

4.90 Consider the proportionally damped, dynamically coupled system given by

$$M = \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 49 & -2 \\ -2 & 2 \end{bmatrix}$$

and calculate the mode shapes, natural frequencies, and damping ratios.

Solution: Use the Toolbox or any of the codes. A Mathcad solution is shown:

$$M := \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix} \qquad K := \begin{bmatrix} 49 & -2 \\ -2 & 2 \end{bmatrix} \qquad C := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$L := \text{cholesky}(M)$$

$$Kh := L^{-1}\cdot K\cdot \left(L^T\right)^{-1} \quad Kh = \begin{bmatrix} 5.444 & 1.218 \\ 1.218 & 2.431 \end{bmatrix} \qquad Ch := L^{-1}\cdot\left[C\cdot\left(L^T\right)^{-1}\right]$$

$$Ch = \begin{bmatrix} 0.333 & -0.236 \\ -0.236 & 0.917 \end{bmatrix} \qquad \text{eigenvals}(Kh) = \begin{bmatrix} 5.875 \\ 2 \end{bmatrix}$$

$$\lambda_1 := 2 \qquad \omega_1 := \sqrt{\lambda_1} \qquad \omega_1 = 1.414 \qquad \lambda_2 := 5.875 \qquad \omega_2 := \sqrt{\lambda_2} \qquad \omega_2 = 2.424$$

$$v_1 := \text{eigenvec}(Kh, \lambda_1) \quad v_1 = \begin{bmatrix} -0.333 \\ 0.943 \end{bmatrix} \quad v_2 := \text{eigenvec}(Kh, \lambda_2) \quad v_2 = \begin{bmatrix} 0.943 \\ 0.333 \end{bmatrix}$$

$$P := \text{augment}(v_1, v_2) \qquad \Lambda c := P^T\cdot Ch\cdot P \qquad \Lambda c = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$\zeta 1 := \frac{\Lambda c_{0,0}}{2\cdot\omega_1} \qquad \zeta 1 = 0.354 \qquad \zeta 2 := \frac{\Lambda c_{1,1}}{2\cdot\omega_2} \qquad \zeta 2 = 0.052$$

Computing the mode shapes from the eigenvectors yields:

$$M := \begin{pmatrix} 9 & -1 \\ -1 & 1 \end{pmatrix} \qquad R := \text{cholesky}(M)$$

$$R^{-1} = \begin{pmatrix} 0.333 & 0 \\ 0.118 & 1.061 \end{pmatrix}$$

$$u1 := R^{-1} \cdot \begin{pmatrix} -0.333 \\ 0.943 \end{pmatrix} \quad u1 = \begin{pmatrix} -0.111 \\ 0.961 \end{pmatrix} \qquad u2 := R^{-1} \cdot \begin{pmatrix} 0.934 \\ 0.333 \end{pmatrix} \quad u2 = \begin{pmatrix} 0.311 \\ 0.463 \end{pmatrix}$$